Komiya, K.
Osaka J. Math.
38 (2001), 239–249

EQUIVARIANT K-THEORETIC EULER CLASSES AND MAPS OF REPRESENTATION SPHERES

Dedicated to Professor Fuichi Uchida on his 60th birthday

KATSUHIRO KOMIYA

(Received April 1, 1999)

0. Introduction

Let G be a compact Lie group, and U a unitary representation of G. The unit sphere SU of U is called a *representation sphere* of G. In this paper we study G-maps between finite dimensional representation spheres of G.

In Atiyah and Tall [2], Bartsch [3], tom Dieck and Petrie [4], Komiya [5], [6], Liulevicious [7], and Marzantowicz [8], the equivariant K-theory is successfully employed for the study of G-maps. In [2] and [4], that is employed for the study of degrees of G-maps between representation spheres. In [3], [5], [6] and [7], that is employed to obtain necessary conditions for the existence of G-maps. In [8], the equivariant K-theoretic Lefschetz number is defined.

The main tool in this paper is also the equivariant K-theory. We give the definitions of the Thom class $tU \in K_G(U)$ of U and the Euler class $eU \in K_G(\mathrm{pt}) = R(G)$, and then show that if there exists a G-map $f : SU \to SW$ between representation spheres SU and SW then $eW = z(f) \cdot eU$ for some element $z(f) \in R(G)$ (Theorem 1.2). Using this equality, we show that if G is connected then the degree of f is uniquely determined only by U and W (Theorem 4.1).

If G is compact abelian, then the degree of f is more explicitly discussed. Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ be the circle group of complex numbers with absolute value 1, and $\mathbb{Z}_n$ be the cyclic group of order n considered as a subgroup of S^1. For any integer i, let $V_i = \mathbb{C}$ be a complex representation of S^1 and $\mathbb{Z}_n$ given by $(z, v) \mapsto z^i v$ for $z \in S^1$ (or $\mathbb{Z}_n$) and $v \in V_i$. A compact abelian group G decomposes into a cartesian product

$$G = T^k \times \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_l},$$

where $T^k = S^1 \times \cdots \times S^1$, the cartesian product of k copies of S^1. Letting γ be a sequence $(a_1, \ldots, a_k, b_1, \ldots, b_l)$ of integers, denote by V_γ the tensor product

$$V_{a_1} \otimes \cdots \otimes V_{a_k} \otimes V_{b_1} \otimes \cdots \otimes V_{b_l},$$

which is considered as a representation of G. Let Γ be the set of sequences

$$\gamma = (a_1, \dots, a_k, b_1, \dots, b_l)$$

of integers a_i $(1 \le i \le k)$ and b_j with $0 \le b_j \le n_j - 1$ $(1 \le j \le l)$. The set $\{V_\gamma \mid \gamma \in \Gamma\}$ gives a complete set of irreducible unitary representations of G, and any unitary representation U of G decomposes into a direct sum

$$U = \bigoplus_{\gamma \in \Gamma} V_\gamma^{u(\gamma)},$$

where $u(\gamma)$ is a nonnegative integer and $V_\gamma^{u(\gamma)}$ denotes the direct sum of $u(\gamma)$ copies of V_γ.

Let

$$W = \bigoplus_{\gamma \in \Gamma} V_\gamma^{w(\gamma)}$$

be a second unitary representation of G with $\dim U = \dim W$. Let $|\gamma| = a_1 + \cdots + a_k + b_1 + \cdots + b_l$ for $\gamma = (a_1, \dots, a_k, b_1, \dots, b_l)$. If there exists a G-map $f : SU \to SW$, then we obtain

$$\deg f \cdot \prod_{\gamma \in \Gamma} |\gamma|^{u(\gamma)} = \prod_{\gamma \in \Gamma} |\gamma|^{w(\gamma)} + nr$$

for $n = \text{g.c.d.}\{n_1, \dots, n_l\}$ and some integer r (Theorem 4.2).

In Marzantowicz [9, Theorems 2.2, 2.5], he obtained the same kind of results as this for $G = T^k$ or $G = \mathbb{Z}_p^k (= \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p)$ with p prime. To obtain the result, he employed the Borel cohomology theory. In the Borel cohomology theory the Euler class resides in $H_G^*(\mathrm{pt}; \mathcal{K}) = H^*(BG; \mathcal{K})$. The choice of the coefficients $\mathcal{K}$ practically depends on G considered. For example, if $G = T^k$ then $\mathcal{K}$ is $\mathbb{Z}$, and if $G = \mathbb{Z}_p^k$ then $\mathcal{K}$ is $\mathbb{Z}_p$. This means that the Borel cohomological Euler classes are not so systematically treated as the equivariant K-theoretic Euler classes. In fact, [9] treated the cases of $G = T^k$ and $G = \mathbb{Z}_p^k$ separately, but in this paper we can simultaneously treat all of compact abelian groups as in Theorem 4.2.

As long as the author knows, the results about G-maps obtained by using the Borel cohomology theory are almost only for the cases of $G = T^k$ and $G = \mathbb{Z}_p^k$.

There seems one more advantage of the equivariant K-theoretic Euler classes. If $G = T^k$ for example, then a unitary representation U of G decomposes into a direct sum

$$U = \bigoplus_{\Gamma} (V_{a_1} \otimes \cdots \otimes V_{a_k})^{u(a_1, \dots, a_k)}$$

where the direct sum $\oplus_\Gamma$ is taken over all sequences $(a_1, \dots, a_k)$ of integers. Denote by $e'U$ the Borel cohomological Euler class of U. We see

$$eU = \prod_{\Gamma}(1 - x_1^{a_1} \cdots x_k^{a_k})^{u(a_1,\dots,a_k)} \quad \text{in} \quad R(T^k) = \mathbb{Z}[x_1, \dots, x_k]_L,$$
$$e'U = \prod_{\Gamma}(a_1x_1 + \cdots + a_kx_k)^{u(a_1,\dots,a_k)} \quad \text{in} \quad H^*(BT^k; \mathbb{Z}) = \mathbb{Z}[x_1, \dots, x_k],$$

where $\mathbb{Z}[\cdots]_L$ denotes the Laurent polynomial ring and $\mathbb{Z}[\cdots]$ the ordinary polynomial ring. The degree of eU (as a polynomial) is higher than that of $e'U$. This means that eU contains much more information than $e'U$.

In the final section of this paper we show that the K-theoretic Euler class distinguishes the isomorphism class of real representations of S^1 (Theorem 5.1). But for the Borel cohomology theory we easily have examples of two nonisomorphic representations U and W of S^1 with $e'U = e'W$.

1. Thom classes and Euler classes

Let G be a compact Lie group. The equivariant K-ring $K_G(X)$ of a *compact* G-space X is defined to be the Grothendieck ring of the isomorphism classes of complex G-vector bundles over X (see Atiyah [1], Segal [10]). If X has a distinguished base point x_0, then $\tilde{K}_G(X)$ is the kernel of the homomorphism $i^* : K_G(X) \to K_G(x_0)$ induced from the inclusion $i : x_0 \to X$. If X is noncompact but locally compact, then $K_G(X)$ is defined to be $\tilde{K}_G(X^+)$, where X^+ is the one-point compactification of X.

Given a complex G-vector bundle $E \to X$ over a compact G-space X, there is the Thom isomorphism $\Psi : K_G(X) \to K_G(E)$. In what follows we only consider the case in which X is the one-point space $\{\text{pt}\}$. In this case the total space of a G-vector bundle is a complex representation U of G, and the Thom isomorphism is an isomorphism $\Psi : R(G) \to K_G(U)$, since $K_G(\text{pt})$ is canonically identified with the complex representation ring $R(G)$ of G. The *Thom class* $tU \in K_G(U)$ of U is defined as $tU = \Psi(1)$, where $1 \in R(G)$ is the 1-dimensional trivial representation. The *Euler class* $eU \in R(G)$ is defined as $eU = s^*(tU)$, where $s^* : K_G(U) \to K_G(\text{pt}) = R(G)$ is the homomorphism induced from the zero section $s : \{\text{pt}\} \to U$. eU can also be given as $eU = \sum_i (-1)^i \Lambda^i U$, where $\Lambda^i U$ is the i-th exterior power of U. The Euler class is multiplicative, i.e., $e(U \oplus W) = eU \cdot eW$ for two complex representations U and W.

Let $f : SU \to SW$ be a G-map between representation spheres. The radial extension of f induces a homomorphism $f^* : K_G(W) \to K_G(U)$. Since $K_G(U)$ is a free $R(G)$-module over just one generator tU, there exists a unique element $z(f) \in R(G)$ such that $f^*(tW) = z(f) \cdot tU$.

We consider an element of $R(G)$ as a class function $G \to \mathbb{C}$, and observe $z(f)(1)$, where 1 is the identity element of G.

Theorem 1.1 (cf. Atiyah-Tall [2, IV. §1]). *Let $f : SU \to SW$ be a G-map between representation spheres of a compact Lie group G. Then*

$$z(f)(1) = \begin{cases} \deg f & \text{if} \quad \dim U = \dim W \\ 0 & \text{if} \quad \dim U \neq \dim W \end{cases}$$

where $\deg f$ is the Brouwer degree of f with respect to the canonical orientation of SU and SW induced from the complex structure.

Proof. Let DU denote the unit disk of U. Consider the following commutative diagram:

$$\begin{array}{ccc} K_G(W) = K_G(DW, SW) & \xrightarrow{f_1^*} & K_G(DU, SU) = K_G(U) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ K(DW, SW) & \xrightarrow{f_2^*} & K(DU, SU) \\ {\scriptstyle ch}\downarrow & & \downarrow{\scriptstyle ch} \\ \mathbb{Z} \cong H^{ev}(DW, SW; \mathbb{Z}) & \xrightarrow{f_3^*} & H^{ev}(DU, SU; \mathbb{Z}) \cong \mathbb{Z}, \end{array}$$

where f_i^* is the homomorphism induced from f, F is the homomorphism which forgets the G-action, ch is the Chern character which has its image in the cohomology with integer coefficients in this case, and $H^{ev}(\ \ )$ denotes the direct sum of even dimensional parts.

We see

$$\begin{aligned} ch \circ F \circ f_1^*(tW) &= ch \circ F(z(f) \cdot tU) \\ &= ch(z(f)(1) \cdot F(tU)) \\ &= z(f)(1), \end{aligned}$$

since $F(tU)$ is the canonical generator of $K(DU, SU) \cong \mathbb{Z}$, and on the other hand we also see

$$\begin{aligned} f_3^* \circ ch \circ F(tW) &= f_3^*(1) \\ &= \begin{cases} \deg f & \text{if} \quad \dim U = \dim W \\ 0 & \text{if} \quad \dim U \neq \dim W. \end{cases} \end{aligned}$$

By the commutativity of the diagram we can obtain the theorem. □

Given a G-map $f : SU \to SW$, there is a commutative diagram:

$$\begin{array}{ccc} K_G(W) & \xrightarrow{f^*} & K_G(U) \\ & {\scriptstyle s^*}\searrow \quad \swarrow{\scriptstyle s^*} & \\ & K_G(\mathrm{pt}) = R(G) & \end{array}$$

From this we see in $R(G)$,

$$eW = s^*(tW) = s^* f^*(tW) = z(f) \cdot eU.$$

Thus we obtain

Theorem 1.2. *If there exists a G-map $f : SU \to SW$ between reprensentation spheres of a compact Lie group G, we obtain in $R(G)$,*

$$eW = z(f) \cdot eU.$$

Corollary 1.3. *If $eU = 0$ and $eW \neq 0$, then there exists no G-map $SU \to SW$.*

The converse of Theorem 1.2 does not hold. We give an example in the following section. That shows that the answer to the question by Marzantowicz [9, Problem 2.6] is negative in K-theoretic version.

2. Representation rings of compact abelian groups

The complex representation ring $R(G)$ of a compact abelian group $G = T^k \times \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_l}$ is isomorphic to $\mathbb{Z}[x_1, \ldots, x_k, y_1, \ldots, y_l]_L / Y$, where $\mathbb{Z}[x_1, \ldots, x_k, y_1, \ldots, y_l]_L$ is the Laurent polynomial ring with indeterminates $x_1, \ldots, x_k, y_1, \ldots, y_l$, and Y is the ideal generated by $1 - y_j^{n_j}$ $(1 \leq j \leq l)$. For a Laurent polynomial

$$\varphi = \varphi(x_1, \ldots, x_k, y_1, \ldots, y_l) \in \mathbb{Z}[x_1, \ldots, x_k, y_1, \ldots, y_l]_L,$$

we denote by $[\varphi]$ the element in $\mathbb{Z}[x_1, \ldots, x_k, y_1, \ldots, y_l]_L / Y$ represented by φ. Through the isomorphism $R(G) \cong \mathbb{Z}[x_1, \ldots, x_k, y_1, \ldots, y_l]_L / Y$,

$$\begin{aligned} V_\gamma &= \left[x_1^{a_1} \cdots x_k^{a_k} y_1^{b_1} \cdots y_l^{b_l}\right], \\ eV_\gamma &= \left[1 - x_1^{a_1} \cdots x_k^{a_k} y_1^{b_1} \cdots y_l^{b_l}\right] \end{aligned}$$

for $\gamma = (a_1, \ldots, a_k, b_1, \ldots, b_l)$.

EXAMPLE 2.1 (in which the converse of Theorem 1.2 does not hold). Consider the group $\mathbb{Z}_2 \times \mathbb{Z}_2$, where $\mathbb{Z}_2 = \{\pm 1\}$ is the cyclic group of order 2. We see

$$R(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong \mathbb{Z}[x, y] / \left(1 - x^2, 1 - y^2\right)$$

where $(1-x^2, 1-y^2)$ is the ideal generated by $1-x^2$ and $1-y^2$. For $i, j \in \{0,1\}$, $V_{(i,j)}$ is the one-dimensional complex representation on which $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts in such a way that $((s,t), v) \mapsto s^i t^j v$ for $(s,t) \in \mathbb{Z}_2 \times \mathbb{Z}_2$, $v \in V_{(i,j)} = \mathbb{C}$. Let $U = V_{(1,0)} \oplus V_{(1,0)}$ and $W = V_{(1,0)} \oplus V_{(0,1)} \oplus V_{(1,1)}$. We see in $R(\mathbb{Z}_2 \times \mathbb{Z}_2)$

$$\begin{aligned} eU &= eV_{(1,0)} \cdot eV_{(1,0)} = [(1-x)^2] = [2(1-x)], \\ eW &= eV_{(1,0)} \cdot eV_{(0,1)} \cdot eV_{(1,1)} = [(1-x)(1-y)(1-xy)] = 0. \end{aligned}$$

Thus eU devides eW, but we can see that there can not exist a $\mathbb{Z}_2 \times \mathbb{Z}_2$-map $f : SU \to SW$. Assume that such $f : SU \to SW$ exists, and let $H_1 = \mathbb{Z}_2 \times \{1\}$, $H_2 = \{1\} \times \mathbb{Z}_2$. Restricting f to the fixed point set of the H_2-action, we obtain an H_1-map $f^{H_2} : S(U^{H_2}) \to S(W^{H_2})$. Since $U^{H_2} = V_{(1,0)} \oplus V_{(1,0)}$ and $W^{H_2} = V_{(1,0)}$, the existence of f^{H_2} contradicts the Borsuk-Ulam theorem.

Lemma 2.2. *Let $G = T^k \times \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_l}$, and let*

$$\varphi = \varphi(x_1, \ldots, x_k, y_1, \ldots, y_l) \in \mathbb{Z}[x_1, \ldots, x_k, y_1, \ldots, y_l]_L$$

be a representative of

$$z \in R(G) \cong \mathbb{Z}[x_1, \ldots, x_k, y_1, \ldots, y_l]_L / Y,$$

i.e., $z = [\varphi]$. Then

$$z(1) = \varphi(1, \cdots, 1, 1, \cdots, 1),$$

where 1 *in the left hand side is the identity element of G, and* 1*'s in the right hand side are the numerical* 1.

Proof. There are complex representations U and W of G such that $z = U - W$. Decompose U and W into the direct sums of irreducible representations,

$$U = \bigoplus_{\gamma \in \Gamma} V_\gamma^{u(\gamma)} \quad \text{and} \quad W = \bigoplus_{\gamma \in \Gamma} V_\gamma^{w(\gamma)},$$

and let

$$\begin{aligned} \varphi_1 &= \varphi_1(x_1, \ldots, x_k, y_1, \ldots, y_l) = \sum_{\gamma \in \Gamma} u(\gamma)(\boldsymbol{xy})^\gamma, \\ \varphi_2 &= \varphi_2(x_1, \ldots, x_k, y_1, \ldots, y_l) = \sum_{\gamma \in \Gamma} w(\gamma)(\boldsymbol{xy})^\gamma, \end{aligned}$$

where $(\boldsymbol{xy})^\gamma = x_1^{a_1} \cdots x_k^{a_k} y_1^{b_1} \cdots b_l^{b_l}$ for $\gamma = (a_1, \ldots, a_k, b_1, \ldots, b_l)$. Then $z = [\varphi_1] - [\varphi_2]$ in $\mathbb{Z}[x_1, \ldots, x_k, y_1, \ldots, y_l]_L / Y$, and

$$z(1) = \dim U - \dim W$$

$$= \varphi_1(1, \cdots, 1, 1, \cdots, 1) - \varphi_2(1, \cdots, 1, 1, \cdots, 1).$$

Since

$$\varphi_1 - \varphi_2 = \varphi + \sum_{j=1}^{l} \psi_j \cdot \left(1 - y_j^{n_j}\right)$$

for some $\psi_j \in \mathbb{Z}[x_1, \ldots, x_k, y_1, \ldots, y_l]_L$, we obtain $z(1) = \varphi(1, \cdots, 1, 1, \cdots, 1)$. □

3. Nonvanishing of Euler classess

In this section we give a necessary and sufficient condition for the nonvanishing of Euler classes of representations of G. We first consider the case $G = \mathbb{Z}_n$. The set $\{V_i \mid 0 \leq i \leq n-1\}$ gives a complete set of irreducible complex representations of $\mathbb{Z}_n$, and $R(\mathbb{Z}_n)$ is isomorphic to $\mathbb{Z}[x]/(1-x^n)$. If U is a complex representation of $\mathbb{Z}_n$ decomposed into a direct sum $\oplus_{i=0}^{n-1} V_i^{u_i}$ for nonnegative integers u_i, then we see

$$eU = \prod_{i=0}^{n-1} (eV_i)^{u_i} = \left[\prod_{i=0}^{n-1} (1 - x^i)^{u_i}\right]. \tag{3.1}$$

Lemma 3.2. *For a complex representation $U = \oplus_{i=0}^{n-1} V_i^{u_i}$ of $\mathbb{Z}_n$, $eU \neq 0$ in $R(\mathbb{Z}_n)$ if and only if $U^{\mathbb{Z}_n} = \{0\}$, i.e., $u_0 = 0$.*

Proof. From (3.1) we easily see that $eU \neq 0$ implies $u_0 = 0$. Conversely assume $u_0 = 0$. Then

$$eU = \left[\prod_{i=1}^{n-1} (1 - x^i)^{u_i}\right] \quad \text{in } \mathbb{Z}[x]/(1 - x^n).$$

If $eU = 0$, then there is $\varphi(x) \in \mathbb{Z}[x]$ such that

$$\prod_{i=1}^{n-1} (1 - x^i)^{u_i} = \varphi(x)(1 - x^n) \quad \text{in } \mathbb{Z}[x]. \tag{3.3}$$

We see

$$1 - x^i = -\prod_{j|i} \Phi_j(x),$$

where the product $\prod_{j|i}$ is taken over all divisors j of i, and $\Phi_j(x)$ is the jth cyclotomic polynomial which is irreducible in $\mathbb{Z}[x]$. From this the right hand side of (3.3) contains $\Phi_n(x)$ as a factor, but the left hand side does not. This is a contradiction. □

For a subgroup H of G we obtain a homomorphism $\gamma_H : R(G) \to R(H)$ by restricting the action of G to H.

Lemma 3.4. *For any compact Lie group G,*

$$\gamma = \bigoplus_H \gamma_H : R(G) \to \bigoplus_H R(H)$$

is injective, where the direct sum $\oplus_H$ is taken over all finite cyclic subgroups H of G.

Proof. Assume that $\gamma(a) = 0$ for $a \in R(G)$. Then $\gamma_H(a) = 0$ in $R(H)$ for any finite cyclic subgroup H of G. To show $a = 0$, considering a as a class function, we will show $a(g) = 0$ for any $g \in G$. To show this we divide into two cases. First, if g is of finite order, i.e., $g^n = 1$ for some integer n, then g generates the cyclic group $\langle g \rangle$ of order n. By the assumption, we have $\gamma_{\langle g \rangle}(a) = 0$ in $R(\langle g \rangle)$. Hence

$$a(g) = \gamma_{\langle g \rangle}(a)(g) = 0.$$

Second, if g is of infinite order, then there exists a sequence of elements of finite order which converges to g. From the continuity of class function, we see $a(g) = 0$. □

We obtain the following theorem by Lemmas 3.2 and 3.4.

Theorem 3.5. *For a complex representation U of a compact Lie group G, $eU \neq 0$ in $R(G)$ if and only if there exists a cyclic subgroup $\mathbb{Z}_n \leq G$ such that $U^{\mathbb{Z}_n} = \{0\}$.*

4. Degrees of G-maps

In this section we discuss the degree of a G-map between representation spheres.

Theorem 4.1. *Let U and W be unitary representations of a compact connected Lie group G with $\dim U = \dim W$ and $eU \neq 0$. If there exists a G-map $f : SU \to SW$, then*
(i) $\deg f$ *is uniquely determined only by U and W, and*
(ii) *in particular,* $\deg f = 0$ *if $eW = 0$.*

Proof. (i) Assume $g : SU \to SW$ is another G-map. From Theorem 1.2 we have $eW = z(f) \cdot eU$ and $eW = z(g) \cdot eU$, and hence $(z(f) - z(g)) \cdot eU = 0$ in $R(G)$. This implies $z(f) = z(g)$ since the restricting homomorohism $\gamma_{T^k} : R(G) \to R(T^k)$ is injective where T^k is a maximal torus of G, and since $R(T^k)$ has no zero divisor. Hence Theorem 1.1 shows $\deg f = z(f)(1) = z(g)(1) = \deg g$.
(ii) If $eW = 0$, we have $z(f) \cdot eU = 0$ and hence $z(f) = 0$. This implies $\deg f = z(f)(1) = 0$. □

If G is compact abelian, the degree can be more explicitly discussed in the following theorem.

Theorem 4.2. *Let G be a compact abelian group, say, $G = T^k \times \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_l}$. Let*

$$U = \bigoplus_{\gamma\in\Gamma} V_\gamma^{u(\gamma)} \quad \text{and} \quad W = \bigoplus_{\gamma\in\Gamma} V_\gamma^{w(\gamma)}$$

be unitary representations of G with $\dim U = \dim W$. If there exists a G-map $f : SU \to SW$, then

$$\deg f \cdot \prod_{\gamma\in\Gamma} |\gamma|^{u(\gamma)} = \prod_{\gamma\in\Gamma} |\gamma|^{w(\gamma)} + nr$$

for $n = \text{g.c.d.}\{n_1, \dots, n_l\}$ and some integer r.

Proof. Theorem 1.2 gives $z(f)\cdot eU = eW$ in $R(G) \cong \mathbb{Z}[x_1, \dots, x_k, y_1, \dots, y_l]_L/Y$. Let $\varphi \in \mathbb{Z}[x_1, \dots, x_k, y_1, \dots, y_l]_L$ be a representative of $z(f)$, i.e., $z(f) = [\varphi]$. The equality $z(f) \cdot eU = eW$ gives the following equality in $\mathbb{Z}[x_1, \dots, x_k, y_1, \dots, y_l]_L$,

$$\varphi \cdot \prod_{\gamma\in\Gamma} \bigl(1 - (\boldsymbol{xy})^\gamma\bigr)^{u(\gamma)} = \prod_{\gamma\in\Gamma} \bigl(1 - (\boldsymbol{xy})^\gamma\bigr)^{w(\gamma)} + \sum_{j=1}^{l} \psi_j \cdot \bigl(1 - y_j^{n_j}\bigr)$$

for some $\psi_j \in \mathbb{Z}[x_1, \dots, x_k, y_1, \dots, y_l]_L$. Substituting X for all of $x_1, \dots, x_k, y_1, \dots, y_l$, we obtain

$$\text{(4.3)} \qquad \bar{\varphi}(X) \prod_{\gamma\in\Gamma} \bigl(1 - X^{|\gamma|}\bigr)^{u(\gamma)} = \prod_{\gamma\in\Gamma} \bigl(1 - X^{|\gamma|}\bigr)^{w(\gamma)} + \sum_{j=1}^{l} \bar{\psi}_j(X)\bigl(1 - X^{n_j}\bigr)$$

in $\mathbb{Z}[X]_L$, where $\bar{\varphi}(X) = \varphi(X, \dots, X)$, $\bar{\psi}_j(X) = \psi_j(X, \dots, X)$. We note that

$$\prod_{\gamma\in\Gamma} \bigl(1 - X^{|\gamma|}\bigr)^{u(\gamma)} = (1 - X)^{\sum_{\gamma\in\Gamma} u(\gamma)} \prod_{\gamma\in\Gamma} \bigl(1 + X + X^2 + \cdots + X^{|\gamma|-1}\bigr)^{u(\gamma)},$$

and

$$\sum_{\gamma\in\Gamma} u(\gamma) = \dim U = \dim W = \sum_{\gamma\in\Gamma} w(\gamma).$$

Thus, dividing (4.3) by $(1 - X)^{\dim U}$, we obtain

$$\bar{\varphi}(X) \prod_{\gamma\in\Gamma} \bigl(1 + X + X^2 + \cdots + X^{|\gamma|-1}\bigr)^{u(\gamma)} = \prod_{\gamma\in\Gamma} \bigl(1 + X + X^2 + \cdots + X^{|\gamma|-1}\bigr)^{w(\gamma)} + A(X),$$

where

$$A(X) = \frac{\sum_{j=1}^{l} \bar{\psi}_j(X)(1 - X^{n_j})}{(1 - X)^{\dim U}} \in \mathbb{Z}[X]_L$$
$$= (1 + X + X^2 + \cdots + X^{n-1})B(X) \quad \text{for some } B(X) \in \mathbb{Z}[X]_L.$$

Substituting 1 for X, we obtain the desired equality since $\bar{\varphi}(1) = \deg f$ from Theorem 1.1 and Lemma 2.2. □

REMARK 4.4. In Theorem 4.2, if G is connected, i.e., $G = T^k$, then $n = 0$. If G is finite, i.e., $G = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_l}$, and if $eU \neq 0$, then we see $\prod_{\gamma \in \Gamma} |\gamma|^{u(\gamma)} \neq 0$ since $u(0, \ldots, 0) = 0$ from Theorem 3.5. If G is not finite, then $\prod_{\gamma \in \Gamma} |\gamma|^{u(\gamma)}$ can be zero even if $eU \neq 0$. For example, the representation $U = V_1 \otimes V_{-1}$ of $G = T^2$ gives an example for this. Essentially the same point as this is missed in [9]. If the last statement of [9, Theorem 2.2] would be correct, $\deg f$ would be always positive. This is incorrect.

5. Euler classes for representations of S^1

In this section we show that the Euler class distinguishes the isomorphism class of real representations of S^1. The set $\{V_i \mid i \in \mathbb{Z}\}$ gives a complete set of irreducible complex representations of S^1. We note that $R(S^1) \cong \mathbb{Z}[x]_L$, and that $V_i \cong V_{-i}$ as real representations.

Theorem 5.1. *Let U and W be two complex representations of S^1. If $eU = eW \neq 0$ in $R(S^1)$, then $U \cong W$ as real representations.*

Proof. Decompose U and W into direct sums of irreducible representations as

$$U = \bigoplus_{i \in \mathbb{Z}} V_i^{u_i} \quad \text{and} \quad W = \bigoplus_{i \in \mathbb{Z}} V_i^{w_i}.$$

Here $u_0 = 0$ and $w_0 = 0$ since $eU \neq 0$ and $eW \neq 0$. Noting that $1 - x^i = -x^i(1 - x^{-i})$ in $\mathbb{Z}[x]_L$, we have

$$eU = (-1)^a x^b \prod_{i>0} (1 - x^i)^{u_i + u_{-i}},$$

where $a = \sum_{i<0} u_i$ and $b = \sum_{i<0} i u_i$. Factoring $1 - x^i$ into the cyclotomic polynomials $\Phi_j(x)$, we have

$$eU = (-1)^{a'} x^b \prod_{j>0} \Phi_j(x)^{c_j},$$

where $a' = a + \sum_{i>0}(u_i + u_{-i})$ and $c_j = \sum_{j|i}(u_i + u_{-i})$ is the sum of $u_i + u_{-i}$ with i a multiple of j. Similarly we have

$$eW = (-1)^r x^s \prod_{j>0} \Phi_j(x)^{t_j}$$

for some integers r, s, and $t_j = \sum_{j|i}(w_i + w_{-i})$. Since $\mathbb{Z}[x]_L$ is a unique factorization domain, $eU = eW$ implies $c_j = t_j$, i.e.,

$$\sum_{j|i}(u_i + u_{-i}) = \sum_{j|i}(w_i + w_{-i})$$

for all $j > 0$. From this we have $u_i + u_{-i} = w_i + w_{-i}$ for all $i > 0$, and hence $U \cong W$ as real representations. □

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Department of Mathematics
Yamaguchi University
Yamaguchi 753-8512
Japan
e-mail: komiya@po.cc.yamagichi-u.ac.jp