Fujino, O.
Osaka J. Math.
43 (2006), 275–281

TORIC VARIETIES WHOSE CANONICAL DIVISORS ARE DIVISIBLE BY THEIR DIMENSIONS

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(Received January 18, 2005)

Abstract

We totally classify the projective toric varieties whose canonical divisors are divisible by their dimensions. In Appendix, we show that Reid's toric Mori theory implies Mabuchi's characterization of the projective space for toric varieties.

1. Introduction

In [5, Section 6], Akio Hattori and Mikiya Masuda determined the structures of n-dimensional *non-singular complete* toric varieties whose first Chern classes are divisible by n or $n+1$ as applications of their theory. Their results are as follows:

Theorem 1.1 (cf. [5, Corollaries 6.4, 6.8]). *Let M be a* complete non-singular *toric variety of dimension n.*
(A) *If $c_1(M)$ is divisible by $n+1$, then M is isomorphic to the projective space $\mathbb{P}^n$ as a toric variety.*
(B) *If $c_1(M)$ is divisible by n, then M is isomorphic to an $(n-1)$-dimensional projective space bundle over $\mathbb{P}^1$ as a toric variety.*
For the more precise statements, see [5, Corollaries 6.6, 6.8].

These results seem to be toric geometric analogues of Kobayashi-Ochiai's theorems (see [6]). In [6], they characterized n-dimensional Fano manifolds whose first Chern classes are divisible by n or $n+1$. Before we state the main theorem of this paper, let us recall the following theorem, which is a direct consequence of the main theorem of [1].

Theorem 1.2. *Let X be an n-dimensional* projective *toric variety and $B=\sum_j d_jB_j$ a $\mathbb{Q}$-divisor on X, where B_j is a torus invariant prime divisor and $0\le d_j\le 1$ for every j. Assume that K_X+B is $\mathbb{Q}$-Cartier, not nef, and $-(K_X+B)\equiv ND$ for some Cartier divisor D on X, where N is a positive rational number. Then,* [1, Theorem 0.1] *implies $N\le n+1$. Furthermore, $N=n+1$ if and only if $X\simeq\mathbb{P}^n$, $B=0$,*

2000 Mathematics Subject Classification. Primary 14M25; Secondary 14E30.
Partially supported by The Sumitomo Foundation.

and $\mathcal{O}_X(D) \simeq \mathcal{O}_{\mathbb{P}^n}(1)$. *More generally,* $N > n$ *implies that* $X \simeq \mathbb{P}^n$, $\sum_j d_j < 1$, *and* $\mathcal{O}_X(D) \simeq \mathcal{O}_{\mathbb{P}^n}(1)$.

Obviously, Theorem 1.2 is much stronger than Theorem 1.1 (A) for *projective* toric varieties. Note that we do not assume that X is *non-singular* in Theorem 1.2. Unfortunately, we need the *projectivity* assumption for our proof since it depends on the toric Mori theory. In this short paper, we try to generalize Theorem 1.1 (B) for *projective* toric varieties without any assumptions about singularities. The next theorem is the main theorem of this paper.

Theorem 1.3. *Let* X *be an* n*-dimensional* projecitve *toric variety such that* K_X *is* $\mathbb{Q}$*-Cartier. Assume that* $K_X \equiv nD$ *for some Cartier divisor* D *on* X*. Then, we can determine the structure of* X*. More precisely, if* X *is non-singular, then* X *has a* $\mathbb{P}^{n-1}$*-bundle structure over* $\mathbb{P}^1$*. If* X *is singular, then* X *is* $\mathbb{P}(1, 1, 2, \dots, 2)$ *or the toric variety constructed in* Theorem 3.4. *For the more precise statements, see* Theorems 3.2 *and* 3.4 *below.*

This paper is not self-contained. It heavily relies on my previous paper: [1]. As we said before, we need the *projectivity* assumption for our proof since it depends on the toric Mori theory. I do not know if our results are true or not without this assumption. In general, if X is non-projective, then the Kleiman-Mori cone $\overline{NE}(X)$ may have little information (see [2], [3], and [9]). After I circulated the preliminary version of this paper, Akio Hattori obtained Theorem 3.2 below for $\mathbb{Q}$-factorial *complete* (not necessarily projective) toric varieties on a slightly stronger assumption that $-K_X \sim nD$. His proof depends on the theory of orbifold elliptic genera. For the details, see [4, Corollary 5.9]. Finally, in Appendix, we show that Reid's toric Mori theory implies Mabuchi's characterization of the projective space for toric varieties (see Theorem 4.1). We freely use the notation in [1]. We will work over an algebraically closed field k throughout this note.

We summarize the contents of this paper. In Section 2, we investigate $\mathbb{Q}$-factorial toric Fano varieties with $\rho = 1$ that have long extremal rays. It is a generalization of [1, Proposition 2.9]. Section 3 is the main part of this paper. Here, we classify the toric varieties whose canonical divisors are divisible by their dimensions. Section 4 is an appendix, where we treat Mabuchi's characterization of the projective space for toric varieties.

NOTATION. The symbol $\equiv$ denotes the numerical equivalence for $\mathbb{Q}$-Cartier divisors.

2. $\mathbb{Q}$-factorial toric Fano varieties with $\rho = 1$

We use the same notation as in [1, 2.8]. The following proposition is a key result in this note. It is a slight generalization of [1, Proposition 2.9]. We recommend the reader to see [1, Section 2] before reading this section.

Proposition 2.1. *Let X be an n-dimensional $\mathbb{Q}$-factorial toric Fano variety with Picard number one. If $X \not\simeq \mathbb{P}^n$ and $-K_X \cdot V(\mu_{l,m}) \geq n$ for every pair (l, m), then $X \simeq \mathbb{P}(1, 1, 2, \ldots, 2)$.*

Proof. It is obvious that $n \geq 2$. By the assumption, we have

$$-K_X \cdot V(\mu_{k,n+1}) = \frac{1}{a_{n+1}} \left(\sum_{i=1}^{n+1} a_i \right) \frac{\operatorname{mult}(\mu_{k,n+1})}{\operatorname{mult}(\sigma_k)} \geq n$$

for $1 \leq k \leq n$. Thus

$$(n+1)a_{n+1} \geq \sum_{i=1}^{n+1} a_i \geq \frac{\operatorname{mult}(\sigma_k)}{\operatorname{mult}(\mu_{k,n+1})} n a_{n+1}$$

for every k. Since

$$\frac{\operatorname{mult}(\sigma_k)}{\operatorname{mult}(\mu_{k,n+1})} \in \mathbb{Z}_{>0},$$

we have $\operatorname{mult}(\sigma_k) = \operatorname{mult}(\mu_{k,n+1})$ for every k. This implies that a_k divides a_{n+1} for all k.

Claim 1. $a_1 = a_2 = 1$, $a_3 = \cdots = a_{n+1} = 2$.

Proof of Claim 1. If $a_1 = a_{n+1}$, then $a_1 = a_2 = \cdots = a_{n+1} = 1$ since we assumed $a_1 \leq \cdots \leq a_{n+1}$. This and $-K_X \cdot V(\mu_{l,m}) \geq n$ for every (l, m) imply that $X \simeq \mathbb{P}^n$. See the proof of [1, Proposition 2.9]. Thus, we have $a_1 \neq a_{n+1}$. It follows from this fact that $a_2 \neq a_{n+1}$ since v_1 is primitive and $\sum_i a_i v_i = 0$. In this case,

$$-K_X \cdot V(\mu_{k,n+1}) = \frac{1}{a_{n+1}} \left(\sum_{i=1}^{n+1} a_i \right) \geq n$$

implies $a_1 = a_2 = 1$, $a_3 = \cdots = a_{n+1} = 2$. We note that

$$\frac{a_i}{a_{n+1}} \leq \frac{1}{2}$$

for $i = 1, 2$ and $a_i \leq a_{n+1}$ for $3 \leq i \leq n$. □

Claim 2. $\operatorname{mult}(\sigma_1) = \operatorname{mult}(\sigma_2) = 1$, *that is, σ_1 and σ_2 are non-singular cones.*

Proof of Claim 2. It is sufficient to prove $\operatorname{mult}(\sigma_1) = 1$. We note that $\operatorname{mult}(\mu_{1,l}) = \operatorname{mult}(\sigma_1)$ for $3 \leq l \leq n+1$ and v_2 is primitive imply that all the lattice points included in

$$\left\{ \sum_{i=2}^{n+1} t_i v_i \;\middle|\; 0 \leq t_i \leq 1 \right\} \subset N_{\mathbb{R}}$$

are vertices. Thus, $\operatorname{mult}(\sigma_1) = 1$. □

Therefore, $\{v_1, v_2, \ldots, v_{n+1}\}$ spans the lattice $N \simeq \mathbb{Z}^n$. Thus, we obtain $X \simeq \mathbb{P}(1, 1, 2, 2, \ldots, 2)$, a weighted projective space. □

REMARK 2.2. Let $X \simeq \mathbb{P}(1, 1, 2, 2, \cdots, 2)$. Then it is not difficult to see that $V(v_i)$ is a torus invariant Cartier divisor and $K_X \sim -nV(v_i)$ for $3 \leq i \leq n+1$.

3. Main Theorems

In this section, we classify the structures of the $\mathbb{Q}$-Gorenstein projective toric varieties X with $-K_X \equiv nD$. Before we go to the classification, let us note the following lemma. The proof is easy.

Lemma 3.1 (Numerical equivalence and $\mathbb{Q}$-linear equivalence). *Let X be a projective toric variety and D a Cartier divisor on X. Then $D \equiv 0$ if and only if $D \sim 0$.*

Let D_1 and D_2 be $\mathbb{Q}$-Cartier divisors on X. Then $D_1 \equiv D_2$ if and only if $D_1 \sim_{\mathbb{Q}} D_2$.

First, we decide the structures of X under the assumption that X is $\mathbb{Q}$-factorial and $-K_X \equiv nD$, where $n = \dim X \geq 2$.

Theorem 3.2 ($\mathbb{Q}$-factorial case). *Let X be a $\mathbb{Q}$-factorial projective toric variety with* $\dim X = n \geq 2$. *Let D be a Cartier divisor on X. If $-K_X \equiv nD$, then the one of the following holds.*

(1) *$X \simeq \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(q_1) \oplus \mathcal{O}(q_2) \oplus \cdots \oplus \mathcal{O}(q_n))$ such that $\sum_{i=1}^n q_i = 2$. In this case, $\mathcal{O}_X(D) \simeq \mathcal{O}_{\mathbb{P}}(1)$, where $\mathcal{O}_{\mathbb{P}}(1)$ is the tautological line bundle of $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(q_1) \oplus \mathcal{O}(q_2) \oplus \cdots \oplus \mathcal{O}(q_n))$. Note that X is non-singular and $\rho(X) = 2$.*

(2) *$X \simeq \mathbb{P}(1, 1, 2, 2, \ldots, 2)$, and D is a torus invariant prime Cartier divisor on X, see* Remark 2.2. *Note that X is singular and $\rho(X) = 1$.*

Proof. Since K_X is not nef, there exists a K_X-negative extremal ray R. Its length is obviously $\geq n$. This means that $-K_X \cdot C \geq n$ for every integral curve C such that $[C] \in R$. So, $\beta = 0$ or 1 in the proof of the theorem in [1] (see [1, p.558–559]). If $\beta = 1$, then it can be checked easily that $\alpha = 0$ (see [1, p.558]). In this case, there exists a contraction $\varphi\colon X \longrightarrow \mathbb{P}^1$ such that the general fibers are $\mathbb{P}^{n-1}$. Therefore, $F \cdot D^{n-1} = 1$ for any fiber F since φ is flat. Thus, every fiber is reduced and isomorphic to $\mathbb{P}^{n-1}$.

So, we obtain $X \simeq \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(q_1)\oplus\mathcal{O}(q_2)\oplus\cdots\oplus\mathcal{O}(q_n))$. We can assume that $0 < \sum q_i \le n$ without loss of generality. Since $\mathcal{O}(K_X) \simeq \varphi^*\mathcal{O}_{\mathbb{P}^1}\big(\sum q_i - 2\big)(-n)$ and $-K_X \equiv nD$, we have $\sum q_i = 2$. Therefore, $\mathcal{O}_X(K_X) \simeq \mathcal{O}_{\mathbb{P}}(-n)$. We finish the proof when $\beta = 1$. When $\beta = 0$, it is obvious that $\alpha = 0$ and $\rho(X) = 1$. Then this case follows from Proposition 2.1. □

REMARK 3.3. Take $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}\oplus\cdots\oplus\mathcal{O}\oplus\mathcal{O}(2))$, which is a special case of (1) in Theorem 3.2. Then, the Picard number $\rho(X) = 2$. So, $NE(X)$ has two rays. One ray R corresponds to the $\mathbb{P}^{n-1}$-bundle structure $X \longrightarrow \mathbb{P}^1$. Another ray Q corresponds to the contraction $\varphi := \varphi_Q\colon X \longrightarrow \mathbb{P}(1, 1, 2, \ldots, 2)$. We note that K_X is φ-numerically trivial and that φ contracts a divisor $\mathbb{P}^1 \times \mathbb{P}^{n-2} \simeq \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}\oplus\cdots\oplus\mathcal{O}) \subset X$. Thus, φ is a crepant resolution of $\mathbb{P}(1, 1, 2, \ldots, 2)$.

Next, we investigate the structures of X when X is not $\mathbb{Q}$-factorial and $-K_X \equiv nD$. In the following theorem, it is obvious that $n \ge 3$. It is because every toric surface is $\mathbb{Q}$-factorial.

Theorem 3.4 (non-$\mathbb{Q}$-factorial case). *Let X be a non-$\mathbb{Q}$-factorial projective toric variety with* $\dim X = n \ge 3$. *Assume that X is $\mathbb{Q}$-Gorenstein and $-K_X \equiv nD$ for some Cartier divisor on X. We put $Y = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}\oplus\cdots\oplus\mathcal{O}\oplus\mathcal{O}(1)\oplus\mathcal{O}(1))$. Then X is the target space of the flopping contraction $\varphi\colon Y \longrightarrow X$. Note that φ contracts $\mathbb{P}^1 \times \mathbb{P}^{n-3} \simeq \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}\oplus\cdots\oplus\mathcal{O}) \subset \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}\oplus\cdots\mathcal{O}\oplus\mathcal{O}(1)\oplus\mathcal{O}(1))$. In this case, $\rho(X) = 1$ and X is Gorenstein.*

Proof. We take a small projective toric $\mathbb{Q}$-factorialization $f\colon Y \longrightarrow X$ (see [1, Corollary 5.9]). Since Y is $\mathbb{Q}$-factorial, $K_Y \equiv nf^*D$, and $\rho(Y) \ge 2$, we have $Y \simeq \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(q_1)\oplus\mathcal{O}(q_2)\oplus\cdots\oplus\mathcal{O}(q_n))$ with $\sum q_i = 2$. Since $\rho(Y) = 2$, $NE(Y)$ has two rays R and Q. One ray R corresponds to the $\mathbb{P}^{n-1}$-bundle structure $Y \longrightarrow \mathbb{P}^1$. Another ray Q corresponds to the flopping contraction $\varphi := \varphi_Q\colon Y \longrightarrow X$. Note that Q is spanned by one of the sections $C_i := \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(q_i)) \subset Y$ for $1 \le i \le n$. It is because all extremal rays are spanned by torus invariant curves. We can assume that $q_1 \le q_2 \le \cdots \le q_n$ without loss of generality. Since $\sum q_i = 2$, we have $q_1 \le 0$. Note that $K_X \cdot R < 0$ and $K_X \cdot C_i = -nq_i$. If $q_1 < 0$, then Q is spanned by some C_{i_0} with $K_X \cdot C_{i_0} = -nq_{i_0} > 0$. It is because $NE(Y)$ is spanned by R and Q and $K_X \cdot R < 0$. Since φ_Q is a flopping contraction, we obtain $q_1 \ge 0$. Therefore, $(q_1, q_2, \ldots, q_n) = (0, 0, \ldots, 0, 1, 1)$ (see also Remark 3.3). It is not difficult to see that the target space of the flopping contraction $\varphi_Q\colon Y \longrightarrow X$ has the desired properties. □

4. Appendix

In this section, we show that Mabuchi's characterization of the projective space for toric varieties (cf. [7, Theorem 4.1]) easily follows from [8]. We can skip Step 2 in the proof of [7, Theorem 4.1] by applying [8, (2.10) Corollary].

Theorem 4.1 (cf. [7, Theorem 4.1]). *Let V be an n-dimensional complete non-singular toric variety. Assume that the normal bundle of each torus invariant divisor is ample. Then $V \simeq \mathbb{P}^n$.*

Proof. We note that V is projective since it has ample line bundles. Let Δ be the fan corresponding to V. Take an extremal ray R of $NE(V)$. Let C be a torus invariant integral curve such that the numerical equivalence class of C is in R. Let $\langle v_1, \ldots, v_{n-1} \rangle \in \Delta$ be the $(n-1)$-dimensional cone corresponding to C. Take two n-dimensional cones $\langle v_1, \ldots, v_{n-1}, v_n \rangle$ and $\langle v_1, \ldots, v_{n-1}, v_{n+1} \rangle$ from Δ. Thus we have $\sum_{i=1}^{n-1} a_i v_i + v_n + v_{n-1} = 0$. Note that V is non-singular. We put $D_i := V(v_i)$ for every i. Since $\mathcal{O}_{D_i}(D_i)$ is ample, we obtain that $a_i = D_1 \cdots D_{i-1} \cdot D_i^2 \cdot D_{i+1} \cdots D_{n-1} > 0$ for every i. Thus, n-dimensional cones $\langle v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n, v_{n+1} \rangle \in \Delta$ for $1 \leq i \leq n-1$ (see [8, (2.10) Corollary]). Therefore, $a_i = 1$ for all i since V is non-singular. So, we obtain that $V \simeq \mathbb{P}^n$. □

The following corollary is obvious by Theorem 4.1.

Corollary 4.2. *Let V be an n-dimensional complete non-singular toric variety. Then $V \simeq \mathbb{P}^n$ if and only if the tangent bundle T_V is ample.*

ACKNOWLEDGMENTS. I would like to thank Doctor Hiroshi Sato, who informed me of the paper [5].

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