S. Shimomura
Nagoya Math. J.
Vol. 192 (2008), 89–109

EQUI-DISTRIBUTION OF VALUES FOR THE THIRD AND THE FIFTH PAINLEVÉ TRANSCENDENTS

SHUN SHIMOMURA

Abstract. We show equi-distribution properties of values for the third and the fifth Painlevé transcendents in a sectorial domain. For our purpose we define a characteristic function of sectorial domain type by employing value distribution theory in a half plane. Some special cases admit analogues of Borel exceptional values. Similar results are obtained for modified versions of these Painlevé transcendents, which are of infinite growth order.

§1. Introduction

For a meromorphic function $f(z)$ in $\mathbb{C}$, the proximity, the counting and the characteristic functions are given by

$$\begin{aligned} m(r,f) &:= \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\phi})|\, d\phi, \\ N(r,f) &:= \int_0^r \frac{1}{t}(n(t,f)-n(0,f))\, dt + n(0,f)\log r, \\ T(r,f) &:= m(r,f)+N(r,f) \end{aligned}$$

with $\log^+ x := \max\{\log x, 0\}$ $(x>0)$, respectively, where $n(t,f)$ denotes the number of poles of $f(z)$ in the disc $|z|\le t$, each counted according to its multiplicity (see [2], [4], [5]). Moreover, for $a\in\mathbb{C}\cup\{\infty\}$, we write

$$m(r,a,f) := \begin{cases} m(r,1/(f-a)) & \text{if } a\in\mathbb{C}, \\ m(r,f) & \text{if } a=\infty, \end{cases}$$

and in like manner we use the notation $N(r,a,f)$ denoting the counting function. Such abbreviation is also used for the proximity and the counting functions in a sectorial domain, which will be defined in Section 2. The

Received September 4, 2007.
Revised February 18, 2008.
2000 Mathematics Subject Classification: 34M55, 34M05, 30D35.

growth order and the exponent of convergence for a-points ($a \in \mathbb{C} \cup \{\infty\}$) are defined by

$$\varrho(f) := \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}, \quad \sigma(a,f) := \limsup_{r\to\infty} \frac{\log N(r,a,f)}{\log r},$$

respectively. Suppose that $0 < \varrho(f) < \infty$. Then, by the second main theorem ([2], [4], [5], [21]), for any $a \in \mathbb{C} \cup \{\infty\}$ with possible two exceptions, we have $\sigma(a,f) = \varrho(f)$ implying equi-distribution of values. Such an exceptional value a satisfying $\sigma(a,f) < \varrho(f)$ is called a *Borel exceptional value*.

Let us consider Painlevé equations

$$\text{(I)} \qquad w'' = 6w^2 + z,$$

$$\text{(II)} \qquad w'' = 2w^3 + zw + \alpha,$$

$$\text{(III}'\text{)} \qquad w'' = \frac{(w')^2}{w} - \frac{w'}{z} + \frac{1}{z^2}(\alpha w^2 + \gamma w^3) + \frac{\beta}{z} + \frac{\delta}{w},$$

$$\text{(IV)} \qquad w'' = \frac{(w')^2}{2w} + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w},$$

$$\text{(V)} \qquad w'' = \Big(\frac{1}{2w} + \frac{1}{w-1}\Big)(w')^2 - \frac{w'}{z} + \frac{(w-1)^2}{z^2}\Big(\alpha w + \frac{\beta}{w}\Big) + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1}$$

($' = d/dz$), where α, β, γ, δ are complex parameters. All the solutions of (I), (II) and (IV) (respectively, (III$'$) and (V)) are meromorphic in the whole complex plane $\mathbb{C}$ (respectively, the universal covering of $\mathbb{C} \setminus \{0\}$). In (III$'$) and (V), replacing z by e^z, we get modified versions of them:

$$\text{(III}'_0\text{)} \qquad w'' = \frac{(w')^2}{w} + \alpha w^2 + \gamma w^3 + \beta e^z + \frac{\delta e^{2z}}{w},$$

$$\text{(V}_0\text{)} \qquad w'' = \Big(\frac{1}{2w} + \frac{1}{w-1}\Big)(w')^2 + (w-1)^2\Big(\alpha w + \frac{\beta}{w}\Big) + \gamma e^z w + \frac{\delta e^{2z} w(w+1)}{w-1},$$

whose solutions are meromorphic in $\mathbb{C}$.

For solutions of (I), (II) and (IV), equi-distribution properties of values immediately follow from the finiteness of their growth order ([1], [13], [14], [19]). Each solution $w_{\mathrm{I}}(z)$ of (I) is transcendental, and satisfies $\varrho(w_{\mathrm{I}}) = 5/2$

([7], [14], [15], [19]), which together with the well-known Clunie lemma implies $m(r,a,w_{\mathrm{I}}) = O(\log r)$ for every $a \in \mathbb{C} \cup \{\infty\}$. Observing that

$$\log N(r,a,w_{\mathrm{I}}) - \log T(r,w_{\mathrm{I}}) = \log\Big(1 - \frac{m(r,a,w_{\mathrm{I}}) + O(1)}{T(r,w_{\mathrm{I}})}\Big) = o(1),$$

we have the following:

Theorem A. *For every $a \in \mathbb{C} \cup \{\infty\}$, each solution $w_{\mathrm{I}}(z)$ of* (I) *satisfies $\sigma(a,w_{\mathrm{I}}) = \varrho(w_{\mathrm{I}}) = 5/2$, namely, $w_{\mathrm{I}}(z)$ admits no Borel exceptional values.*

For (II) (respectively, (IV)) each transcendental solution $w_{\mathrm{II}}(z)$ (respectively, $w_{\mathrm{IV}}(z)$) satisfies $3/2 \le \varrho(w_{\mathrm{II}}) \le 3$ (respectively, $2 \le \varrho(w_{\mathrm{IV}}) \le 4$) ([3], [14], [17], [19], [20]). Equation (IV) with $\beta = 0$ admits a family of solutions $\mathcal{V}^{\pm}_{\mathrm{IV}} := \{v^{\pm}_c(z) \mid c \in \mathbb{C}\}$ with

$$v^{\pm}_c(z) := \exp(\mp z^2)\Big(c \pm \int_0^z \exp(\mp \tau^2)\,d\tau\Big)^{-1}$$

satisfying $w' = \mp(w^2 + 2zw)$ as well, if and only if $\alpha = \pm 1$ ([18]). Using an estimate for $m(r,a,w_{\mathrm{II}})$ (respectively, $m(r,a,w_{\mathrm{IV}})$) ([1], [18]), we immediately obtain the following:

Theorem B. (i) *Each transcendental solution of* (II) *admits no Borel exceptional values.*

(ii) *Let $w_{\mathrm{IV}}(z)$ be a transcendental solution of* (IV). *If $w_{\mathrm{IV}} \notin \mathcal{V}^{\pm}_{\mathrm{IV}}$, then $w_{\mathrm{IV}}(z)$ admits no Borel exceptional values. If $w_{\mathrm{IV}} \in \mathcal{V}^{\pm}_{\mathrm{IV}}$, then $w_{\mathrm{IV}}(z)$ admits the Borel exceptional value* 0.

The purpose of this paper is to show equi-distribution properties of values for the third and the fifth Painlevé transcendents. General solutions of (III′) and (V) are not necessarily single-valued in $\mathbb{C} \setminus \{0\}$, and should be considered in a sector around $z = \infty$. To examine these solutions, we define a characteristic function of sectorial domain type (see Section 2.2.1) by employing value distribution theory in a half plane developed by [6], [21], which is surveyed in [22]. All the solutions of (III'_0) and (V_0) are meromorphic in $\mathbb{C}$, but they are not necessarily of finite order. For them we consider the iterative growth order. Our results are given under growth condition (2.1) or (2.7) on solutions. For certain families of solutions, this condition is checked by using their asymptotic expressions along a line (see Examples 2.1, 2.2 and 2.3).

In addition to the standard notation of value distribution theory, we write $\varphi(r) \ll \psi(r)$ or $\psi(r) \gg \varphi(r)$ if $\varphi(r) = O(\psi(r))$ as $r \to \infty$; and $\varphi(r) \asymp \psi(r)$ if $\varphi(r) \ll \psi(r)$ and $\psi(r) \ll \varphi(r)$ are simultaneously valid.

§2. Main results

If $\gamma = \delta = 0$ (respectively, if $\beta = \delta = 0$ or if $\alpha = \gamma = 0$), then (V) (respectively, (III′)) is solvable by quadrature ([11], [12]). In what follows, we impose the conditions

$$\begin{aligned} &(\gamma, \delta) \neq (0,0) && \text{on (V) and } (\mathrm{V}_0); \\ &(\beta, \delta) \neq (0,0) \text{ and } (\alpha, \gamma) \neq (0,0) && \text{on (III}') \text{ and } (\mathrm{III}'_0). \end{aligned}$$

2.1. Equations (V_0) and (III'_0)

We call a solution $w(z)$ of (V_0) or (III'_0) *admissible*, if

$$\frac{r}{T(r,w)} \to 0 \quad \text{as } r \to \infty. \tag{2.1}$$

It is known that, under the condition

$$\alpha = 0, \quad -4\beta\delta + (\gamma \pm (-2\delta)^{1/2})^2 = 0, \tag{2.2}$$

equation (V_0) admits a family of solutions $\mathcal{V}_0 := \{\chi^0_\pm(\gamma, \delta; c, z) \mid c \in \mathbb{C}\}$, where

$$\begin{aligned} \chi^0_\pm(\gamma, \delta; c, z) &:= \exp(\kappa_\pm z \mp (-2\delta)^{1/2} e^z) \\ &\qquad \times \left(c - \kappa_\pm \int_0^z \exp(-\kappa_\pm \tau \pm (-2\delta)^{1/2} e^\tau)\, d\tau \right), \\ \kappa_\pm &:= 1 \pm \gamma(-2\delta)^{-1/2} \end{aligned}$$

([12, §2]). Furthermore, under the condition

$$\beta = 0, \quad 4\alpha\delta + (-\gamma \pm (-2\delta)^{1/2})^2 = 0, \tag{2.3}$$

equation (V_0) admits a family of solutions $\mathcal{W}_0 := \{1/\chi^0_\pm(-\gamma, \delta; c, z) \mid c \in \mathbb{C}\}$. It is easy to see that $\chi^0_\pm$ and $1/\chi^0_\pm$ are admissible.

THEOREM 2.1. *Let $w(z)$ be an admissible solution of* (V_0)*. If $w \notin \mathcal{V}_0 \cup \mathcal{W}_0$, then, for every $a \in \mathbb{C} \cup \{\infty\}$, we have $\sigma_0(a, w) = \varrho_0(w) < \infty$, where*

$$\sigma_0(a,w) := \limsup_{r\to\infty} \frac{\log\log N(r,a,w)}{\log r}, \quad \varrho_0(w) := \limsup_{r\to\infty} \frac{\log\log T(r,w)}{\log r}.$$

If $w \in \mathcal{V}_0$ (respectively, $w \in \mathcal{W}_0$), then $\sigma_0(a,w) = \varrho_0(w) < \infty$ holds for every $a \in \mathbb{C}$ (respectively, $a \in \mathbb{C} \cup \{\infty\} \setminus \{0\}$), and $w(z)$ admits no poles (respectively, no zeros).

THEOREM 2.2. *Let $w(z)$ be an admissible solution of* (III'_0)*. Then $\sigma_0(a,w) = \varrho_0(w) < \infty$ holds for every $a \in \mathbb{C} \cup \{\infty\}$.*

Remark 2.1. If $\varrho_0 = \varrho_0(w) > 0$, then the relation $\sigma_0(a,w) = \varrho_0(w)$ implies

$$\sum_{\nu=1}^{\infty} \exp(-|z_\nu(a)|^{\varrho_0-\varepsilon}) = \infty \quad \text{and} \quad \sum_{\nu=1}^{\infty} \exp(-|z_\nu(a)|^{\varrho_0+\varepsilon}) < \infty$$

for any $\varepsilon > 0$, where $z_\nu(a)$ ($\nu \in \mathbb{N}$) denote the a-points of $w(z)$.

Remark 2.2. For $w \in \mathcal{V}_0$ (respectively, $w \in \mathcal{W}_0$), the value ∞ (respectively, 0) may be regarded as a Borel exceptional value in a sense of iterative growth order.

EXAMPLE 2.1. Suppose that $\beta = 0$, $\delta > 0$, $\alpha, \gamma \in \mathbb{R}$. Then (V_0) admits a two parameter family of solutions expressible in the form

$$\varphi_0(R_0, \Theta_0, z) = R_0(1+o(1))e^{-z}\cos^2\big(\sqrt{\delta/2}\,e^z - C(R_0)z + \Theta_0 + o(1)\big),$$
$$C(R_0) = (\gamma/4)\sqrt{2/\delta} - \sqrt{\delta/2}\,R_0, \quad R_0 > 0, \quad \Theta_0 \in \mathbb{R}$$

as $z \to \infty$ along the positive real axis ([12]). This implies $N(r, 1/\varphi_0) \gg e^r$, and hence φ_0 is admissible. By this estimate and Lemma 3.2, we have $\log T(r, \varphi_0) \asymp r$. If $\gamma \neq 0$, then $\varphi_0 \notin \mathcal{V}_0 \cup \mathcal{W}_0$, and $\sigma_0(a, \varphi_0) = \varrho_0(\varphi_0) = 1$ for every $a \in \mathbb{C} \cup \{\infty\}$.

2.2. Equations (V) and (III′)

2.2.1. Notation

To state our results for (V) and (III′), we define the notation of value distribution in a sectorial domain. Suppose that $f(z)$ is meromorphic in a domain containing the half plane $\operatorname{Im} z \ge 0$. Write for $r > 1$

$$\begin{aligned} m_{\mathrm{H}}(r,f) &:= \frac{1}{2\pi}\int_{\arcsin(r^{-1})}^{\pi-\arcsin(r^{-1})} \log^{+}\bigl|f(re^{i\phi}\sin\phi)\bigr|\,\frac{d\phi}{r\sin^2\phi}\,, \\ N_{\mathrm{H}}(r,f) &:= \int_1^r \frac{n_{\mathrm{H}}(t,f)}{t^2}\,dt, \\ T_{\mathrm{H}}(r,f) &:= m_{\mathrm{H}}(r,f) + N_{\mathrm{H}}(r,f), \end{aligned}$$

where $n_{\mathrm{H}}(t,f)$ denotes the number of poles of $f(z)$ in the set

$$\Omega_0(t) := \bigl\{ z = \tau e^{i\phi} \bigm| 0 < \phi < \pi,\ 1 < \tau \le t\sin\phi \bigr\} \tag{2.4}$$

([6], [21], [22]).

Remark 2.3. Suppose in addition that $f(z) = O(|z|^L)$ $(L > 0)$ as $|z| \to \infty$ in the half plane $\operatorname{Im} z \ge 0$. Then by definition $T_{\mathrm{H}}(r,f) = O(\log r)$ as $r \to \infty$.

Let $w(z)$ be a solution of (V) or (III′) on the universal covering of $\mathbb{C}\backslash\{0\}$. Given $\theta_0 \in \mathbb{R}$ and $\lambda > 0$, we set

$$w_{\lambda}^{\theta_0}(\zeta) := w(e^{i\theta_0}(e^{-\pi i/2}(i+\zeta))^{\lambda}) = w(e^{(\theta_0-\lambda\pi/2)i}(i+\zeta)^{\lambda})$$

representing $w(z)$ in

$$\Omega(\theta_0,\lambda) := \bigl\{ z = e^{(\theta_0-\lambda\pi/2)i}(i+\zeta)^{\lambda} \bigm| \operatorname{Im}\zeta \ge 0,\ |\zeta| > 1 \bigr\}, \tag{2.5}$$

where the branch of $(e^{-\pi i/2}(i+\zeta))^{\lambda}$ is taken so that $\arg((e^{-\pi i/2}(i+xi))^{\lambda}) = 0$ for $x \ge 0$. Note that, for any $\varepsilon > 0$, there exists a number $\rho_\varepsilon > 1$ satisfying

$$\begin{aligned} \bigl\{ z \bigm| |\arg z - \theta_0| < \lambda\pi/2 - \varepsilon,\ |z| > \rho_\varepsilon \bigr\} &\subset \Omega(\theta_0,\lambda) \\ &\subset \bigl\{ z \bigm| |\arg z - \theta_0| < \lambda\pi/2,\ |z| > 1 \bigr\}, \end{aligned}$$

which implies that $\Omega(\theta_0,\lambda)$ is essentially equivalent to a sectorial domain. Clearly $w(z)$ is meromorphic in a domain containing $\Omega(\theta_0,\lambda)$. Note that the arc $\zeta = r^{1/\lambda}e^{i\phi}\sin\phi$, $|\zeta| > 1$ $(0 < \phi < \pi)$ is mapped to a curve expressed as $z = e^{(\theta_0-\lambda\pi/2)i} r e^{i\lambda\phi}(\sin^\lambda \phi + O(r^{-1/\lambda}))$ in $\Omega(\theta_0, \lambda)$ for a sufficiently large number r. Taking these facts into account, we define, for $r > 1$, the proximity, the counting and the characteristic functions in $\Omega(\theta_0, \lambda)$ by

$$\begin{aligned}
m_\lambda^{\theta_0}(r, w) &:= m_{\mathrm{H}}(r^{1/\lambda}, w_\lambda^{\theta_0}) \\
&= \frac{1}{2\pi} \int_{\arcsin(r^{-1/\lambda})}^{\pi-\arcsin(r^{-1/\lambda})} \log^+ \bigl| w_\lambda^{\theta_0}(r^{1/\lambda} e^{i\phi} \sin\phi) \bigr| \frac{d\phi}{r^{1/\lambda} \sin^2 \phi} \\
&= \frac{1}{2\pi} \int_{\arcsin(r^{-1/\lambda})}^{\pi-\arcsin(r^{-1/\lambda})} \log^+ \bigl| w(e^{(\theta_0-\lambda\pi/2)i}(i + r^{1/\lambda} e^{i\phi} \sin\phi)^\lambda) \bigr| \frac{d\phi}{r^{1/\lambda} \sin^2 \phi}, \\
N_\lambda^{\theta_0}(r, w) &:= N_{\mathrm{H}}(r^{1/\lambda}, w_\lambda^{\theta_0}) = \int_1^{r^{1/\lambda}} \frac{n_{\mathrm{H}}(t, w_\lambda^{\theta_0})}{t^2}\, dt = \frac{1}{\lambda} \int_1^r \frac{n_\lambda^{\theta_0}(t, w)}{t^{1+1/\lambda}}\, dt, \\
T_\lambda^{\theta_0}(r, w) &:= m_\lambda^{\theta_0}(r, w) + N_\lambda^{\theta_0}(r, w),
\end{aligned}$$

where $n_\lambda^{\theta_0}(t, w)$ denotes the number of poles of $w(z)$ in the set

$$\Omega(\theta_0, \lambda, t) := \bigl\{ z = e^{(\theta_0-\lambda\pi/2)i}(i + \zeta)^\lambda \bigm| \zeta \in \Omega_0(t^{1/\lambda}) \bigr\}. \tag{2.6}$$

Remark 2.4. Our characteristic function $T_\lambda^{\theta_0}(r, f)$ of sectorial domain type is somewhat different from that of [22]. Our sector has the vertex $e^{\theta_0 i} \neq 0$, and no restriction is imposed on the opening angle $\lambda\pi/2$.

2.2.2. Statement of results

We call a solution $w(z)$ of (V) or (III′) *admissible in* $\Omega(\theta_0, \lambda)$, if

$$\frac{\log r}{T_\lambda^{\theta_0}(r, w)} \to 0 \quad \text{as } r \to \infty. \tag{2.7}$$

Under condition (2.2) (respectively, (2.3)), equation (V) admits a family of solutions $\mathcal{V} := \{\chi_\pm(\gamma, \delta; c, z) \mid c \in \mathbb{C}\}$ (respectively, $\mathcal{W} := \{1/\chi_\pm(-\gamma, \delta; c, z) \mid c \in \mathbb{C}\}$) with

$$\begin{aligned}
\chi_\pm(\gamma, \delta; c, z) &:= z^{\kappa_\pm} \exp(\mp(-2\delta)^{1/2} z) \\
&\qquad \times \left(c - \kappa_\pm \int_1^z \tau^{-1-\kappa_\pm} \exp(\pm(-2\delta)^{1/2}\tau)\, d\tau \right), \\
\kappa_\pm &= 1 \pm \gamma(-2\delta)^{-1/2}.
\end{aligned}$$

Theorem 2.3. *Suppose that a solution* $w(z)$ *of* (V) *is admissible in* $\Omega(\theta_0, \lambda)$ *for some* $\theta_0 \in \mathbb{R}$ *and* $\lambda > 0$. *If* $w \notin \mathcal{V} \cup \mathcal{W}$, *then, for every* $a \in \mathbb{C} \cup \{\infty\}$, *we have* $\sigma_\lambda^{\theta_0}(a, w) = \varrho_\lambda^{\theta_0}(w) < \infty$, *where*

$$\sigma_\lambda^{\theta_0}(a, w) := \limsup_{r\to\infty} \frac{\log N_\lambda^{\theta_0}(r, a, w)}{\log r}, \quad \varrho_\lambda^{\theta_0}(w) := \limsup_{r\to\infty} \frac{\log T_\lambda^{\theta_0}(r, w)}{\log r}.$$

If $w \in \mathcal{V}$ (*respectively,* $w \in \mathcal{W}$), *then* $\sigma_\lambda^{\theta_0}(a, w) = \varrho_\lambda^{\theta_0}(w) < \infty$ *holds for every* $a \in \mathbb{C}$ (*respectively,* $a \in \mathbb{C} \cup \{\infty\} \setminus \{0\}$), *and* $w(z)$ *admits no poles* (*respectively, no zeros*) *in* $\Omega(\theta_0, \lambda)$.

Theorem 2.4. *Suppose that a solution* $w(z)$ *of* (III′) *is admissible in* $\Omega(\theta_0, \lambda)$ *for some* $\theta_0 \in \mathbb{R}$ *and* $\lambda > 0$. *Then* $\sigma_\lambda^{\theta_0}(a, w) = \varrho_\lambda^{\theta_0}(w) < \infty$ *holds for every* $a \in \mathbb{C} \cup \{\infty\}$.

Remark 2.5. If $n_\lambda^{\theta_0}(t, w) \gg t^{1/\lambda+\varepsilon_0}$ for some $\varepsilon_0 > 0$, then $\varrho_\lambda^{\theta_0} = \varrho_\lambda^{\theta_0}(w) > 0$. Then the relation $\sigma_\lambda^{\theta_0}(a, w) = \varrho_\lambda^{\theta_0}(w)$ implies, for any $\varepsilon > 0$,

$$\sum_{\Omega(\theta_0,\lambda)} |z_\nu(a)|^{-\varrho_\lambda^{\theta_0}+\varepsilon} = \infty \quad \text{and} \quad \sum_{\Omega(\theta_0,\lambda)} |z_\nu(a)|^{-\varrho_\lambda^{\theta_0}-\varepsilon} < \infty.$$

Here $z_\nu(a)$ ($\nu \in \mathbb{N}$) denote the a-points in $\Omega(\theta_0, \lambda)$, and the summation $\sum_{\Omega(\theta_0,\lambda)} := \lim_{r\to\infty} \sum_{\Omega(\theta_0,\lambda,r)}$ ranges over the interior of $\Omega(\theta_0, \lambda)$.

Remark 2.6. For $w \in \mathcal{V}$ (respectively, $w \in \mathcal{W}$), the value ∞ (respectively, 0) may be regarded as an analogue of Borel exceptional value.

Example 2.2. Under the condition $\beta = 0$, $\delta > 0$, α, $\gamma \in \mathbb{R}$, equation (V) admits a two parameter family of solutions expressible in the form

$$\psi_0(R_0, \Theta_0, z) = R_0(1 + o(1))z^{-1} \cos^2\big(\sqrt{\delta/2}\, z - C(R_0) \log z + \Theta_0 + o(1)\big)$$

as $z \to \infty$ along the positive real axis (cf. Example 2.1, [10, Theorem I]). Suppose that $\lambda > 1$. Then we have $n_\lambda^0(r, 1/\psi_0) \gg r$ implying $T_\lambda^0(r, \psi_0) \geq N_\lambda^0(r, 1/\psi_0) + O(1) \gg r^{1-1/\lambda}$ (cf. Lemma 4.2), and hence ψ_0 is admissible in $\Omega(0, \lambda)$. Using Proposition 5.4, for some $\Lambda < \infty$, we have $T_\lambda^0(r, \psi_0) \ll r^\Lambda$. If $\gamma \neq 0$, then $\psi_0 \notin \mathcal{V} \cup \mathcal{W}$, and $0 < 1 - 1/\lambda \leq \sigma_\lambda^0(a, \psi_0) = \varrho_\lambda^0(\psi_0) < \infty$ holds for every $a \in \mathbb{C} \cup \{\infty\}$. This implies equi-distribution of all values for ψ_0 in $\Omega(0, \lambda)$ if $\lambda > 1$.

EXAMPLE 2.3. Under the condition $\beta = \delta = 0$, $\gamma < 0$, $\alpha \in \mathbb{R}$, equation (V) admits a two parameter family of solutions expressible in the form

$$\psi_0^*(R_0, \Theta_0, z) = R_0(1 + o(1))z^{-1/2} \times \cos^2\big(\sqrt{-2\gamma}\, z^{1/2} + \sqrt{-\gamma/32}\, R_0 \log z + \Theta_0 + o(1)\big)$$

as $z \to \infty$ along the positive real axis ([10, Theorem II]). This expression implies $n_\lambda^0(r, 1/\psi_0^*) \gg r^{1/2}$. Note that $\psi_0^* \notin \mathcal{V} \cup \mathcal{W}$. If $\lambda > 2$, then ψ_0^* is admissible in $\Omega(0, \lambda)$, and satisfies $0 < 1/2 - 1/\lambda \le \sigma_\lambda^0(a, \psi_0^*) = \varrho_\lambda^0(\psi_0^*) < \infty$ for every $a \in \mathbb{C} \cup \{\infty\}$ implying equi-distribution of all values for ψ_0^* in $\Omega(0, \lambda)$ with $\lambda > 2$.

§3. Proofs of Theorems 2.1 and 2.2

Let $f(z)$ be a meromorphic function in $\mathbb{C}$. Then

$$m(r, f'/f) \ll \log T(2r, f) + \log r \tag{3.1}$$

as $r \to \infty$ ([2, Lemma 2.3], [4, Satz 9.3]). This fact implies that the error term $S_0(r, \phi)$ in [12, Theorem 2.1] may be replaced by $O(\log T(2r, \phi) + \log r)$. Hence we immediately obtain the following:

LEMMA 3.1. *Let $w(z)$ be a solution of* (V_0) *such that $w \notin \mathcal{V}_0 \cup \mathcal{W}_0$. Then, for every $a \in \mathbb{C} \cup \{\infty\}$, we have*

$$m(r, a, w) \le (1/2)T(r, w) + O(\log T(2r, w) + \log r) \tag{3.2}$$

as $r \to \infty$. If $w \in \mathcal{V}_0$ (respectively, $w \in \mathcal{W}_0$), then (3.2) *holds for every $a \in \mathbb{C}$ (respectively, $a \in \mathbb{C} \cup \{\infty\} \setminus \{0\}$).*

Furthermore, we note the following ([16, Theorem 1.1]):

LEMMA 3.2. *Let $w(z)$ be a solution of* (V_0) *or* (III_0'). *Then $T(r, w) = O(e^{\Lambda r})$, where $\Lambda = \Lambda_{\alpha,\beta,\gamma,\delta}$ is a positive number independent of $w(z)$.*

Suppose that $w \notin \mathcal{V}_0 \cup \mathcal{W}_0$ is admissible, and that $a \in \mathbb{C} \cup \{\infty\}$. Then using these lemmas and (2.1), we have $m(r, a, w)/T(r, w) \le 1/2 +$

$O(r/T(r,w)) = 1/2 + o(1)$, and hence

$$\begin{aligned}\frac{\log\log N(r,a,w)}{\log r} &= \frac{1}{\log r}\log\log\bigl(T(r,w) - m(r,a,w) + O(1)\bigr)\\ &= \frac{1}{\log r}\log\biggl(\log T(r,w) + \log\Bigl(1 - \frac{m(r,a,w)+O(1)}{T(r,w)}\Bigr)\biggr)\\ &= \frac{1}{\log r}\log\Bigl(\log T(r,w) + O(1)\Bigr)\\ &= \frac{\log\log T(r,w)}{\log r} + O\Bigl(\frac{1}{\log r}\Bigr)\end{aligned}$$

as $r \to \infty$, which implies $\sigma_0(a,w) = \varrho_0(w)$. In this way we obtain Theorem 2.1.

Theorem 2.2 is proved by using the following lemma, which is obtained from (3.1) and [11, Theorem 2.1].

Lemma 3.3. *Let $w(z)$ be a solution of* (III'_0). *Then, for every $a \in \mathbb{C} \cup \{\infty\}$, we have $m(r,a,w) \ll \log T(2r,w) + \log r$.*

§4. Value distribution in a half plane

We review several facts on value distribution theory in a half plane ([6], [21], [22]). In what follows suppose that $f(z)$ is meromorphic in a domain containing the half plane $\mathrm{Im}\, z \geq 0$. A Poisson-Jensen type formula for the half plane ([6, p. 331], [22, Theorem 2.1.2]) is given by

Lemma 4.1. *We have for $r > 1$*

$$N_{\mathrm{H}}(r,1/f) - N_{\mathrm{H}}(r,f) = \frac{1}{2\pi}\int_{\arcsin(r^{-1})}^{\pi-\arcsin(r^{-1})} \log|f(re^{i\phi}\sin\phi)|\,\frac{d\phi}{r\sin^2\phi} + C_{f,r}$$

with

$$|C_{f,r}| \leq \frac{1}{2\pi}\int_0^\pi \bigl(\bigl|\log|f(e^{i\phi})|\bigr| + \bigl|\arg f(e^{i\phi})\bigr|\bigr)\,d\phi.$$

From this lemma the first main theorem follows ([6, (12)], [22, Theorem 2.1.4]):

Lemma 4.2. *For every $a \in \mathbb{C}$, $T_{\mathrm{H}}(r,1/(f-a)) = T_{\mathrm{H}}(r,f) + O(1)$.*

For each $\theta \in [0, 2\pi]$, applying Lemma 4.1 to $f(z) - e^{i\theta}$, we have

$$\begin{aligned} &N_{\mathrm{H}}(r, e^{i\theta}, f) - N_{\mathrm{H}}(r, f) \\ &\quad - \frac{1}{2\pi}\int_{\arcsin(r^{-1})}^{\pi-\arcsin(r^{-1})} \log\bigl|f(re^{i\phi}\sin\phi) - e^{i\theta}\bigr| \frac{d\phi}{r\sin^2\phi} = h(\theta), \end{aligned}$$

with

$$|h(\theta)| \le \frac{1}{2\pi}\int_0^{\pi} \bigl(\bigl|\log|f(e^{i\phi}) - e^{i\theta}|\bigr| + \bigl|\arg(f(e^{i\phi}) - e^{i\theta})\bigr|\bigr)\, d\phi.$$

Integrating from $\theta = 0$ to 2π, and observing that

$$\frac{1}{2\pi}\int_0^{2\pi} \log|a - e^{i\theta}|\, d\theta = \log^+|a| \quad (a \in \mathbb{C}),$$

we have

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} N_{\mathrm{H}}(r, e^{i\theta}, f)\, d\theta - N_{\mathrm{H}}(r, f) \\ &\quad - \frac{1}{2\pi}\int_{\arcsin(r^{-1})}^{\pi-\arcsin(r^{-1})} \log^+\bigl|f(re^{i\phi}\sin\,\phi)\bigr| \frac{d\phi}{r\sin^2\phi} \\ &\qquad = \frac{1}{2\pi}\int_0^{2\pi} N_{\mathrm{H}}(r, e^{i\theta}, f)\, d\theta - T_{\mathrm{H}}(r, f) = \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\, d\theta. \end{aligned}$$

This implies the identity of Cartan type:

Lemma 4.3. *We have*

$$T_{\mathrm{H}}(r, f) = \frac{1}{2\pi}\int_0^{2\pi} N_{\mathrm{H}}(r, e^{i\theta}, f)\, d\theta + C^*_{f,r}\,,$$

where $C^*_{f,r} = O(1)$ *as* $r \to \infty$.

The logarithmic derivative of $f(z)$ is estimated as follows ([6, p. 332], [22, Theorem 2.1.7]):

Lemma 4.4. *For each* $k \in \mathbb{N}$, $m_{\mathrm{H}}(r, f^{(k)}/f) \ll \log^+ T_{\mathrm{H}}(r, f) + \log r$ *as* $r \to \infty$ *outside a possible exceptional set of finite linear measure. In particular, if* $T_{\mathrm{H}}(r, f) = O(r^{\rho_0})$ *for some* $\rho_0 < \infty$, *then* $m_{\mathrm{H}}(r, f^{(k)}/f) = O(\log r)$ *as* $r \to \infty$.

The following two lemmas are regarded as half plane versions of results due to Clunie and Mohon'ko-Mohon'ko, respectively, for the whole complex plane ([1, Lemmas B.11 and B.12], [5, Lemma 2.4.2 and Proposition 9.2.3]).

Lemma 4.5. *Suppose that* $f(z)^{q+1} = Q(z, f(z))$ $(q \in \mathbb{N})$, *where* $Q(z,u)$ *is a polynomial in* u *and its derivatives whose coefficients* $a_\mu(z)$ $(\mu \in M)$ *are meromorphic in a domain containing the half plane* $\operatorname{Im} z \geq 0$. *Suppose that the total degree with respect to* u *and its derivatives does not exceed* q. *Then*

$$m_{\mathrm{H}}(r,f) \ll \sum_{\mu \in M} m_{\mathrm{H}}(r, a_\mu) + \log^+ T_{\mathrm{H}}(r,f) + \log r \tag{4.1}$$

as $r \to \infty$ *outside a possible exceptional set of finite linear measure. Moreover if* $T_{\mathrm{H}}(r,f) = O(r^{\rho_0})$ $(\rho_0 < \infty)$, *then the right hand member of* (4.1) *may be replaced by* $\sum_{\mu \in M} m_{\mathrm{H}}(r, a_\mu) + \log r$ *as* $r \to \infty$.

Proof. We write $Q(z,u) = \sum_{\mu \in M} a_\mu(z) u^{\mu_0} (u')^{\mu_1} \cdots (u^{(l)})^{\mu_l}$, where each $\mu = (\mu_0, \mu_1, \ldots, \mu_l) \in (\mathbb{N} \cup \{0\})^{l+1}$ satisfies $\sum_{j=0}^{l} \mu_j \leq q$. Put

$$I_0(r) = \big\{ \phi \in [\arcsin(r^{-1}), \pi - \arcsin(r^{-1})] \;\big|\; |f(re^{i\phi}\sin\phi)| \geq 1 \big\}.$$

Then we have, uniformly for $\phi \in I_0(r)$,

$$\begin{aligned} &\log^+ \big| f(re^{i\phi}\sin\phi) \big| = \log^+ |f^{-q} Q(z,f)| \\ &\leq \log^+ \Big(\sum_{\mu \in M} |a_\mu| \Big| \frac{f'}{f} \Big|^{\mu_1} \cdots \Big| \frac{f^{(l)}}{f} \Big|^{\mu_l} \Big) \ll \sum_{\mu \in M} \log^+ |a_\mu| + \sum_{j=1}^{l} \log^+ \Big| \frac{f^{(j)}}{f} \Big|. \end{aligned}$$

Substituting this into

$$m_{\mathrm{H}}(r,f) = \frac{1}{2\pi} \int_{I_0(r)} \log^+ |f(re^{i\phi}\sin\phi)| \, \frac{d\phi}{r \sin^2 \phi}$$

and using Lemma 4.4, we obtain (4.1). ∎

Lemma 4.6. *Let* $F(z,u)$ *be a polynomial in* u *and its derivatives whose coefficients* $b_\nu(z)$ $(\nu \in N)$ *are meromorphic in a domain containing the half plane* $\operatorname{Im} z \geq 0$. *Suppose that* $F(z, f(z)) = 0$, *and let* a *be a complex number such that* $F(z,a) \not\equiv 0$. *Then*

$$m_{\mathrm{H}}(r,a,f) \ll \sum_{\nu \in N} T_{\mathrm{H}}(r, b_\nu) + \log^+ T_{\mathrm{H}}(r,f) + \log r \tag{4.2}$$

as $r \to \infty$ outside a possible exceptional set of finite linear measure. Moreover if $T_{\rm H}(r,f) = O(r^{\rho_0})$ $(\rho_0 < \infty)$, then the right hand member of (4.2) *may be replaced by $\sum_{\nu\in N} T_{\rm H}(r, b_\nu) + \log r$ as $r \to \infty$.*

Proof. Since $g := f - a$ satisfies $F(z, g + a) = 0$, we may write

$$-F(z,a) = F(z, g+a) - F(z,a) = \sum_{1\le|\iota|\le d_0} \tilde{b}_\iota(z) g^{\iota_0}(g')^{\iota_1}\cdots(g^{(l)})^{\iota_l}$$

$(\iota = (\iota_0, \iota_1, \dots, \iota_l)$, $|\iota| = \sum_{j=0}^{l} \iota_j)$ for some $d_0 \in \mathbb{N}$, where $\tilde{b}_\iota$ $(\iota \in N')$ are polynomials in b_ν such that $\log^+|\tilde{b}_\iota| \ll \sum_{\nu\in N}\log^+|b_\nu|$. Set

$$I_0^*(r) = \bigl\{\phi \in [\arcsin(r^{-1}), \pi - \arcsin(r^{-1})] \bigm| |g(re^{i\phi}\sin\phi)| \le 1\bigr\}.$$

If $\phi \in I_0^*(r)$, then

$$\log^+|1/g(re^{i\phi}\sin\phi)| \ll \log^+|F(z,a)^{-1}| + \sum_{\nu\in N}\log^+|b_\nu| + \sum_{j=1}^{l}\log^+\Bigl|\frac{g^{(j)}}{g}\Bigr|.$$

Substituting this into

$$m_{\rm H}(r, 1/g) = \frac{1}{2\pi}\int_{I_0^*(r)} \log^+|1/g(re^{i\theta}\sin\phi)|\,\frac{d\phi}{r\sin^2\phi}$$

and using $m_{\rm H}(r, 1/F(z,a)) \ll \sum_{\nu\in N} T_{\rm H}(r, b_\nu) + \log r$, we obtain (4.2). □

§5. Proofs of Theorems 2.3 and 2.4

Theorems 2.3 and 2.4 immediately follow from the propositions:

PROPOSITION 5.1. *Let θ_0 and λ be numbers satisfying $\theta_0 \in \mathbb{R}$ and $\lambda > 0$. Let $w(z)$ ($\not\equiv$ const.) be a solution of* (V). *Then, for every $a \in \mathbb{C}\cup\{\infty\}$, we have $m_\lambda^{\theta_0}(r,a,w) = O(\log r)$ except in the cases below*:

(1) *if $\alpha = 0$, $w \notin \mathcal{V}$, then $m_\lambda^{\theta_0}(r,w) \le (1/2)T_\lambda^{\theta_0}(r,w) + O(\log r)$;*

(2) *if $\beta = 0$, $w \notin \mathcal{W}$, then $m_\lambda^{\theta_0}(r,1/w) \le (1/2)T_\lambda^{\theta_0}(r,w) + O(\log r)$;*

(3) *if $\alpha+\beta = 0$, $\gamma = 0$, $\delta \ne 0$, then $m_\lambda^{\theta_0}(r,-1,w) \le (1/2)T_\lambda^{\theta_0}(r,w) + O(\log r)$;*

(4) *if $w \in \mathcal{V}$ (respectively, $w \in \mathcal{W}$), then $m_\lambda^{\theta_0}(r,w) = T_\lambda^{\theta_0}(r,w)$ (respectively, $m_\lambda^{\theta_0}(r,1/w) = T_\lambda^{\theta_0}(r,w) + O(1)$).*

PROPOSITION 5.2. *Let θ_0 and λ be numbers as in Proposition 5.1. Let $w(z)$ ($\not\equiv$ const.) be a solution of* (III′). *Then, for every $a \in \mathbb{C}\cup\{\infty\}$, we have $m_\lambda^{\theta_0}(r,a,w) = O(\log r)$.*

5.1. Proof of Proposition 5.1

Note that $w(z)$ ($\not\equiv$ const.) is meromorphic in a domain containing $\Omega(\theta_0, \lambda)$, and that $w_\lambda^{\theta_0}(\zeta) = w(e^{(\theta_0-\lambda\pi/2)i}(i+\zeta)^\lambda)$ is meromorphic in a domain H_0 containing the half plane $\operatorname{Im}\zeta \geq 0$. Substitution of $z = e^{(\theta_0-\lambda\pi/2)i}(i+\zeta)^\lambda$ and $w = 1 - 1/v$ into (V) yields the following:

Lemma 5.3. *In H_0, $u = w_\lambda^{\theta_0}(\zeta)$ and $v = 1/(1 - w_\lambda^{\theta_0}(\zeta))$ satisfy*

(5.1)
$$2u(u-1)u_{\zeta\zeta} - (3u-1)(u_\zeta)^2 + \frac{2u(u-1)u_\zeta}{i+\zeta} - \frac{2\lambda^2(u-1)^3}{(i+\zeta)^2}(\alpha u^2+\beta) - \frac{2\gamma\lambda^2 e^{(\theta_0-\lambda\pi/2)i}}{(i+\zeta)^{2-\lambda}}u^2(u-1) - \frac{2\delta\lambda^2 e^{2(\theta_0-\lambda\pi/2)i}}{(i+\zeta)^{2-2\lambda}}u^2(u+1) = 0$$

($_\zeta = d/d\zeta$), *and*

$$2(v-1)(2(v_\zeta)^2 - vv_{\zeta\zeta}) - (2v-3)(v_\zeta)^2 - \frac{2v(v-1)v_\zeta}{i+\zeta} + \frac{2\lambda^2}{(i+\zeta)^2}(\alpha(v-1)^2+\beta v^2) + \frac{2\gamma\lambda^2 e^{(\theta_0-\lambda\pi/2)i}}{(i+\zeta)^{2-\lambda}}v^2(v-1)^2 - \frac{2\delta\lambda^2 e^{2(\theta_0-\lambda\pi/2)i}}{(i+\zeta)^{2-2\lambda}}v^2(v-1)^2(2v-1) = 0, \tag{5.2}$$

respectively.

Note that $\widetilde{w}(s) = w(e^s)$ is a solution of (V_0), and that

$$\left\{e^s \,\middle|\, \log(1/2) \leq \operatorname{Re}s \leq \log(2t),\ |\operatorname{Im}s - \theta_0| \leq \lambda\pi/2\right\} \supset \left\{z \,\middle|\, 1 \leq |z| \leq 2t,\ |\arg z - \theta_0| \leq \lambda\pi/2\right\} \supset \Omega(\theta_0,\lambda,t).$$

From Lemma 3.2 it follows that

$$N_\lambda^{\theta_0}(r,1,w) = \frac{1}{\lambda}\int_1^r \frac{n_\lambda^{\theta_0}(t,1,w)}{t^{1+1/\lambda}}\,dt \ll \int_1^r \frac{n(\log t+1+\lambda\pi,1,\widetilde{w})}{t^{1+1/\lambda}}\,dt \ll \int_{1+\lambda\pi}^{\log r+1+\lambda\pi} \frac{n(\rho,1,\widetilde{w})}{e^{\rho/\lambda}}\,d\rho \ll N(\log r+1+\lambda\pi,1,\widetilde{w})+1 \ll T(\log r+1+\lambda\pi,\widetilde{w})+1 \ll r^\Lambda. \tag{5.3}$$

We remark that the constant Λ in (5.3) can be chosen so that it is independent of α, β, γ, δ, because (5.3) is considered in the sector ([8], [9]).

Since $(\gamma,\delta)\neq(0,0)$, using Remark 2.3, Lemma 4.5 and (5.2), we have

$$\begin{aligned} m_\lambda^{\theta_0}(r,1,w) &= m_{\mathrm{H}}(r^{1/\lambda},1,w_\lambda^{\theta_0}) \\ &\ll \log r + \log^+ T_{\mathrm{H}}(r^{1/\lambda},w_\lambda^{\theta_0}) \ll \log r + \log^+ T_\lambda^{\theta_0}(r,w) \end{aligned}$$

as $r\to\infty$ outside an exceptional set E_0 whose linear measure $|E_0|$ is finite. Observing Lemma 4.2 and (5.3), we have

$$T_\lambda^{\theta_0}(r,w) = N_\lambda^{\theta_0}(r,1,w) + m_\lambda^{\theta_0}(r,1,w) + O(1) \ll r^\Lambda + \log^+ T_\lambda^{\theta_0}(r,w) \tag{5.4}$$

as $r\to\infty$ outside E_0. Suppose that $T_\lambda^{\theta_0}(r,w)$ is unbounded. Then Lemma 4.3 implies $T_\lambda^{\theta_0}(r,w)\to\infty$ as $r\to\infty$, and hence by (5.4) we have $T_\lambda^{\theta_0}(r,w)\le K_0 r^\Lambda$ for $r\notin E_0$, where K_0 is some positive number independent of r. For each $r>1$ we may choose $r_*\notin E_0$ such that $r<r_*<r+2|E_0|$. By Lemma 4.3 again,

$$T_\lambda^{\theta_0}(r,w)\le T_\lambda^{\theta_0}(r_*,w)+K_1\le K_0 r_*^\Lambda + K_1 \le K_0(r+2|E_0|)^\Lambda + K_1 \ll r^\Lambda$$

for $r>1$, where K_1 is some positive number independent of r. Thus we obtain

Proposition 5.4. *Under the same supposition as in Proposition* 5.1, *we have* $T_\lambda^{\theta_0}(r,w)=O(r^\Lambda)$ *as* $r\to\infty$.

Using Lemmas 4.5, 4.6, 5.3 and Proposition 5.4, by the same argument as in the proofs of [12, Propositions 4.1 and 4.3], we conclude the following:

Proposition 5.5. (i) *If* $\alpha\neq 0$, *then* $m_\lambda^{\theta_0}(r,w)=O(\log r)$ *as* $r\to\infty$.

(ii) *For every* $a\in\mathbb{C}$, *we have* $m_\lambda^{\theta_0}(r,a,w)=O(\log r)$ *as* $r\to\infty$ *except in the cases below*:

(a) $a=-1$, $\beta\neq 0$, $\alpha+\beta=0$, $\gamma=0$, $\delta\neq 0$;

(b) $a=0$, $\beta=0$, $(\alpha,\gamma)\neq(0,0)$;

(c) $a=-1,0$, $\alpha=\beta=\gamma=0$, $\delta\neq 0$.

To prove Proposition 5.1, we treat the following exceptional cases:

(A) $\alpha=0$ or $\beta=0$; (B) $\alpha+\beta=0$, $\gamma=0$, $\delta\neq 0$.

5.1.1. Case (A)

Suppose that $\alpha = 0$. Then (V) is written in the form

$$\frac{(zw')^2}{w(w-1)} + \frac{2\beta(w-1)}{w} + \frac{2\delta z^2 w}{w-1}$$
$$= (w-1)\Big[U_0 - \frac{2\gamma z}{w-1} + 2\int_{z_0}^{z}\Big(\frac{\gamma}{w(t)-1} + \frac{2\delta t w(t)}{(w(t)-1)^2}\Big)dt\Big]$$

($U_0 \neq \infty$, $z_0 \in \Omega(\theta_0, \lambda)$) (cf. [12, (4.1), (4.2)]). Hence, $w_\lambda^{\theta_0}(\zeta)$ ($\zeta \in H_0$) satisfies

(5.5)
$$\Phi(\zeta) = \frac{(i+\zeta)^2((w_\lambda^{\theta_0})_\zeta)^2}{\lambda^2 w_\lambda^{\theta_0}(w_\lambda^{\theta_0}-1)} + \frac{2\beta(w_\lambda^{\theta_0}-1)}{w_\lambda^{\theta_0}} + \frac{2\delta e^{2(\theta_0-\lambda\pi/2)i}(i+\zeta)^{2\lambda}w_\lambda^{\theta_0}}{w_\lambda^{\theta_0}-1}$$

(5.6)
$$= e^{(\theta_0-\lambda\pi/2)i}(w_\lambda^{\theta_0}-1)\Big[U_0^* - \frac{2\gamma(i+\zeta)^\lambda}{w_\lambda^{\theta_0}-1}$$
$$+ 2\int_{\zeta_0}^{\zeta}\Big(\frac{\gamma}{w_\lambda^{\theta_0}(\tau)-1} + \frac{2\delta e^{(\theta_0-\lambda\pi/2)i}(i+\tau)^\lambda w_\lambda^{\theta_0}(\tau)}{(w_\lambda^{\theta_0}(\tau)-1)^2}\Big)\lambda(i+\tau)^{\lambda-1}\,d\tau\Big]$$

($U_0^* \neq \infty$, $\operatorname{Im}\zeta_0 \geq 0$). We note the following ([12, Lemma 3.3]):

LEMMA 5.6. *If $w_\lambda^{\theta_0}(\zeta_1) = 1$, then, around $\zeta = \zeta_1$,*

$$w_\lambda^{\theta_0}(\zeta) = \begin{cases} 1 \pm (-2\delta)^{1/2}\lambda e^{(\theta_0-\lambda\pi/2)i}(i+\zeta_1)^{\lambda-1}(\zeta-\zeta_1) + \cdots & \text{if } \delta \neq 0, \\ 1 - (\gamma/2)\lambda^2 e^{(\theta_0-\lambda\pi/2)i}(i+\zeta_1)^{\lambda-2}(\zeta-\zeta_1)^2 + \cdots & \text{if } \delta = 0. \end{cases}$$

Suppose that $\alpha = 0$, $\delta \neq 0$. We put $\Psi(\zeta) := z^{-1}\Phi(\zeta) = e^{-(\theta_0-\lambda\pi/2)i}(i+\zeta)^{-\lambda}\Phi(\zeta)$. Let ζ_1 satisfy $w_\lambda^{\theta_0}(\zeta_1) = 1$. By Lemma 5.6 and (5.6) which is meromorphic at ζ_1, we have $\Psi(\zeta_1) = a_0^\pm = -2(\gamma \pm (-2\delta)^{1/2})$. Suppose that

$$\Psi(\zeta) \not\equiv a_0^\pm\,. \tag{5.7}$$

Then

(5.8)
$$N_\lambda^{\theta_0}(r, 1, w) = N_{\mathrm{H}}(r^{1/\lambda}, 1, w_\lambda^{\theta_0})$$
$$\leq N_{\mathrm{H}}(r^{1/\lambda}, a_0^-, \Psi) + N_{\mathrm{H}}(r^{1/\lambda}, a_0^+, \Psi) \leq 2T_{\mathrm{H}}(r^{1/\lambda}, \Psi) + O(1).$$

Note that (5.6) is holomorphic at every zero of $w_\lambda^{\theta_0}(\zeta)$ as well. Every pole of $\Psi(\zeta)$ must be a pole of $w_\lambda^{\theta_0}(\zeta)$, whose multiplicity is not less than that of $\Psi(\zeta)$. Hence

$$\begin{aligned}(5.9)\quad T_{\mathrm{H}}(r^{1/\lambda},\Psi) &= N_{\mathrm{H}}(r^{1/\lambda},\Psi)+m_{\mathrm{H}}(r^{1/\lambda},\Psi)\\ &\le N_{\mathrm{H}}(r^{1/\lambda},w_\lambda^{\theta_0})+m_{\mathrm{H}}(r^{1/\lambda},\Psi)=N_\lambda^{\theta_0}(r,w)+m_{\mathrm{H}}(r^{1/\lambda},\Psi).\end{aligned}$$

By (5.5), Lemma 4.4, Propositions 5.4 and 5.5, we have

$$\begin{aligned}(5.10)\quad m_{\mathrm{H}}(r^{1/\lambda},\Psi) &\le m_{\mathrm{H}}(r^{1/\lambda},1,w_\lambda^{\theta_0})+m_{\mathrm{H}}(r^{1/\lambda},\beta/w_\lambda^{\theta_0})+O(\log r)\\ &= m_\lambda^{\theta_0}(r,1,w)+m_\lambda^{\theta_0}(r,\beta/w)+O(\log r)=O(\log r).\end{aligned}$$

Note that $N_\lambda^{\theta_0}(r,1,w)=T_\lambda^{\theta_0}(r,w)+O(\log r)$. Combining (5.8), (5.9) and (5.10), we obtain

$$(5.11)\qquad m_\lambda^{\theta_0}(r,w)\le (1/2)T_\lambda^{\theta_0}(r,w)+O(\log r)$$

under condition (5.7).

Suppose that $\alpha=\delta=0$, $\gamma\ne 0$. Then

$$\begin{aligned}\Phi(\zeta) &= \frac{(i+\zeta)^2((w_\lambda^{\theta_0})_\zeta)^2}{\lambda^2 w_\lambda^{\theta_0}(w_\lambda^{\theta_0}-1)}+\frac{2\beta(w_\lambda^{\theta_0}-1)}{w_\lambda^{\theta_0}}\\ &= e^{(\theta_0-\lambda\pi/2)i}(w_\lambda^{\theta_0}-1)\Big[U_0^*-\frac{2\gamma(i+\zeta)^\lambda}{w_\lambda^{\theta_0}-1}+2\int_{\zeta_0}^{\zeta}\frac{\gamma\lambda(i+\tau)^{\lambda-1}}{w_\lambda^{\theta_0}(\tau)-1}\,d\tau\Big].\end{aligned}$$

For every 1-point $\zeta=\zeta_1$ of $w_\lambda^{\theta_0}(\zeta)$, which is double (cf. Lemma 5.6), we have $\Psi(\zeta_1)=-2\gamma$. If $\Psi(\zeta)\equiv -2\gamma$, then

$$U_0^*+2\int_{\zeta_0}^{\zeta}\frac{\gamma\lambda(i+\tau)^{\lambda-1}}{w_\lambda^{\theta_0}(\tau)-1}\,d\tau\equiv 0,$$

implying $w_\lambda^{\theta_0}(\zeta)\equiv\infty$, which is a contradiction. Hence

$$(5.12)\qquad \Psi(\zeta)\not\equiv -2\gamma.$$

Then

$$N_\lambda^{\theta_0}(r,1,w)=N_{\mathrm{H}}(r^{1/\lambda},1,w_\lambda^{\theta_0})\le 2N_{\mathrm{H}}(r^{1/\lambda},-2\gamma,\Psi)\le 2T_{\mathrm{H}}(r^{1/\lambda},\Psi)+O(1).$$

By the same argument as above, we derive (5.11) under condition (5.12).

Finally, under the condition $\alpha = 0$, $\delta \neq 0$, suppose the contrary to (5.7), namely

$$\Psi(\zeta) \equiv a_0^{\pm} = -2(\gamma \pm (-2\delta)^{1/2}). \tag{5.13}$$

Then

$$\begin{aligned}\frac{a_0^{\pm}(i+\zeta)^{\lambda}}{w_{\lambda}^{\theta_0}-1} &= U_0^* - \frac{2\gamma(i+\zeta)^{\lambda}}{w_{\lambda}^{\theta_0}-1} \\ &+ 2\int_{\zeta_0}^{\zeta}\Big(\frac{\gamma}{w_{\lambda}^{\theta_0}(\tau)-1} + \frac{2\delta e^{(\theta_0-\lambda\pi/2)i}(i+\tau)^{\lambda}w_{\lambda}^{\theta_0}(\tau)}{(w_{\lambda}^{\theta_0}(\tau)-1)^2}\Big)\lambda(i+\tau)^{\lambda-1}\,d\tau,\end{aligned} \tag{5.14}$$

and

$$\begin{aligned}\frac{(i+\zeta)^2((w_{\lambda}^{\theta_0})_{\zeta})^2}{\lambda^2 w_{\lambda}^{\theta_0}(w_{\lambda}^{\theta_0}-1)} &+ \frac{2\beta(w_{\lambda}^{\theta_0}-1)}{w_{\lambda}^{\theta_0}} \\ &+ \frac{2\delta e^{2(\theta_0-\lambda\pi/2)i}(i+\zeta)^{2\lambda}w_{\lambda}^{\theta_0}}{w_{\lambda}^{\theta_0}-1} = a_0^{\pm}e^{(\theta_0-\lambda\pi/2)i}(i+\zeta)^{\lambda}.\end{aligned} \tag{5.15}$$

From (5.14), we have

$$\mp 2(-2\delta)^{1/2}zw' = -4\delta zw + a_0^{\pm}(w-1) \tag{5.16}$$

with $z = e^{(\theta_0-\lambda\pi/2)i}(i+\zeta)^{\lambda}$, which is compatible with (5.15) if and only if (2.2) and $\alpha = 0$ hold. Conversely, under condition (2.2) with $\alpha = 0$, if $w(z)$ satisfies (5.16), then $w_{\lambda}^{\theta_0}(\zeta)$ satisfies (5.13). Solving (5.16) under (2.2), we obtain the family of solutions $\mathcal{V}$. Thus we have

Proposition 5.7. *If $\alpha = 0$, and if $w \notin \mathcal{V}$, then $m_{\lambda}^{\theta_0}(r, w) \leq (1/2)$ $T_{\lambda}^{\theta_0}(r, w) + O(\log r)$.*

Observing that $W(z) = 1/w(z)$ satisfies (V) with $(-\beta, -\alpha, -\gamma, \delta)$, we have

Proposition 5.8. *If $\beta = 0$, and if $w \notin \mathcal{W}$, then $m_{\lambda}^{\theta_0}(r, 1/w) \leq (1/2)$ $T_{\lambda}^{\theta_0}(r, w) + O(\log r)$.*

5.1.2. Case (B)

Under the condition $\alpha+\beta=0$, $\gamma=0$, $\delta\ne 0$, we have for $\zeta\in H_0$

$$\widetilde{\Phi}(\zeta)=\frac{(i+\zeta)^2((w_\lambda^{\theta_0})_\zeta)^2}{\lambda^2 w_\lambda^{\theta_0}(w_\lambda^{\theta_0}+1)(w_\lambda^{\theta_0}-1)} - \frac{2\alpha((w_\lambda^{\theta_0})^2-1)}{w_\lambda^{\theta_0}} + \frac{\delta e^{2(\theta_0-\lambda\pi/2)i}(i+\zeta)^{2\lambda}(w_\lambda^{\theta_0}+1)}{2(w_\lambda^{\theta_0}-1)} \tag{5.17}$$

$$= \frac{w_\lambda^{\theta_0}-1}{w_\lambda^{\theta_0}+1}\Big[U_0-4\alpha+\frac{\delta}{2}e^{2(\theta_0-\lambda\pi/2)i}(i+\zeta)^{2\lambda} + 4\delta\int_{\zeta_0}^{\zeta}\frac{\lambda e^{2(\theta_0-\lambda\pi/2)i}(i+\tau)^{2\lambda-1}w_\lambda^{\theta_0}(\tau)}{(w_\lambda^{\theta_0}(\tau)-1)^2}\,d\tau\Big] \tag{5.18}$$

(cf. [12, (4.14), (4.15)]). By Lemma 5.6, if $w_\lambda^{\theta_0}(\zeta_1)=1$, then the function $\widetilde{\Psi}(\zeta):=e^{-(\theta_0-\lambda\pi/2)i}(i+\zeta)^{-\lambda}\widetilde{\Phi}(\zeta)$ satisfies $\widetilde{\Psi}(\zeta_1)=b_\pm=\pm(-2\delta)^{1/2}$. Suppose that

$$\widetilde{\Psi}(\zeta)\not\equiv b_\pm\,. \tag{5.19}$$

Then

$$N_\lambda^{\theta_0}(r,1,w)=N_{\mathrm{H}}(r^{1/\lambda},1,w_\lambda^{\theta_0}) \le N_{\mathrm{H}}(r^{1/\lambda},b_-,\widetilde{\Psi})+N_{\mathrm{H}}(r^{1/\lambda},b_+,\widetilde{\Psi})\le 2T_{\mathrm{H}}(r^{1/\lambda},\widetilde{\Psi})+O(1).$$

From (5.17) it follows that

$$m_{\mathrm{H}}(r^{1/\lambda},\widetilde{\Psi})\le m_{\mathrm{H}}(r^{1/\lambda},\alpha w_\lambda^{\theta_0})+m_{\mathrm{H}}(r^{1/\lambda},\beta/w_\lambda^{\theta_0}) + m_{\mathrm{H}}(r^{1/\lambda},1,w_\lambda^{\theta_0})+O(\log r)=O(\log r).$$

Note that $w_\lambda^{\theta_0}(\zeta_2)=-1$ implies $(w_\lambda^{\theta_0})_\zeta(\zeta_2)\ne 0$, since $\alpha+\beta=\gamma=0$. By (5.18), every pole of $\widetilde{\Psi}$ is a (-1)-point of $w_\lambda^{\theta_0}$, and is simple. Then we have $N_{\mathrm{H}}(r^{1/\lambda},\widetilde{\Psi})\le N_{\mathrm{H}}(r^{1/\lambda},-1,w_\lambda^{\theta_0})$, and hence

$$m_\lambda^{\theta_0}(r,-1,w)\le(1/2)T_\lambda^{\theta_0}(r,w)+O(\log r) \tag{5.20}$$

under condition (5.19). In case $\widetilde{\Psi}(\zeta)\equiv b_\pm$, from (5.17) and (5.18), we obtain a cubic relation with respect to $w_\lambda^{\theta_0}+1$ expressed as $f((i+\zeta)^\lambda, w_\lambda^{\theta_0}+1)=0$, $f(X,Y)\in\mathbb{C}[X,Y]$, $\deg_Y f=3$ (cf. [12, §4.3]). At any rate this case also implies (5.20). In this way we have

Proposition 5.9. *If $\alpha+\beta=0$, $\gamma=0$, $\delta\ne 0$, then $m_\lambda^{\theta_0}(r,-1,w)\le(1/2)T_\lambda^{\theta_0}(r,w)+O(\log r)$.*

5.1.3. Completion of the proof

Since each exceptional case in Proposition 5.5 is included in Propositions 5.7, 5.8 and 5.9, we obtain Proposition 5.1.

5.2. Proof of Proposition 5.2

For a solution $w(z)$ of (III′), the function $w_\lambda^{\theta_0}(\zeta)$ ($\zeta \in H_0$) satisfies

$$u_{\zeta\zeta} = \frac{(u_\zeta)^2}{u} - \frac{u_\zeta}{i+\zeta} + \frac{\lambda^2}{(i+\zeta)^2}(\alpha u^2 + \gamma u^3) + \frac{\beta\lambda^2 e^{(\theta_0-\lambda\pi/2)i}}{(i+\zeta)^{2-\lambda}} + \frac{\delta\lambda^2 e^{2(\theta_0-\lambda\pi/2)i}}{(i+\zeta)^{2-2\lambda}u}. \tag{5.21}$$

From this equation, we obtain $m_\lambda^{\theta_0}(r,w) \ll \log r + \log^+ T_\lambda^{\theta_0}(r,w)$ outside a possible exceptional set of finite linear measure. Using

$$T_\lambda^{\theta_0}(r,w) = m_\lambda^{\theta_0}(r,w) + N_\lambda^{\theta_0}(r,w) \ll r^\Lambda + \log^+ T_\lambda^{\theta_0}(r,w)$$

instead of (5.4), we similarly derive $T_\lambda^{\theta_0}(r,w) = O(r^\Lambda)$ and $m_\lambda^{\theta_0}(r,w) = O(\log r)$ as $r \to \infty$. Since $u \equiv a$ ($\neq 0$) is not a solution of (5.21), by Lemma 4.6 we have $m_\lambda^{\theta_0}(r,a,w) = O(\log r)$ as $r \to \infty$ for every $a \in \mathbb{C} \setminus \{0\}$. Using the equation with respect to $W(z) = z/w(z)$, we obtain $m_\lambda^{\theta_0}(r,1/w) = O(\log r)$ (cf. [11, §4]). This completes the proof of Proposition 5.2.

References

[1] V. I. Gromak, I. Laine and S. Shimomura, Painlevé Differential Equations in the Complex Plane, Walter de Gruyter, Berlin, New York, 2002.

[2] W. K. Hayman, Meromorphic Functions, Clarendon Press, Oxford, 1964.

[3] A. Hinkkanen and I. Laine, *Growth results for Painlevé transcendents*, Math. Proc. Cambridge Philos. Soc., **137** (2004), 645–655.

[4] G. Jank and L. Volkmann, Einführung in die Theorie der ganzen und meromorphen Funktionen mit Anwendungen auf Differentialgleichungen, Birkhäuser, Basel, Boston, 1985.

[5] I. Laine, Nevanlinna Theory and Complex Differential Equations, Walter de Gruyter, Berlin, New York, 1993.

[6] B. Ja. Levin and I. V. Ostrovskii, *On the dependence of the growth of an entire function and the distribution of the zeros of its derivatives*, Amer. Math. Soc. Transl., AMS, Vol. **32** (1963), 322–357.

[7] E. Mues and R. Redheffer, *On the growth of logarithmic derivatives*, J. London Math. Soc., **8** (1974), 412–425.

[8] Y. Sasaki, *Value distribution of the fifth Painlevé transcendents in sectorial domains*, J. Math. Anal. Appl., **330** (2007), 817–828.

[9] Y. Sasaki, *Value distribution of the third Painlevé transcendents in sectorial domains*, preprint.
[10] S. Shimomura, *On solutions of the fifth Painlevé equation on the positive real axis II*, Funkcial. Ekvac., **30** (1987), 203–224.
[11] S. Shimomura, *Value distribution of Painlevé transcendents of the third kind*, Complex Variables, **40** (1999), 51–62.
[12] S. Shimomura, *Value distribution of Painlevé transcendents of the fifth kind*, Result. Math., **38** (2000), 348–361.
[13] S. Shimomura, *The first, the second and the fourth Painlevé transcendents are of finite order*, Proc. Japan Acad., Ser. A, **77** (2001), 42–45.
[14] S. Shimomura, *Growth of the first, the second and the fourth Painlevé transcendents*, Math. Proc. Cambridge Philos. Soc., **134** (2003), 259–269.
[15] S. Shimomura, *Lower estimates for the growth of Painlevé transcendents*, Funkcial. Ekvac., **46** (2003), 287–295.
[16] S. Shimomura, *Growth of modified Painlevé transcendents of the fifth and the third kind*, Forum Math., **16** (2004), 231–247.
[17] S. Shimomura, *Lower estimates for the growth of the fourth and the second Painlevé transcendents*, Proc. Edinburgh Math. Soc., **47** (2004), 231–249.
[18] N. Steinmetz, *Zur Wertverteilung der Lösungen der vierten Painlevéschen Differentialgleichung*, Math. Z., **181** (1982), 553–561.
[19] N. Steinmetz, *Value distribution of Painlevé transcendents*, Israel J. Math., **128** (2002), 29–52.
[20] N. Steinmetz, *Boutroux's method vs. re-scaling lower estimates for the orders of growth of the second and fourth Painlevé transcendents*, Port. Math., **61** (2004), 369–374.
[21] M. Tsuji, *On Borel's directions of meromorphic functions of finite order*, Tohoku Math. J., **2** (1950), 97–112.
[22] S. Wang, *On the sectorial oscillation theory of* $f'' + A(z)f = 0$, Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes **92** (1994).

Department of Mathematics
Keio University
3-14-1, Hiyoshi, Kohoku-ku
Yokohama 223-8522
Japan
shimomur@math.keio.ac.jp