

RENORMALIZATION OF THE LOCAL TIME FOR THE d -DIMENSIONAL FRACTIONAL BROWNIAN MOTION WITH N PARAMETERS

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Abstract. We study the asymptotic behavior in Sobolev norm of the local time of the d -dimensional fractional Brownian motion with N -parameters when the space variable tends to zero, both for the fixed time case and when simultaneously time tends to infinity and space variable to zero.

§1. Introduction

Let $B^H = \{B_t^H : t \geq 0\}$ be a standard fractional Brownian motion (fBm for brevity) with Hurst parameter $H \in (0, 1)$. It is well known that this process is a centered Gaussian process which admits an integral representation of the form

$$B_t^H = \int_0^t K_H(t, s) dW_s,$$

where W is a standard Wiener process. The kernel $K_H(t, s)$ is given, for $s < t$, by

$$(1) \quad K_H(t, s) = c_H(t-s)^\mu - \mu c_H \int_s^t (r-s)^{\mu-1} \left(1 - \left(\frac{s}{r}\right)^{-\mu}\right) dr,$$

with c_H being a constant and $\mu = H - \frac{1}{2}$.

The covariance function of B_t^H can be represented as

$$R_H(s, t) = \mathbb{E}(B_s^H B_t^H) = \int_0^{s \wedge t} K_H(t, r) K_H(s, r) dr,$$

Received June 10, 2005.

Revised January 25, 2006.

1991 Mathematics Subject Classification: 60H05, 60H10.

¹Supported by the Spanish *Ministerio de Educación, Cultura y Deporte*: Grant SB2000-0060.

²Supported by the Spanish *Ministerio de Educación, Cultura y Deporte*: Grant BFM2000-0009 and by the Catalan CIRIT: Grant 2001SGR00174.

and has the explicit form

$$R_H(s, t) = \frac{1}{2}(s^{2H} + t^{2H} - |t - s|^{2H}).$$

A very good survey about the fBm is the paper of Nualart [5].

For $\overline{H} = (H_1, \dots, H_N)$ the $(N, 1)$ -fBm is defined as

$$B_t^{\overline{H}} = \int_{[0,t]} K_{\overline{H}}(t, s) dW_s,$$

where $K_{\overline{H}}(t, s) = \bigotimes_{j=1}^N K_{H_j}(t_j, s_j)$, $s, t \in \mathbb{R}_+^N$ and W is a standard N -parameter Brownian motion. Its covariance function is

$$R_{\overline{H}}(s, t) = \mathbb{E}(B_s^{\overline{H}} B_t^{\overline{H}}) = \prod_{j=1}^N R_{H_j}(s_j, t_j).$$

Finally given the $N \times d$ -matrix $\overline{H} = (\overline{H}_1, \dots, \overline{H}_d)$ where for $i = 1, \dots, d$ and $j = 1, \dots, N$, $\overline{H}_i = (H_{i,1}, \dots, H_{i,N})$ is a column vector and $H_{i,j} \in (0, 1)$, the N -parameter, d -dimensional fractional Brownian motion ((N, d) -fBm for brevity) is defined by $B^{\overline{H}} = (B_t^{\overline{H}_1}, \dots, B_t^{\overline{H}_d})_{t \in \mathbb{R}_+^N}$ where its components are independent and for every $i = 1, \dots, d$, $B^{\overline{H}_i}$ is a $(N, 1)$ -fBm with Hurst parameter \overline{H}_i .

For any $t \in \mathbb{R}_+^N$ and $x \in \mathbb{R}^d$, the local time $L(t, x)$ of the (N, d) -fBm can be defined as the density of the occupation measure μ_t , defined as

$$\mu_t(A) = \int_{[0,t]} \mathbb{1}_A(B_s^{\overline{H}}) ds, \quad A \in \mathcal{B}(\mathbb{R}^d).$$

Formally, we can write

$$L(t, x) = \int_{[0,t]} \delta_x(B_s^{\overline{H}}) ds$$

where δ_x denotes the Dirac function and $\delta_x(B_s^{\overline{H}})$ is therefore a distribution in the Watanabe sense (see [6]).

This local time for (N, d) -fBm has been studied by Xiao and Zhang [7], Hu and Oksendal [2] and Eddahbi et al. [1] between others.

The aim of this paper is to study the asymptotic behavior of $L(t, x)$ when $|x|$, the euclidean norm of x in \mathbb{R}^d , goes to 0, both for a fixed time and

when the time goes to infinity, and we renormalize his Sobolev norm. We generalize the results of [3] from the (N, d) -standard Brownian motion to the (N, d) -fractional Brownian motion. In the standard Brownian motion case, the covariance function is simply $R_{\frac{1}{2}}(s, t) = s \wedge t$. Here, the control of the covariance function $R_H(s, t)$ for $H \neq \frac{1}{2}$ is the main difficulty.

Section 2 is devoted to the presentation of the problem. In particular we review from [1] the chaotic decomposition of the local time $L(t, x)$ as a functional of the (N, d) -fBm and its regularity in terms of Sobolev-Watanabe norms. In Section 3 we present a list of auxiliary lemmas. Section 4 is devoted to the presentation and proof of the main result, namely the asymptotic behavior of this local time, for fixed t , in the case $H_{i,j} = H$, $\forall i, j$, when $|x|$ goes to 0. In Section 5 we extend the result to the case $\underline{t} := t_1 \cdots t_N$ going to infinity.

§2. Preliminaries and statement of the problem

If F is a square integrable Brownian random variable, it can be represented by its Wiener chaos expansion

$$F = \sum_{n=0}^{\infty} I_n(f_n),$$

where $I_n(f_n)$ denotes the multiple Itô stochastic integral of the symmetric kernel $f_n \in L^2(\mathbb{R}_+^n)$ with respect to the Wiener process W .

If \mathbf{L} is the Ornstein-Uhlenbeck operator

$$\mathbf{L}F = - \sum_{n=0}^{\infty} n I_n(f_n),$$

$p \in (1, \infty)$ and $\alpha \in \mathbb{R}$, we define the Sobolev-Watanabe spaces $\mathbb{D}^{\alpha,p}$ as the closure of the set of polynomial random variables with respect to the norm

$$\|F\|_{\alpha,p} = \|(\mathbf{Id} - \mathbf{L})^{\frac{\alpha}{2}} F\|_{L^p(\Omega)},$$

where \mathbf{Id} stands for the identity mapping.

We denote by D the chaotic derivative operator. It acts on multiple Itô stochastic integrals as

$$D_t(I_n(f_n)) = n I_{n-1}(f_n(\cdot, t)),$$

and is continuous from $\mathbb{D}^{\alpha,p}$ into $\mathbb{D}^{\alpha-1,p}(L^2(\mathbb{R}_+))$.

It is known that a Brownian random variable F belongs to $\mathbb{D}^{\alpha,2}$ if and only if its chaotic decomposition $\sum_{n=0}^{\infty} I_n(f_n)$ satisfies

$$\sum_{n=0}^{\infty} (1+n)^{\alpha} \|I_n(f_n)\|_2^2 < \infty,$$

where $\|I_n(f_n)\|_2^2 = n! \|f_n\|_2^2$.

Set $\mathbb{D}^{\infty,2} = \bigcap_{\alpha \in \mathbb{R}} \mathbb{D}^{\alpha,2}$. If $F \in \mathbb{D}^{\infty,2}$, we can compute its chaos expansion using the Stroock formula

$$F = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(\mathbb{E}(D^n F)).$$

For a complete survey of this subjects we refer the reader to the book of Watanabe [6].

Let $p_{\varepsilon}(x)$ be the centered Gaussian kernel with variance $\varepsilon > 0$. Consider also, for $x \in \mathbb{R}^d$ and $\varepsilon > 0$, the Gaussian kernel on \mathbb{R}^d given by

$$p_{\varepsilon}^d(x) = \prod_{i=1}^d p_{\varepsilon}(x_i), \quad x = (x_1, \dots, x_d).$$

We denote by \mathbf{H}_n the n -th Hermite polynomial, defined for $n \geq 1$, by

$$\mathbf{H}_n(x) = \frac{(-1)^n}{n!} \exp\left(\frac{x^2}{2}\right) \frac{d^n}{dx^n} \left(\exp\left(-\frac{x^2}{2}\right)\right), \quad x \in \mathbb{R}$$

and $\mathbf{H}_0(x) = 1$.

As we proved in [1] the chaotic decomposition of the local time of the (N, d) -fBm is

$$L(t, x) = \sum_{n_1, \dots, n_d \geq 0} \int_{[0,t]} \prod_{i=1}^d \frac{p_{\underline{s}^2 \overline{H}_i}(x_i)}{\underline{s}^{n_i \overline{H}_i}} \mathbf{H}_{n_i}\left(\frac{x_i}{\underline{s} \overline{H}_i}\right) I_{n_i}^i(K_{\overline{H}_i}(s, \cdot)^{\otimes n_i}) ds,$$

provided that $\sum_{j=1}^N \frac{1}{H_j^*} > d$, where $t \in \mathbb{R}_+^N$, $x \in \mathbb{R}^d \setminus \{0\}$, $\underline{s} = s_1 \cdots s_N$, $\underline{s}^{\overline{H}_i} = \prod_{j=1}^N s_j^{H_{i,j}}$ and $H_j^* = \max\{H_{i,j}, i = 1, \dots, d\}$. The integrals $I_{n_i}^i$ denotes the multiple Itô stochastic integrals with respect to the independent N -parameter Wiener processes W^i .

Moreover, in [1] we proved that this functional belongs to the space $\mathbb{D}^{\alpha,2}$ if

$$\alpha < \sum_{j=1}^N \frac{1}{2H_j^*} - \frac{d}{2}.$$

If all $H_{i,j} = H$, this expression becomes $\alpha < \frac{N}{2H} - \frac{d}{2}$, and then a sufficient condition for the local time to be in $L^2(\Omega)$ is $N > Hd$. Observe that this sufficient condition is also founded in Xiao and Zhang [7]. From now on we will suppose always this condition.

Recall that if $H = \frac{1}{2}$, $\sum_{j=1}^N \frac{1}{2H_j^2} - \frac{d}{2} = N - \frac{d}{2}$, which is the same condition obtained in [3] for the N -parameter Wiener process in \mathbb{R}^d .

§3. Auxiliary lemmas

LEMMA 1. *If $\frac{1}{4} \leq \beta \leq \frac{1}{2}$ we have*

$$\sup_{x \in \mathbb{R}} |\sqrt{n!} \mathbf{H}_n(x) e^{-\beta x^2}| \leq c(n \vee 1)^{-\frac{8\beta-1}{12}}.$$

Proof. This result is proved in [4]. \square

Remark 2. The factor $\sqrt{n!}$ appears because we do not use the same definition of Hermite polynomials as in [4].

LEMMA 3. *Let $d \geq 1$ and $\nu \in (0, 1)$. We can choose a universal constant c such that for any $m \geq 1$,*

$$\sum_{n_1 + \dots + n_d = m} \prod_{i=1}^d (n_i \vee 1)^{-\nu} \leq cm^{d(1-\nu)-1}.$$

Proof. This result is proved in [4]. \square

LEMMA 4. *Let γ and a be positive constants and $b \in \mathbb{R}$. Set $\alpha = \frac{b-1}{a}$. Then*

$$\int_{[0,1]^N} \exp\left(-\frac{\gamma}{\underline{s}^a}\right) \frac{ds}{\underline{s}^b} = \frac{1}{(N-1)!} \left(\frac{1}{a}\right)^N \gamma^{-\alpha} g_{N-1}(\gamma, \alpha)$$

where

$$g_{N-1}(\gamma, \alpha) := \int_{\gamma}^{\infty} t^{\alpha-1} e^{-t} \left(\log \frac{t}{\gamma}\right)^{N-1} dt.$$

Proof. Using the change of variables $u_1 = s_1 \cdots s_N, u_2 = s_2 \cdots s_N, \dots, u_N = s_N$, with Jacobi determinant $\frac{1}{u_2 \cdots u_N}$, we obtain

$$\begin{aligned} \int_{[0,1]^N} \exp\left(-\frac{\gamma}{\underline{s}^a}\right) \frac{ds}{\underline{s}^b} &= \int_{\{0 \leq u_1 \leq \dots \leq u_N \leq 1\}} \frac{1}{u_1^b} \exp\left(-\frac{\gamma}{u_1^a}\right) \frac{du_N \cdots du_2}{u_N \cdots u_2} du_1 \\ &= \frac{1}{(N-1)!} \int_0^1 \left(\log \frac{1}{r}\right)^{N-1} \frac{1}{r^b} \exp\left(-\frac{\gamma}{r^a}\right) dr, \end{aligned}$$

and making the change of variable $\gamma r^{-a} = t$ we get the desired result. \square

LEMMA 5. *The function*

$$Q_H(z) = \begin{cases} \frac{R_H(1, z)}{z^H} & \text{if } z \in (0, 1] \\ 0 & \text{if } z = 0, \end{cases}$$

has the following properties:

1. *It is strictly increasing and it continuously maps $[0, 1]$ onto $[0, 1]$. Moreover, $Q_H(1) = 1$.*
2. *For fixed $\delta \in (0, 1)$ and for any $z \in [0, 1 - \delta]$, it satisfies the inequality*

$$Q_H(z) \leq c(H, \delta)z^G,$$

where $G = H \wedge (1 - H)$.

3. *For fixed $\delta \in (0, 1)$ and $\beta > 0$, it satisfies the inequality*

$$\int_{1-\delta}^1 Q_H(z)^\beta dz \leq \frac{c(H, \delta)}{\beta^{\frac{1}{2H}}}.$$

Proof. The proof of parts 1 and 3 are done in [1].

For the part 2 we have

$$Q_H(z) = \frac{1 - (1 - z)^{2H}}{2z^H} + \frac{z^H}{2}.$$

Using Taylor expansion, $1 - (1 - z)^{2H} = 2H(1 - \theta)^{2H-1}z$ with $0 \leq \theta \leq z$.

If $H \geq \frac{1}{2}$, we have $1 - (1 - z)^{2H} \leq 2Hz$, and therefore $Q_H(z) \leq Hz^{1-H} + \frac{1}{2}z^H \leq c_1z^{1-H}$, c_1 being a positive constant.

If $H < \frac{1}{2}$ and $z \in [0, 1 - \delta]$, we have $1 - (1 - z)^{2H} \leq 2H\delta^{2H-1}z$, and then $Q_H(z) \leq H\delta^{2H-1}z^{1-H} + \frac{1}{2}z^H \leq c_2z^H$, c_2 being another positive constant. \square

In what follows, for every $x > 0$ and $\gamma \geq 0$, we denote the complementary incomplete Gamma function as

$$\Gamma(x, \gamma) = \int_\gamma^\infty e^{-t}t^{x-1} dt.$$

In particular $\Gamma(x) := \Gamma(x, 0)$ and $\Gamma(x, \gamma) \leq \Gamma(x)$.

LEMMA 6. *The function*

$$g_{N-1}(\gamma, \alpha) := \int_{\gamma}^{\infty} t^{\alpha-1} e^{-t} \left(\log \frac{t}{\gamma} \right)^{N-1} dt, \quad N \in \mathbb{N}$$

has the following behavior when γ tends to 0:

1. If $\alpha > 0$ and $N \geq 2$, $g_{N-1}(\gamma, \alpha) = (\log \frac{1}{\gamma})^{N-1} \Gamma(\gamma, \alpha) + \mathcal{O}((\log \frac{1}{\gamma})^{N-2})$. If $N = 1$, it tends to $\Gamma(\alpha)$
2. If $\alpha = 0$ and $N \geq 2$, $g_{N-1}(\gamma, \alpha) = e^{-\gamma} \frac{1}{N} (\log \frac{1}{\gamma})^N + \mathcal{O}((\log \frac{1}{\gamma})^{N-1})$. If $N = 1$, it behaves as $e^{-\gamma} \ln \frac{1}{\gamma}$
3. If $\alpha < 0$ and $N \geq 2$, $g_{N-1}(\gamma, \alpha) = \gamma^{\alpha} \left(\frac{\Gamma(N)}{|\alpha|^N} + o(\gamma) \right)$. If $N = 1$, its behavior is as $\frac{1}{|\alpha|} \gamma^{\alpha}$

Proof. The cases with $N = 1$ are straightforward. For $N \geq 2$, note that

$$(2) \quad \left(\log \frac{t}{\gamma} \right)^{N-1} = \sum_{k=0}^{N-1} \binom{N-1}{k} \left(\log \frac{1}{\gamma} \right)^{N-1-k} (\log t)^k.$$

Then,

- If $\alpha > 0$, the function

$$t \mapsto t^{\frac{\alpha}{2}-1} e^{-t} (\log t)^k$$

is always integrable on $[0, \infty)$ for any $k \in \mathbb{N} \cup \{0\}$. Therefore,

$$g_{N-1}(\gamma, \alpha) = \left(\log \frac{1}{\gamma} \right)^{N-1} \Gamma(\gamma, \alpha) + \mathcal{O} \left(\left(\log \frac{1}{\gamma} \right)^{N-2} \right).$$

- If $\alpha = 0$, we need to estimate the integral

$$g_{N-1}(\gamma, 0) = \int_{\gamma}^{\infty} t^{-1} e^{-t} \left(\log \frac{t}{\gamma} \right)^{N-1} dt.$$

Integrating by parts we obtain

$$\begin{aligned} g_{N-1}(\gamma, 0) &= \frac{1}{N} \int_{\gamma}^{\infty} e^{-t} \left(\log \frac{t}{\gamma} \right)^N dt \\ &= \frac{e^{-\gamma}}{N} \left(\log \frac{1}{\gamma} \right)^N + \mathcal{O} \left(\left(\log \frac{1}{\gamma} \right)^{N-1} \right). \end{aligned}$$

- If $\alpha < 0$, making the change of variable $s = -\alpha \log(\frac{t}{\gamma})$, the result follows immediately. \square

§4. Renormalization of the local time for fixed t

The main purpose of this section is to study the asymptotic behavior of $L(t, x)$, for $t \in \mathbb{R}_+^N$ and $x \in \mathbb{R}^d \setminus \{0\}$, when $|x| \rightarrow 0$. In the case $dH \geq 1$ it has a singularity. An interesting question is to renormalize the local time, that means, to find a deterministic function $f(t, x)$ such that $f(t, x)L(t, x)$ converge to 1 in some precise sense. We will do it with respect the norm $\|\cdot\|_{\alpha,2}$. Then we will obtain a function $f(t, x)$ such that $\|f(t, x)L(t, x)\|_{\alpha,2}$ converges to 1 when $|x| \rightarrow 0$, both for fixed t and when $\underline{t} = t_1 \cdots t_N \rightarrow \infty$.

Recall the expression of the $\mathbb{D}^{\alpha,2}$ -norm of the local time $L(t, x)$. For the sake of simplicity we take $t = \tilde{1} := (1, \dots, 1)$.

We have

$$(3) \quad \|L(\tilde{1}, x)\|_{\alpha,2}^2 = \sum_{m=0}^{\infty} (1+m)^\alpha A_m(x),$$

where

$$A_m(x) = \sum_{n_1+\dots+n_d=m} \left\| \int_{[0,1]^N} \prod_{i=1}^d \frac{p_{\underline{s}^{2\overline{H}_i}}(x_i)}{\underline{s}^{n_i \overline{H}_i}} \times \mathbf{H}_{n_i} \left(\frac{x_i}{\underline{s}^{\overline{H}_i}} \right) I_{n_i}^i(K_{\overline{H}_i}(s, \cdot)^{\otimes n_i}) ds \right\|_{L^2(\Omega)}^2,$$

and as

$$E(I_{n_i}^i(K_{\overline{H}_i}(u, \cdot)^{\otimes n_i}) I_{n_j}^j(K_{\overline{H}_j}(v, \cdot)^{\otimes n_j})) = \delta_{ij} n_i! (R_{\overline{H}_i}(u, v))^{n_i},$$

$$A_m(x) = \sum_{n_1+\dots+n_d=m} \int_{[0,1]^N} du \int_{[0,1]^N} dv \prod_{i=1}^d \left(\prod_{j=1}^N \frac{R_{H_{i,j}}(u_j, v_j)}{(u_j v_j)^{H_{i,j}}} \right)^{n_i} \times n_i! \mathbf{H}_{n_i} \left(\frac{x_i}{\underline{u}^{\overline{H}_i}} \right) \mathbf{H}_{n_i} \left(\frac{x_i}{\underline{v}^{\overline{H}_i}} \right) p_{\underline{u}^{2\overline{H}_i}}(x_i) p_{\underline{v}^{2\overline{H}_i}}(x_i),$$

and in particular

$$A_0(x) = \left(\int_{[0,1]^N} ds \prod_{i=1}^d \frac{1}{(2\pi \prod_{j=1}^N s_j^{2H_{i,j}})^{\frac{1}{2}}} \exp\left(-\frac{x_i^2}{2 \prod_{j=1}^N s_j^{2H_{i,j}}}\right) \right)^2.$$

In all this section we confine our attention to the situation where $H_{i,j} = H$ for all $(i, j) \in \{1, \dots, d\} \times \{1, \dots, N\}$, and use the notation B^H for $B^{\overline{H}}$.

Observe that in this particular case

$$A_0(x) = \frac{1}{(2\pi)^d} \left(\int_{[0,1]^N} \frac{1}{\underline{s}^{dH}} \exp\left(-\frac{|x|^2}{2\underline{s}^{2H}}\right) ds \right)^2,$$

and

$$A_m(x) = \sum_{n_1+\dots+n_d=m} \int_{[0,1]^N} \int_{[0,1]^N} \left(\prod_{j=1}^N \frac{R_H(u_j, v_j)}{(u_j v_j)^H} \right)^m \\ \times \prod_{i=1}^d n_i! \mathbf{H}_{n_i} \left(\frac{x_i}{\underline{u}^H} \right) \mathbf{H}_{n_i} \left(\frac{x_i}{\underline{v}^H} \right) p_{\underline{u}^{2H}}(x_i) p_{\underline{v}^{2H}}(x_i) dudv.$$

Our main result is the following:

THEOREM 7. *Let B^H be (N, d) -fBm. Set $\lambda := d - \frac{1}{H}$. For any $\alpha < \frac{N}{2H} - \frac{d}{2}$ we have:*

1) *If $\lambda > 0$ and $N \geq 2$,*

$$\lim_{|x| \rightarrow 0} \|L(\tilde{1}, x)\|_{\alpha,2} \left(\frac{2^{\frac{\lambda}{2}} (\frac{1}{2H})^N |x|^{-\lambda}}{(2\pi)^{\frac{d}{2}} (N-1)!} \left(\log \frac{2}{|x|^2} \right)^{N-1} \Gamma\left(\frac{\lambda}{2}\right) \right)^{-1} = 1.$$

If $N = 1$, $\|L(1, x)\|_{\alpha,2}$ explodes when x tends to 0, and $\|L(1, x)\|_{\alpha,2} \cdot (|x|^{-\lambda})^{-1}$ is bounded by a positive constant.

2) *If $\lambda = 0$,*

$$\lim_{|x| \rightarrow 0} \|L(\tilde{1}, x)\|_{\alpha,2} \left(\frac{(\frac{1}{2H})^N}{(2\pi)^{\frac{d}{2}} N!} \left(\log \frac{2}{|x|^2} \right)^N \right)^{-1} = 1.$$

3) *If $\lambda < 0$,*

$$\lim_{|x| \rightarrow 0} \|L(\tilde{1}, x)\|_{\alpha,2} = \|L(\tilde{1}, 0)\|_{\alpha,2} = \frac{1}{(2\pi)^{\frac{d}{2}}} \left(\frac{1}{1-Hd} \right)^{\frac{N}{2}} \\ \times \left[\sum_{r=0}^{\infty} (1+2r)^\alpha \sum_{r_1+\dots+r_d=r} \prod_{i=1}^d \frac{(2r_i)!}{(r_i!) 2^{2r_i}} \left(\int_0^1 Q_H(z)^m \frac{dz}{z^{dH}} \right)^N \right]^{\frac{1}{2}} < \infty.$$

Remark 8. This theorem shows that for $\lambda \geq 0$ the local time explodes at the origin and for $\lambda < 0$ it does not. Observe that for the N -Brownian process, that is $H = \frac{1}{2}$, the local time explodes at the origin if and only if $d \geq 2$, and we obtain, with another proof, the same results as Imkeller and Weisz [3].

Proof. The idea of the proof is to show that the convergence of $A_m(x)$ for any $m \geq 1$ when $|x| \rightarrow 0$, is controlled by $A_0(x)$ and then the asymptotic behavior of $L(1, x)$ coincides with the asymptotic behavior of $A_0(x)^{\frac{1}{2}}$.

Define now, for $\gamma > 0$ and $m \geq 0$,

$$B_m(\gamma) = \int_{[0,1]^N} \int_{[0,1]^N} \frac{(\prod_{j=1}^N R_H(u_j, v_j))^m}{(\underline{u} \cdot \underline{v})^{H(m+d)}} \exp\left(-\frac{\gamma}{\underline{u}^{2H}}\right) \exp\left(-\frac{\gamma}{\underline{v}^{2H}}\right) dudv.$$

Clearly,

$$A_0(x) = \frac{1}{(2\pi)^d} B_0\left(\frac{1}{2}|x|^2\right).$$

For $m \geq 1$, choosing $\beta \in [\frac{1}{4}, \frac{1}{2})$, we can write

$$\begin{aligned} A_m(x) &= \frac{1}{(2\pi)^d} \sum_{n_1+\dots+n_d=m} \int_{[0,1]^N} \int_{[0,1]^N} \left(\prod_{j=1}^N \frac{R_H(u_j, v_j)}{(u_j v_j)^H}\right)^m \frac{1}{(\underline{uv})^{dH}} \\ &\quad \times \prod_{i=1}^d \sqrt{n_i!} \mathbf{H}_{n_i}\left(\frac{x_i}{\underline{u}^H}\right) \exp\left\{-\beta \frac{x_i^2}{\underline{u}^{2H}}\right\} \sqrt{n_i!} \mathbf{H}_{n_i}\left(\frac{x_i}{\underline{v}^H}\right) \exp\left\{-\beta \frac{x_i^2}{\underline{v}^{2H}}\right\} \\ &\quad \times \exp\left\{-\left(\frac{1}{2} - \beta\right) \frac{x_i^2}{\underline{u}^{2H}}\right\} \exp\left\{-\left(\frac{1}{2} - \beta\right) \frac{x_i^2}{\underline{v}^{2H}}\right\} dudv, \end{aligned}$$

and applying Lemmas 1 and 3 we obtain

$$A_m(x) \leq c \frac{1}{(2\pi)^d} m^{d(1-\frac{8\beta-1}{6})-1} B_m\left(\left(\frac{1}{2} - \beta\right)|x|^2\right).$$

Then our problem reduces to the study of the asymptotic behavior of B_m .

As $R_H(u_j, v_j) = R_H(1, \frac{v_j}{u_j})u_j^{2H}$, we have

$$\begin{aligned} B_m(\gamma) &= 2^N \int_{[0,1]^N} \int_0^{u_N} \dots \int_0^{u_1} \prod_{j=1}^N \frac{R_H(1, \frac{v_j}{u_j})^m u_j^{2Hm}}{(u_j v_j)^{H(m+d)}} \\ &\quad \times \exp\left(-\frac{\gamma}{\underline{u}^{2H}}\right) \exp\left(-\frac{\gamma}{\underline{v}^{2H}}\right) dv_1 \dots dv_N du. \end{aligned}$$

With the change $\frac{v_j}{u_j} = z_j$, ($j = 1, \dots, N$) and computing iteratively the

previous integral, we find

$$B_m(\gamma) = 2^N \int_{[0,1]^N} \left(\int_{[0,1]^N} \underline{u}^{1-2Hd} \exp\left(\frac{-\kappa(\underline{z})\gamma}{\underline{u}^{2H}}\right) du_1 \cdots du_N \right) \\ \times \prod_{j=1}^N \frac{R_H(1, z_j)^m}{z_j^{H(m+d)}} dz_1 \cdots dz_N$$

where $\kappa(r) = 1 + \frac{1}{r^{2H}}$.

By Lemma 4, with $a = 2H$ and $b = 2Hd - 1$, we have

$$J_N(\gamma, \underline{z}) = \int_{[0,1]^N} \underline{u}^{1-2Hd} \exp\left(\frac{-\kappa(\underline{z})\gamma}{\underline{u}^{2H}}\right) du \\ = c(N, d, H) \gamma^{-\lambda} \int_{\gamma}^{\infty} e^{-s\kappa(\underline{z})} s^{\lambda-1} \left(\log \frac{s}{\gamma}\right)^{N-1} ds,$$

where $\lambda = d - \frac{1}{H} = \frac{b-1}{a}$.

Therefore

$$B_m(\gamma) = c(N, H, d) \gamma^{-\lambda} \int_{\gamma}^{\infty} \int_{[0,1]^N} \prod_{j=1}^N \frac{R_H(1, z_j)^m}{z_j^{Hm}} \cdot \frac{e^{-\frac{s}{z^{2H}}}}{z^{Hd}} \\ \times e^{-s} s^{\lambda-1} \left(\log \frac{s}{\gamma}\right)^{N-1} dz ds.$$

First we will see that for $m > \frac{\lambda H}{G}$, we have

$$(4) \quad B_m(\gamma) \leq c(H, d, N) \gamma^{-\lambda} g_{N-1}(\gamma, \lambda) m^{-\frac{N}{2H}}.$$

Indeed, controlling the exponential by 1, we obtain

$$B_m(\gamma) \leq c(N, H, d) \gamma^{-\lambda} \int_{[0,1]^N} \prod_{j=1}^N \frac{R_H(1, z_j)^m}{z_j^{H(m+d)}} \int_{\gamma}^{\infty} e^{-s} s^{\lambda-1} \left(\log \frac{s}{\gamma}\right)^{N-1} dz ds \\ = c(N, H, d) \gamma^{-\lambda} g_{N-1}(\gamma, \lambda) \left(\int_0^1 Q_H(z)^m \frac{1}{z^{dH}} dz \right)^N,$$

where the function Q_H is introduced in Lemma 5.

Now, choosing $\delta \in (0, 1)$, we have

$$\int_0^1 Q_H(z)^m \frac{1}{z^{dH}} dz \leq \int_0^{1-\delta} Q_H(z)^m \frac{1}{z^{dH}} dz + (1-\delta)^{-dH} \int_{1-\delta}^1 Q_H(z)^m dz.$$

The second summand on the right, using part 3 of Lemma 5, is controlled by $c(H, \delta)m^{-\frac{1}{2H}}$.

For the first summand, if $m > \frac{dH-1}{G} = \frac{\lambda H}{G}$, we fix $\alpha \in (\frac{\lambda H}{G}, m)$, and write

$$\int_0^{1-\delta} Q_H(z)^m \frac{1}{z^{dH}} dz = \int_0^{1-\delta} Q_H(z)^{m-\alpha} Q_H(z)^\alpha \frac{1}{z^{dH}} dz.$$

Using that Q_H is an increasing function and part 2 of Lemma 5, we control this by

$$Q_H(1-\delta)^{m-\alpha} c(H, \delta, \alpha) \int_0^{1-\delta} z^{\alpha G - dH} dz.$$

As $\alpha > \frac{\lambda H}{G}$, the integral that appears in the last expression is a constant that depends on H, d, α and δ .

Therefore, being $Q_H(1-\delta) < 1$, we can estimate this term by

$$c(H, d, \delta, \alpha)m^{-\frac{1}{2H}},$$

and we get (4).

Note that this result is true only for $m > \frac{\lambda H}{G}$. If $\lambda \leq 0$ this covers all cases. But if $\lambda > 0$ the B_m terms with $m \leq \frac{\lambda H}{G}$ are not controlled yet. The following part of the proof will discuss these first terms. From now on in each expression c will be the suitable constant.

Observe that for any $0 < \epsilon < m$, being $Q_H(\cdot) \leq 1$, we have

$$B_m(\gamma) \leq B_\epsilon(\gamma).$$

Now we will see that for $\lambda > 0$,

$$B_\epsilon(\gamma) \leq c(H, d, N)\gamma^{-\lambda}(g_{N-1}(\gamma, \lambda) + g_{N-1}(\gamma, \alpha))$$

where α is some positive constant depending also on ϵ .

Indeed, putting $c = c(N, H, d)$,

$$\begin{aligned} B_\epsilon(\gamma) &= c\gamma^{-\lambda} \int_\gamma^\infty \int_{[0,1]^N} \prod_{j=1}^N Q_H(z_j)^\epsilon \cdot \frac{e^{-\frac{s}{z^{2H}}}}{z^{Hd}} e^{-s} s^{\lambda-1} \left(\log \frac{s}{\gamma}\right)^{N-1} dz ds \\ &= c\gamma^{-\lambda} \int_\gamma^\infty \sum_{k=0}^N \binom{N}{k} \underbrace{\int_0^{1-\delta} \cdots \int_0^{1-\delta}}_k \underbrace{\int_{1-\delta}^1 \cdots \int_{1-\delta}^1}_{N-k} \prod_{j=1}^N Q_H(z_j)^\epsilon \cdot \frac{e^{-\frac{s}{z^{2H}}}}{z^{Hd}} \\ &\quad \times e^{-s} s^{\lambda-1} \left(\log \frac{s}{\gamma}\right)^{N-1} dz_1 \cdots dz_k ds, \end{aligned}$$

because the function

$$\prod_{j=1}^N Q_H(z_j)^\epsilon \cdot \frac{e^{-\frac{s}{z^{2H}}}}{z^{Hd}},$$

is symmetric in z .

Now estimating Q_H and the exponential by 1 in the integrals between $1 - \delta$ and 1, we obtain

$$\begin{aligned} B_\epsilon(\gamma) &\leq c\gamma^{-\lambda} \int_\gamma^\infty \left(\left(\frac{\delta}{(1-\delta)^{dH}} \right)^N + \sum_{k=1}^N \binom{N}{k} \int_0^{1-\delta} \cdots \int_0^{1-\delta} \frac{\delta^{N-k}}{(1-\delta)^{dH(N-k)}} \right. \\ &\quad \left. \times \prod_{j=1}^k Q_H(z_j)^\epsilon \cdot \frac{e^{-\frac{s}{(z_1 \cdots z_k)^{2H}}}}{(z_1 \cdots z_k)^{Hd}} dz_1 \cdots dz_k \right) e^{-s} s^{\lambda-1} \left(\log \frac{s}{\gamma} \right)^{N-1} ds \\ &\leq c\gamma^{-\lambda} \int_\gamma^\infty \left(\left(\frac{\delta}{(1-\delta)^{dH}} \right)^N + \sum_{k=1}^N \binom{N}{k} \int_0^{1-\delta} \cdots \int_0^{1-\delta} \left(\frac{\delta}{(1-\delta)^{dH}} \right)^{N-k} \right. \\ &\quad \left. \times c(H, \delta)^{k\epsilon} \prod_{j=1}^k z_j^{\epsilon G - dH} e^{-\frac{s}{(z_1 \cdots z_k)^{2H}}} dz_1 \cdots dz_k \right) e^{-s} s^{\lambda-1} \left(\log \frac{s}{\gamma} \right)^{N-1} ds, \end{aligned}$$

where we have used part 2 of Lemma 5.

Now, choosing $\epsilon < \frac{dH}{G}$, we can use Lemma 4 with $a = 2H$, $b = -\epsilon G + dH$, $\gamma = s$, $N = k$ and $\alpha = \frac{dH - \epsilon G - 1}{2H}$, to bound the right hand side of the last inequality by

$$\begin{aligned} c\gamma^{-\lambda} \left\{ g_{N-1}(\gamma, \lambda) + \int_\gamma^\infty \sum_{k=1}^N \int_s^\infty \left(\log \frac{t}{s} \right)^{k-1} s^{\frac{\lambda}{2} + \frac{\epsilon G}{2H} - 1} t^{\frac{\lambda}{2} - \frac{\epsilon G}{2H} - 1} \right. \\ \left. \times e^{-t} e^{-s} \left(\log \frac{s}{\gamma} \right)^{N-1} dt ds \right\}, \end{aligned}$$

where c is a constant that depends on $H, d, N, \epsilon, \delta$.

Using the fact that for any $n \geq 1$ and for $t \geq s$ we have

$$\log \frac{t}{s} \leq n \left(\frac{t}{s} \right)^{\frac{1}{n}},$$

and taking $n = M(k - 1)$ for a big M , we obtain

$$\begin{aligned} B_\epsilon(\gamma) &\leq c\gamma^{-\lambda} \left\{ g_{N-1}(\gamma, \lambda) + \int_\gamma^\infty \int_s^\infty s^{\frac{\lambda}{2} + \frac{\epsilon G}{2H} - 1 - \frac{1}{M}} t^{\frac{\lambda}{2} - \frac{\epsilon G}{2H} - 1 + \frac{1}{M}} \right. \\ &\quad \left. \times e^{-t} e^{-s} \left(\log \frac{s}{\gamma} \right)^{N-1} dt ds \right\}, \end{aligned}$$

where now c depends also on M .

As $\epsilon < m < \frac{\lambda H}{G}$, we have $\frac{\lambda}{2} - \frac{\epsilon G}{2H} + \frac{1}{M} > 0$ and

$$B_\epsilon(\gamma) \leq c\gamma^{-\lambda} \left\{ g_{N-1}(\gamma, \lambda) + \int_\gamma^\infty s^{\frac{\lambda}{2} + \frac{\epsilon G}{2H} - 1 - \frac{1}{M}} e^{-s} \Gamma\left(\frac{\lambda}{2} - \frac{\epsilon G}{2H} + \frac{1}{M}, s\right) \times \left(\log \frac{s}{\gamma}\right)^{N-1} ds \right\}.$$

Controlling the complementary incomplete Gamma function by the corresponding Gamma function we obtain

$$\begin{aligned} B_\epsilon(\gamma) &\leq c\gamma^{-\lambda} \left\{ g_{N-1}(\gamma, \lambda) + \int_\gamma^\infty s^{\frac{\lambda}{2} + \frac{\epsilon G}{2H} - 1 - \frac{1}{M}} e^{-s} \left(\log \frac{s}{\gamma}\right)^{N-1} ds \right\} \\ &= c\gamma^{-\lambda} \left\{ g_{N-1}(\gamma, \lambda) + g_{N-1}\left(\gamma, \frac{\lambda}{2} + \frac{\epsilon G}{2H} - \frac{1}{M}\right) \right\}. \end{aligned}$$

Observe that for M sufficiently large

$$\frac{\lambda}{2} + \frac{\epsilon G}{2H} - \frac{1}{M} > 0.$$

Finally for the $m = 0$ case, using Lemma 4, we have immediately, as $\alpha = \frac{\lambda}{2}$

$$B_0(\gamma) = \frac{1}{((N-1)!)^2} \frac{1}{(2H)^{2N}} \gamma^{-\lambda} g_{N-1}\left(\gamma, \frac{\lambda}{2}\right)^2.$$

Therefore we have to separate the cases $\lambda \geq 0$ and $\lambda < 0$.

For $\lambda \geq 0$ we have

$$\|L(\tilde{1}, x)\|_{\alpha,2}^2 = \sum_{m=0}^\infty (1+m)^\alpha A_m(x)$$

- The terms $A_m(x)$ with $m = 1, \dots, [\frac{\lambda H}{G}]$ are controlled by

$$c\gamma^{-\lambda} \left\{ g_{N-1}(\gamma, \lambda) + g_{N-1}\left(\gamma, \frac{\lambda}{2} + \frac{\epsilon G}{2H} - \frac{1}{M}\right) \right\} m^{d(1 - \frac{8\beta-1}{6})-1}$$

where $\gamma = (\frac{1}{2} - \beta)|x|^2$, and ϵ and M satisfy

$$\frac{\lambda}{2} + \frac{\epsilon G}{2H} - \frac{1}{M} > 0.$$

Then, by Lemma 6, part 1, this behaves asymptotically, when $\gamma \downarrow 0$, as $c\gamma^{-\lambda} (\log \frac{1}{\gamma})^{N-1}$ if $\lambda > 0$, and as $c(\log(\frac{1}{\gamma}))^N$ if $\lambda = 0$.

- The terms $A_m(x)$ with $m \geq \lceil \frac{\lambda H}{G} \rceil + 1$ are controlled by

$$cm^{d(1-\frac{8\beta-1}{6})-1}\gamma^{-\lambda}g_{N-1}(\gamma, \lambda)m^{-\frac{N}{2H}}.$$

Then

$$\sum_{m > \frac{\lambda H}{d}} (1+m)^\alpha A_m(x) \leq c \left[\sum_{m > \frac{\lambda H}{d}} m^{d(1-\frac{8\beta-1}{6})-1} m^{-\frac{N}{2H}} (1+m)^\alpha \right] \times \gamma^{-\lambda} g_{N-1}(\gamma, \lambda),$$

and using the fact that $\alpha < \frac{N}{2H} - \frac{d}{2}$, we have that the series between keys is convergent and the asymptotic behavior of the last expression is, by Lemma 6, as $c\gamma^{-\lambda}(\log \frac{1}{\gamma})^{N-1}$ if $\lambda > 0$, and $c(\log \frac{1}{\gamma})^N$ if $\lambda = 0$.

Finally,

$$\begin{aligned} A_0(x) &= \frac{1}{(2\pi)^d} B_0\left(\frac{1}{2}|x|^2\right) \\ &= \frac{1}{(2\pi)^d} \frac{1}{((N-1)!)^2} \frac{1}{(2H)^{2N}} \gamma^{-\lambda} g_{N-1}\left(\gamma, \frac{\lambda}{2}\right)^2, \end{aligned}$$

where $\gamma = \frac{|x|^2}{2}$.

Summarizing, if $N \geq 2$ and $\lambda > 0$, $A_0(x)$ goes to ∞ when $\gamma \downarrow 0$, as $c\gamma^{-\lambda}(\log \frac{1}{\gamma})^{2N-2}$, and this first term dominates the asymptotical behavior because all the rest goes to ∞ more slowly than $\gamma^{-\lambda}(\log \frac{1}{\gamma})^{(N-1)}$.

If $N = 1$ and $\lambda > 0$, as the logarithm disappears, the proof only says that $\|L(1, x)\|_{\alpha, 2}(|x|^{-\lambda})^{-1}$ is bounded by a positive constant.

If $\lambda = 0$, $A_0(x)$ goes to ∞ when $\gamma \downarrow 0$, as $c(\log \frac{1}{\gamma})^{2N}$, and the rest as $c(\log \frac{1}{\gamma})^N$, so also the first term dominates the asymptotical behavior.

Note that we consider $A_0^{\frac{1}{2}}$ in place of A_0 to get the functions that appear in the theorem.

The $\lambda < 0$ case follows directly. As we have seen before,

$$\sum_{m \geq 1} (1+m)^\alpha A_m(x)$$

is controlled by $\gamma^{-\lambda}g_{N-1}(\gamma, \lambda)$, and by Lemma 6, part 3, this term goes to a constant when $\gamma \downarrow 0$.

In this case the norm $\|L(t, x)\|_{\alpha,2}$ is continuous. Therefore we don't have an explosion, and

$$\lim_{|x| \rightarrow 0} \|L(\tilde{1}, x)\|_{\alpha,2} = \|L(\tilde{1}, 0)\|_{\alpha,2} = \left(\sum_{m=0}^{\infty} (1+m)^\alpha A_m(0) \right)^{\frac{1}{2}},$$

where

$$A_m(0) = \frac{1}{(2\pi)^d} \left(\sum_{n_1+\dots+n_d=m} \prod_{i=1}^d n_i! \mathbf{H}_{n_i}(0)^2 \right) B_m(0),$$

and

$$\begin{aligned} B_m(0) &= 2^N \int_{[0,1]^N} \left(\int_{[0,1]^N} u^{1-2Hd} du_1 \dots du_N \right) \prod_{j=1}^N \frac{R_H(1, z_j)^m}{z_j^{H(m+d)}} dz_1 \dots dz_N \\ &= 2^N \left(\int_0^1 u^{1-2Hd} du \right)^N \left(\int_0^1 Q_H(z)^m \frac{dz}{z^{dH}} \right)^N \\ &= \left(\frac{1}{1-Hd} \right)^N \left(\int_0^1 Q_H(z)^m \frac{dz}{z^{dH}} \right)^N. \end{aligned}$$

Note that as $\lambda < 0$, $1 - 2Hd > -1$.

Finally,

$$\begin{aligned} \|L(\tilde{1}, 0)\|_{\alpha,2}^2 &= \frac{1}{(2\pi)^d} \left(\frac{1}{1-Hd} \right)^N \sum_{m=0}^{\infty} (1+m)^\alpha \\ &\quad \times \left(\sum_{n_1+\dots+n_d=m} \prod_{i=1}^d n_i! \cdot \mathbf{H}_{n_i}(0)^2 \right) \left(\int_0^1 Q_H(z)^m \frac{dz}{z^{dH}} \right)^N \\ &= \frac{1}{(2\pi)^d} \left(\frac{1}{1-Hd} \right)^N \sum_{r=0}^{\infty} (1+2r)^\alpha \\ &\quad \times \left(\sum_{r_1+\dots+r_d=r} \prod_{i=1}^d \frac{(2r_i)!}{(r_i!)^2 2^{2r_i}} \right) \left(\int_0^1 Q_H(z)^m \frac{dz}{z^{dH}} \right)^N, \end{aligned}$$

because $\mathbf{H}_{2n}(0) = \frac{1}{2^n n!}$ and $\mathbf{H}_{2n+1}(0) = 0$.

By the continuity of the norm, it is not necessary to prove the convergence of this series.

Remark 9. Xiao and Zhang [7] proved that when $Hd < 1$, that is $\lambda < 0$, B^H has a jointly continuous local time.

§5. Renormalization of the local time when the time tends to infinity

We can also deduce the behavior of the local time $L(t, x)$ when $\underline{t} = t_1 \cdots t_N \rightarrow \infty$ and $|x| \rightarrow 0$. We also have to distinguish the three cases, namely $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$.

The precise result is the following:

THEOREM 10. *Let $\{L(t, x) : t \in [0, \infty)^N, x \in \mathbb{R}^d \setminus \{0\}\}$ be the local time of the (N, d) -fBm B^H . Let $\lambda = d - \frac{1}{H}$. Then the following limits hold for any $\alpha < \frac{N}{2H} - \frac{d}{2}$:*

1) For $\lambda > 0$ and $N \geq 2$,

$$\lim_{\underline{t} \rightarrow \infty, |x| \rightarrow 0} \|L(t, x)\|_{\alpha, 2} \left(\frac{2^{\frac{\lambda}{2}} (\frac{1}{2H})^N |x|^{-\lambda}}{(2\pi)^{\frac{d}{2}} (N-1)!} \left(\log \frac{2\underline{t}^{2H}}{|x|^2} \right)^{N-1} \Gamma\left(\frac{\lambda}{2}\right) \right)^{-1} = 1.$$

If $N = 1$, $\|L(t, x)\|_{\alpha, 2}$ explodes when x tends to 0 and t to ∞ , and independently of $t \in \mathbb{R}_+$, $\|L(t, x)\|_{\alpha, 2} \cdot (|x|^{-\lambda})^{-1}$ is bounded by a positive constant.

2) For $\lambda = 0$,

$$\lim_{\underline{t} \rightarrow \infty, |x| \rightarrow 0} \|L(t, x)\|_{\alpha, 2} \left(\frac{(\frac{1}{2H})^N}{(2\pi)^{\frac{d}{2}} N!} \left(\log \frac{2\underline{t}^{2H}}{|x|^2} \right)^N \right)^{-1} = 1.$$

3) For $\lambda < 0$,

$$\lim_{\underline{t} \rightarrow \infty, |x| \rightarrow 0} \|L(t, x)\|_{\alpha, 2} (\underline{t}^{(1-dH)} \|L(\tilde{1}, 0)\|_{\alpha, 2})^{-1} = 1.$$

Proof. From the Wiener chaos expansion of $L(t, x)$ proved in [1], we can obtain that

$$\|L(t, x)\|_{\alpha, 2}^2 = \underline{t}^{2(1-dH)} \|L(\tilde{1}, \underline{t}^{-H} x)\|_{\alpha, 2}^2,$$

and the conclusion follows from the results of the previous section. □

Acknowledgment. The authors wish to express their thanks to the referee for all his helpful suggestions and comments. Also M. Eddahbi is grateful to the *Centre de Recerca Matemàtica (IEC) and the Department of Mathematics of the UAB, Bellaterra, Barcelona, Spain* for the kind support and hospitality he received while this work was carried out.

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