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HILBERT-KUNZ MULTIPLICITY AND REDUCTION MOD p

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Abstract. We show that the Hilbert-Kunz multiplicities of the reductions to positive characteristics of an irreducible projective curve in characteristic 0 have a well-defined limit as the characteristic tends to infinity.

Let R be a Noetherian ring of prime characteristic $p > 0$ and of dimension d and let $I \subseteq R$ be an ideal of finite colength. Then we recall that the Hilbert-Kunz multiplicity of R with respect to I is defined as

$$e_{HK}(R, I) = \lim_{n\to\infty} \frac{\ell(R/I^{[p^n]})}{p^{nd}},$$

where

$$\begin{aligned} I^{[p^n]} &= n\text{-th Frobenius power of } I \\ &= \text{ideal generated by } p^n\text{-th power of elements of } I \end{aligned}$$

is an ideal of finite colength and $\ell(R/I^{[p^n]})$ denotes the length of the R-module $R/I^{[p^n]}$.

We note that this limit always exists as proved by Monsky. However, unlike Hilbert-Samuel multiplicity, this multiplicity could depend on the characteristic of the ring (see example of [HM] given here in Section 2).

In this paper, we study the behaviour of Hilbert-Kunz multiplicities (abbreviated henceforth to HK multiplicities) of the reductions to positive characteristics of an irreducible projective curve in characteristic 0.

For instance, consider the following question. Let f be a nonzero irreducible homogeneous element in the polynomial ring $\mathbb{Z}[X_1, X_2, \ldots, X_r]$, and for any prime number $p \in \mathbb{Z}$, let $R_p = \mathbb{Z}/p\mathbb{Z}[X_1, X_2, \ldots, X_r]/(f)$ (this is the homogeneous coordinate ring of a projective variety over $\mathbb{Z}/p\mathbb{Z}$). Let

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$e_{HK}(R_p)$ denote the Hilbert-Kunz multiplicity of R_p with respect to the graded maximal ideal. Then one can ask: does $\lim_{p\to\infty} e_{HK}(R_p)$ exist?

This question was first encountered by the author in a survey article [C], Problem 4, Section 5 (see also Remark 4.10 in [B1]). This seems a difficult question in general, as so far, there is no known general formula for HK multiplicity in terms of 'better understood' invariants. There does not seem to even be a heuristic argument as to why the limit should exist, in general, in arbitrary dimensions.

However in the case of a projective curve (equivalently 2 dimensional standard graded ring) over an algebraically closed field of characteristic $p > 0$, one can express HK multiplicity in terms of (i) "standard" invariants of the curve which are constant in a flat family and (ii) normalized slopes of the quotients occuring in a strongly semistable Harder-Narasimhan filtration (HN filtration) (see Definitions 1.2 and 1.9) of the associated vector bundle on the curve (see [B1] and [T1]).

Hence, we may pose the question in the following more general setting. Given a projective curve X defined over a field k of char 0 with a vector bundle V on X, let (A, X_A, V_A) be a spread of the pair (X, V) (details given above Proposition 2.2). For all closed points $s \in \operatorname{Spec} A$, let $V_s = V_A \otimes \overline{k(s)}$. Now for given $k \geq 0$ and each such V_s, let

$$0 \subset F_1^s \subset \cdots \subset F_{t_s}^s \subset F_{t_s+1}^s = F^{k*}V_s$$

be the HN filtration of $F^{k*}V_s$. Denote

$$r_i(F^{k*}V_s) = \operatorname{rank}\left(\frac{F_i^s}{F_{i-1}^s}\right) \text{ and}$$
$$\text{the normalized slope } a_i(F^{k*}V_s) = \frac{1}{p^k}\mu\left(\frac{F_i^s}{F_{i-1}^s}\right).$$

Let $s_0 \in \operatorname{Spec} A$ be the generic point of $\operatorname{Spec} A$. Then the question is:

$$\text{(0.1)} \qquad \text{For given } k \geq 0, \text{ does } \lim_{s\to s_0} \sum_i r_i(F^{k*}V_s) a_i(F^{k*}V_s)^2 \text{ exist?}$$

We approach the question as follows. Following the notation of [L], for a vector bundle V on a nonsingular projective curve X in characteristic p, we attach convex polygons as follows. Consider the HN filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_l \subset E_{l+1} = V$$

of V. For $k \geq 0$, consider the HN filtration

$$0 = F_0 \subset F_1 \subset \cdots \subset F_t \subset F_{t+1} = F^{k*}V$$

of the iterated Frobenius pull back bundle $F^{k*}V$. Let $P(F_i) = (\operatorname{rank} F_i, \deg F_i/p^k)$ in $\mathbb{R}^2$. Let $HNP_{p^k}(V)$ be the convex polygon in $\mathbb{R}^2$ obtained by connecting $P(F_0), \ldots, P(F_{t+1})$ successively by line segments, and connecting the last one with the first one.

Let $p \geq 4(\text{genus}(X) - 1)(\operatorname{rank} V)^3$. Then we prove (Lemma 1.8) that the vertices of $HNP_{p^{k-1}}(V)$ are retained as a subset of the vertices of $HNP_{p^k}(V)$ and hence $HNP_{p^k}(V) \supset HNP_{p^0}(V)$. In particular, for $k \gg 0$, the HN filtration of the bundle $F^{k*}(V)$ is strongly semistable, therefore Theorem 2.7 of [L] comes as a corollary, in this case.

Now, for every vector bundle F_j of the HN filtration of $F^{k*}(V)$, if we denote the slope of the line segment, joining $P(F_{j-1})$ and $P(F_j)$, by $\mu_j(F^{k*}(V))/p^k$ (see Notation 1.4), and if E_i denotes the unique vector bundle occuring in the HN filtration of V such that F_j 'almost descends to' E_i (see Definition 1.12), then we prove (Lemma 1.14) that

$$\mu_j(F^{k*}V)/p^k = \mu_i(V) + O\left(\frac{1}{p}\right).$$

Hence $\lim_{p\to\infty} \operatorname{Area} HNP_{p^k}(V) = \operatorname{Area} HNP_{p^0}(V)$. In both Lemmas 1.8 and 1.14 we make crucial use of a result from the paper [SB] of Shepherd-Barron.

Now, following the notation set up for the question (0.1), if we take a vector bundle F_j^s occuring in the HN filtration of $F^{k*}(V_s)$ such that it almost descends to a vector bundle E_i^s occuring in the HN filtration of V_s then we get

$$a_j(F^{k*}(V_s)) := \frac{\mu_j(F^{k*}V_s)}{p^k} = \mu_i(V_s) + O\left(\frac{1}{p}\right),$$

where $p = \operatorname{char} k(s)$. From this we conclude (Proposition 2.2) that the question (0.1) has an *affirmative* answer.

In particular (Theorem 2.4) the Hilbert-Kunz multiplicities of the reductions to positive characteristics of an irreducible projective curve in characteristic 0 have a well-defined limit as the characteristic tends to ∞. This limit, which is (relatively) an easier invariant to compute, is a *lower bound* for the HK multiplicities of the reductions (mod p), though examples of Han-Monsky show that the convergence is <u>not</u> monotonic as $p \to \infty$, in general (see Remark 2.7).

§1. The HN slope of F^*V in terms of the HN slope of V

Let X be a nonsingular projective curve of genus $g \geq 1$, over an algebraically closed field k of characteristic $p > 0$. We recall the following definitions.

Definition 1.1. Let V be a vector bundle (*i.e.*, locally free coherent sheaf of $\mathcal{O}_X$-modules) on X. We say V is a *semistable* vector bundle on X if, for every subsheaf of $\mathcal{O}_X$-modules $F \subseteq V$, we have

$$\mu(F) := \frac{\deg F}{\operatorname{rank} F} \leq \mu(V),$$

where for a rank r vector bundle V on X we define

$$\deg V = \text{degree of the line bundle } \textstyle\bigwedge^r V \text{ on } X.$$

Definition 1.2. Let V be a vector bundle on X. A filtration of V by vector subbundles

$$0 = E_0 \subset E_1 \subset \cdots \subset E_l \subset E_{l+1} = V \tag{1.1}$$

is a *Harder-Narasimhan filtration* if

(1) the vector bundles $E_1, E_2/E_1, \ldots, E_{l+1}/E_l$ are all semistable.
(2) $\mu(E_1) > \mu(E_2/E_1) > \cdots > \mu(E_{l+1}/E_l)$.

Remark 1.3. For any Harder-Narasimhan filtration (we would call it HN filtration from now onwards), denoted as in Equation (1.1), the following is true (see [HN], Lemma 1.3.7),

(1) the filtration always exists and is unique for given V,
(2) $\mu(E_1) > \mu(E_2) > \cdots > \mu(E_{l+1}) = \mu(V)$,
(3) $\mu(E_i/E_{i-1}) \geq \mu(V) \geq \mu(E_{i+1}/E_i)$, for some $1 \leq i \leq l$.

Notation 1.4. If

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_l \subset E_{l+1} = V$$

is the HN filtration for a vector bundle V on X then we denote

$$\mu_i(V) = \mu\left(\frac{E_i}{E_{i-1}}\right), \quad \mu_{\max}(V) = \mu(E_1) \quad \text{and} \quad \mu_{\min}(V) = \mu\left(\frac{V}{E_l}\right).$$

LEMMA 1.5. *Let V be a vector bundle over X of rank r and let*

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_l \subset E_{l+1} = V$$

be the HN filtration of V. Then

$$r^3 > \frac{r-1}{\mu_i(V) - \mu_{i+1}(V)}.$$

Proof. Let $\mu_i = \mu_i(V)$. Let us denote $\bar{r}_i = \operatorname{rank} E_i/E_{i-1}$ and $\bar{d}_i = \deg E_i/E_{i-1}$. Then

$$\frac{r-1}{\mu_i - \mu_{i+1}} = \frac{r-1}{\bar{d}_i/\bar{r}_i - \bar{d}_{i+1}/\bar{r}_{i+1}} = \frac{(r-1)\bar{r}_i\bar{r}_{i+1}}{\bar{d}_i\bar{r}_{i+1} - \bar{d}_{i+1}\bar{r}_i}.$$

But

$$\mu_i - \mu_{i+1} > 0 \implies \bar{d}_i\bar{r}_{i+1} - \bar{d}_{i+1}\bar{r}_i > 0 \implies \bar{d}_i\bar{r}_{i+1} - \bar{d}_{i+1}\bar{r}_i \geq 1.$$

Therefore

$$\frac{r-1}{\mu_i - \mu_{i+1}} \leq (r-1)\bar{r}_i\bar{r}_{i+1} < r^3.$$

This proves the lemma. □

DEFINITION 1.6. If X is a projective variety defined over an algebraically closed field of characteristic $p > 0$, then the absolute Frobenius morphism $F : X \to X$ is a morphism of schemes which is identity on the underlying set of X and on the underlying sheaf of rings $F^{\#} : \mathcal{O}_X \to \mathcal{O}_X$ is the p^{th} power map.

Remark 1.7. For a vector bundle V on X, the Frobenius pull back F^*V is a vector bundle on X and

$$\operatorname{rank} F^*V = \operatorname{rank} V \quad \text{and} \quad \mu(F^*V) = p\mu(V).$$

We recall the following crucial result by Shepherd-Barron.

COROLLARY 2^p. ([SB]) *If X is a nonsingular projective curve of genus g and if V is a semistable vector bundle on X of rank r such that F^*V is not semistable then*

$$0 < \mu_{\max}(F^*V) - \mu_{\min}(F^*V) \leq (2g-2)(r-1). \qquad \square$$

Now we prove the following crucial lemma.

LEMMA 1.8. *Let V be a vector bundle on X with the HN filtration as in Lemma* 1.5. *Assume that* $\operatorname{char} k = p > 4(g-1)r^3$. *Then,*

$$F^*E_1 \subset F^*E_2 \subset \cdots \subset F^*E_l \subset F^*V$$

is a subfiltration of the HN filtration of F^*V, that is, if

$$0 \subset \widetilde{E}_1 \subset \cdots \subset \widetilde{E}_{l_1+1} = F^*V$$

is the HN filtration of F^*V then for every $1 \le i \le l$ there exists $1 \le j_i \le l_1$ such that $F^*E_i = \widetilde{E}_{j_i}$.

Proof. For each $0 \le i \le l$, let

$$F^*E_i \subset E_{i1} \subset \cdots \subset E_{it_i} \subset F^*E_{i+1}$$

be a filtration of vector bundles on X such that

$$0 \subset \frac{E_{i1}}{F^*E_i} \subset \frac{E_{i2}}{F^*E_i} \subset \cdots \subset \frac{F^*E_{i+1}}{F^*E_i}$$

is the HN filtration of $F^*(E_{i+1}/E_i)$. Now it is enough to prove the

CLAIM.

$$0 \subset E_{01} \subset \cdots \subset E_{0t_0} \subset F^*E_1 \subset \cdots \subset F^*E_i \subset E_{i1} \subset \cdots$$
$$\cdots \subset E_{it_i} \subset F^*E_{i+1} \subset \cdots \subset F^*V$$

is the HN filtration of F^*V.

Proof of the claim. By construction, for $0 \le i \le l$ and for $1 \le j < t_i$, we have

$$\mu\left(\frac{E_{ij}}{E_{i,j-1}}\right) > \mu\left(\frac{E_{i,j+1}}{E_{ij}}\right)$$

and

$$\frac{E_{ij}}{E_{i,j-1}}, \quad \frac{F^*E_i}{E_{i-1,t_{i-1}}} \quad \text{and} \quad \frac{E_{i1}}{F^*E_i}$$

are semistable. Hence, by Definition 1.2, it is enough to prove that

$$\mu\left(\frac{F^*E_i}{E_{i-1,t_{i-1}}}\right) > \mu\left(\frac{E_{i1}}{F^*E_i}\right).$$

Now, by Corollary 2^p of [SB], we have

$$0 \leq \mu_{\max} F^*\left(\frac{E_{i+1}}{E_i}\right) - \mu_{\min} F^*\left(\frac{E_{i+1}}{E_i}\right) \leq (2g-2)(r-1). \tag{1.2}$$

By Remark 1.3, for all $0 \leq i \leq l$, we have

$$\mu_{\max} F^*\left(\frac{E_{i+1}}{E_i}\right) \geq \mu\left(F^*\left(\frac{E_{i+1}}{E_i}\right)\right) \geq \mu_{\min} F^*\left(\frac{E_{i+1}}{E_i}\right).$$

Therefore

$$0 \leq \mu_{\max} F^*\left(\frac{E_{i+1}}{E_i}\right) - \mu\left(F^*\left(\frac{E_{i+1}}{E_i}\right)\right) \leq (2g-2)(r-1).$$

Let $\mu_i = \mu_i(V)$. Then we have

$$0 \leq \mu\left(\frac{E_{i1}}{F^*E_i}\right) - p\mu_{i+1} \leq (2g-2)(r-1). \tag{1.3}$$

Similarly

$$0 \leq \mu\left(F^*\left(\frac{E_i}{E_{i-1}}\right)\right) - \mu_{\min}\left(F^*\left(\frac{E_i}{E_{i-1}}\right)\right) \leq (2g-2)(r-1)$$

which means

$$0 \leq p\mu_i - \mu\left(\frac{F^*E_i}{E_{i-1,t_{i-1}}}\right) \leq (2g-2)(r-1). \tag{1.4}$$

Now, multiplying (1.3) and (1.4) by -1 and adding, we get

$$\begin{aligned} -4(g-1)(r-1) + p(\mu_i - \mu_{i+1}) &\leq \mu\left(\frac{F^*E_i}{E_{i-1,t_{i-1}}}\right) - \mu\left(\frac{E_{i1}}{F^*E_i}\right) \\ &\leq p(\mu_i - \mu_{i+1}). \end{aligned} \tag{1.5}$$

Since $p > 4(g-1)r^3$, Lemma 1.5 implies that

$$-4(g-1)(r-1) + p(\mu_i - \mu_{i+1}) > 0,$$

and hence

$$\mu\left(\frac{F^*E_i}{E_{i-1,t_{i-1}}}\right) > \mu\left(\frac{E_{i1}}{F^*E_i}\right),$$

This proves the claim, and hence the lemma. ∎

Definition 1.9. (1) A vector bundle V on X is strongly semistable if $F^{s*}(V)$ is semistable for every s^{th} iterated power of the absolute Frobenius map $F : X \to X$.

(2) A filtration by subbundles

$$0 = E_0 \subset E_1 \subset \cdots \subset E_l \subset E_{l+1} = V$$

of V is a *strongly semistable HN filtration* if

(a) it is the HN filtration and
(b) $E_1, E_2/E_1, \ldots, E_{l+1}/E_l$ are strongly semistable vector bundles.

Remark 1.10. (1) If the HN filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_l \subset E_{l+1} = V$$

of V is strongly semistable then, for any $k \geq 0$, the filtration

$$0 = E_0 \subset F^{k*}E_1 \subset \cdots \subset F^{k*}E_l \subset F^{k*}E_{l+1} = F^{k*}V$$

is the strongly semistable HN filtration of $F^{k*}V$.

(2) If V is a rank 2 vector bundle on X and is not semistable then its HN filtration will be strongly semistable; as it would be filtered by line bundles, which are always semistable and hence strongly semistable.

Remark 1.11. Note that, if $\operatorname{rank} V = r$ and $\operatorname{char} k = p > 4(g-1)r^3$, then Lemma 1.8 implies that there exists $s \geq 0$ such that the HN filtration of $F^{s*}V$ is strongly semistable. Therefore, Theorem 2.7 of [L] follows in this case.

Definition 1.12. Let E be a vector bundle on X. A vector bundle $F_j \neq 0$ occuring in the HN filtration of $F^{s*}E$ is said to *almost descend* to a bundle E_i occuring in the HN filtration of E if $F_j \subseteq F^{s*}E_i$ and E_i is the smallest bundle in the HN filtration of E, with this property.

Remark 1.13. Note that, if $p > 4(g-1)(\operatorname{rank} E)^3$, then by Lemma 1.8, we have the following transitivity property: if F_j almost descends to a bundle $\widetilde{E}_i$ in the HN filtration of $F^{k*}E$, and $\widetilde{E}_i$ almost descends to a bundle E_t occuring in the HN filtration of E, then F_j almost descends to the bundle E_t.

LEMMA 1.14. *Let E be a vector bundle on X of rank r and let the characteristic p satisfy $p > 4(g-1)r^3$. Let $F_j \neq 0$ be a subbundle in the HN filtration of $F^{s*}E$, which almost descends to a vector bundle E_i occuring in the HN filtration of E. Then*

$$\frac{\mu_j(F^{s*}E)}{p^s} = \mu_i(E) + \frac{C}{p},$$

where $|C| \leq 4(g-1)(r-1)$, and $\mu_j(F^{s*}E)$ and $\mu_i(E)$ are given as in Notation 1.4.

Proof. Let F_{j-1} be the vector bundle on X such that $F_{j-1} \subset F_j$ are two consecutive subbundles of the HN filtration of $F^{s*}E$. Therefore, by Lemma 1.8, there exist two consecutive subbundles $E_{i_1-1} \subset E_{i_1}$ in the HN filtration of $F^{(s-1)*}E$ such that

$$F^*E_{i_1-1} \subseteq F_{j-1} \subset F_j \subseteq F^*E_{i_1}.$$

In particular, we are in the situation that E_{i_1}/E_{i_1-1} is a semistable vector bundle on X and

(1) either $F_{j-1}/F^*E_{i_1-1} = 0$ in $F^*(E_{i_1}/E_{i_1-1})$, and $F_j/F^*E_{i_1-1}$ is the first nonzero vector bundle in the HN filtration of $F^*(E_{i_1}/E_{i_1-1})$ or
(2) $F_{j-1}/F^*E_{i_1-1} \subset F_j/F^*E_{i_1-1}$ are two consecutive subbundles in the HN filtration of $F^*(E_{i_1}/E_{i_1-1})$.

In both the cases, by Definition 1.2, we have

$$\mu_{\min}F^*\left(\frac{E_{i_1}}{E_{i_1-1}}\right) \leq \mu\left(\frac{F_j}{F_{j-1}}\right) \leq \mu_{\max}F^*\left(\frac{E_{i_1}}{E_{i_1-1}}\right).$$

Therefore, Corollary 2^p of [SB] implies

$$-2(g-1)(r-1) \leq \mu_j(F^{s*}(V)) - \mu\left(F^*\left(\frac{E_{i_1}}{E_{i_1-1}}\right)\right) \leq 2(g-1)(r-1).$$

Note that $\mu(F^*(E_{i_1}/E_{i_1-1})) = p\mu_{i_1}(F^{(s-1)*}E)$. Therefore we have

$$\mu_j(F^{s*}E) = p\mu_{i_1}(F^{(s-1)*}E) + C_1,$$

where $|C_1| \leq 2(g-1)(r-1)$.

Note E_{i_1} is a nonzero subbundle in the HN filtration of $F^{(s-1)*}E$ which almost descends to E_i occuring in the HN filtration of E. Hence, inductively one can prove that

$$\mu_{i_1}(F^{(s-1)*}E) = p^{s-1}\mu_i(E) + p^{s-2}C_s + \cdots + C_2,$$

where $|C_2|, \ldots, |C_s| \leq 2(g-1)(r-1)$. Therefore

$$\mu_j(F^{s*}E) = p^s\mu_i(E) + p^{s-1}C_s + \cdots + pC_2 + C_1.$$

Therefore

$$\frac{\mu_j(F^{s*}E)}{p^s} = \mu_i(E) + \frac{1}{p^s}(p^{s-1}C_s + \cdots + pC_2 + C_1).$$

But

$$|(p^{s-1}C_s + \cdots + pC_2 + C_1)| \leq (1 + \cdots + p^{s-1})(2(g-1)(r-1)).$$

Since $(1 + p + \cdots + p^{s-1})/p^{s-1} \leq 2$, we have

$$\frac{|p^{s-1}C_s + \cdots + pC_2 + C_1|}{p^{s-1}} \leq 4(g-1)(r-1).$$

Therefore we conclude that

$$\frac{\mu_j(F^{s*}E)}{p^s} = \mu_i(E) + \frac{C}{p},$$

where $|C| \leq 4(g-1)(r-1)$. This proves the lemma. ∎

Notation 1.15. Henceforth we assume that the characteristic p satisfies $p > 4(g-1)r^3$. We also fix a vector bundle V on X of rank r with the HN filtration

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_l \subset E_{l+1} = V.$$

Let

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_t \subset F_{t+1} = F^{k*}V \tag{1.6}$$

be the HN filtration of $F^{k*}V$, and let

$$r_i(F^{k*}V) = \mathrm{rank}\left(\frac{F_i}{F_{i-1}}\right) \quad \text{and} \quad a_i(F^{k*}V) = \frac{\mu_i(F^{k*}V)}{p^k}.$$

Proposition 1.16. *With the notation as above, where $p > 4(g-1)r^3$, if a vector bundle F_j of the HN filtration of $F^{k*}V$ almost descends to a vector bundle E_i of the HN filtration of V then, for any $m \geq 1$,*

$$a_j(F^{k*}V)^m = \mu_i(V)^m + \frac{C}{p},$$

where $|C| \leq 8gr(\max\{2|\mu_1(V)|, \ldots, 2|\mu_{l+1}(V)|, 2\}^{m-1})$.

Proof. By Lemma 1.14, we have

$$a_j(F^{k*}V) = \mu_i(V) + \frac{c_{ij}}{p},$$

where $|c_{ij}| \leq 4(g-1)(r-1)$. For the sake of abbreviation let us denote $\mu_j(V)$ by μ_j. Therefore

$$a_j(F^{k*}V)^m - \mu_i^m = \binom{m}{1}\mu_i^{m-1}\frac{c_{ij}}{p} + \cdots + \binom{m}{m-1}\mu_i\frac{c_{ij}^{m-1}}{p^{m-1}} + \binom{m}{m}\frac{c_{ij}^m}{p^m}.$$

Hence

$$\begin{aligned}&|a_j(F^{k*}V)^m - \mu_i^m| \\ &\quad \leq \frac{|c_{ij}|}{p}\left[\binom{m}{1}|\mu_i|^{m-1} + \cdots + \binom{m}{m-1}|\mu_i|\frac{|c_{ij}|^{m-2}}{p^{m-2}} + \binom{m}{m}\frac{|c_{ij}|^{m-1}}{p^{m-1}}\right].\end{aligned}$$

Now, as $|c_{ij}|/p \leq 1$, this implies

$$|a_j(F^{k*}V)^m - \mu_i^m| \leq \frac{|c_{ij}|}{p}\left[\binom{m}{1}|\mu_i|^{m-1} + \cdots + \binom{m}{m-1}|\mu_i| + \binom{m}{m}\right].$$

(1) Let $|\mu_i| \leq 1$. Then

$$\begin{aligned}|a_j(F^{k*}V)^m - \mu_i^m| &\leq \frac{|c_{ij}|}{p}\left[\binom{m}{1} + \cdots + \binom{m}{m-1} + \binom{m}{m}\right] \\ &\leq \frac{|c_{ij}|}{p}(2^m - 1) \leq \frac{1}{p}(8gr(2^{m-1})).\end{aligned}$$

(2) Let $|\mu_i| \geq 1$. Then

$$\begin{aligned}|a_j(F^{k*}V)^m - \mu_i^m| &\leq \frac{|c_{ij}||\mu_i|^{m-1}}{p}\left[\binom{m}{1} + \cdots + \binom{m}{m-1} + \binom{m}{m}\right] \\ &\leq \frac{|c_{ij}||\mu_i|^{m-1}}{p}(2^m - 1) \leq \frac{1}{p}(8gr(2|\mu_i|)^{m-1}).\end{aligned}$$

Hence the proposition. ∎

§2. Applications

We extend Notation 1.15 to the case, when the underlying field is of arbitrary characteristic, as follows.

NOTATION 2.1. Let X be a nonsingular curve over an algebraically closed field k and V a vector bundle on X, with HN filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_l \subset E_{l+1} = V.$$

(1) If $\operatorname{char} k = p > 0$, then we define the numbers $\mu_i(F^{k*}V)$, $r_i(F^{k*}V)$ and $a_i(F^{k*}V)$ as in Notations 1.4 and 1.15. Moreover, we choose an integer $s \geq 0$ such that $F^{s*}(V)$ has a strongly semistable HN filtration and we denote

$$\widetilde{a}_i(V) = a_i(F^{s*}(V)) \quad \text{and} \quad \widetilde{r}_i(V) = r_i(F^{s*}(V))$$

(note that, by Remark 1.10, these numbers are independent of the choice of such an s).

(2) If $\operatorname{char} k = 0$, define

$$\widetilde{a}_i(V) = \mu_i(V) = \mu\left(\frac{E_i}{E_{i-1}}\right), \quad \text{and} \quad \widetilde{r}_i(V) = r_i(V) = \operatorname{rank}\left(\frac{E_i}{E_{i-1}}\right).$$

Here we recall a notion of *spread* for the pair (X, V), where X is a nonsingular curve over a field of characteristic 0 and V is a vector bundle on X. For such a pair there exists a finitely generated $\mathbb{Z}$-algebra $A \subseteq k$ and a projective A-scheme X_A over A and coherent, locally free sheaves V_A and

$$E_{1A} \subset \cdots \subset E_{lA} \subset V_A$$

on X_A such that

$$X_A \times_{\operatorname{Spec} A} \operatorname{Spec} k = X \quad \text{and} \quad V_A \otimes_A k = V,$$

and for all closed points $s \in \operatorname{Spec} A$, if

$$V_s = V_A \otimes \overline{k(s)}, \quad \text{and} \quad E_{i(s)} = E_{iA} \otimes \overline{k(s)},$$

then

$$0 \subset E_{1(s)} \subset \cdots \subset E_{l(s)} \subset V_s$$

is the HN filtration of V_s (this follows by an openness property of semistable vector bundles ([Ma])). We call the triple (A, X_A, V_A) a spread of (X, V).

Moreover, if, for the pair (X, V), we have a spread (A, X_A, V_A) as above and $A \subset A' \subset k$, for some finitely generated $\mathbb{Z}$-algebra A' then $(A', X_{A'}, V_{A'})$ satisfy the same properties as (A, X_A, V_A). Hence we may always assume that the spread (A, X_A, V_A) as above is chosen such that A contains a given finitely generated algebra $A_0 \subseteq k$.

Proposition 2.2. *Let $f : X_A \to \operatorname{Spec} A$ be a projective morphism of Noetherian schemes, smooth of relative dimension 1, where A is a finitely generated $\mathbb{Z}$-algebra and is an integral domain. Let $\mathcal{O}_{X_A}(1)$ be an f-very ample invertible sheaf on X_A. Let V_A be a vector bundle on X_A. For $s \in \operatorname{Spec} A$, let $V_s = V_A \otimes_A \overline{k(s)}$ be the induced vector bundle on the smooth projective curve $X_s = X_A \otimes_A \overline{k(s)}$. Let $s_0 = \operatorname{Spec} Q(A)$ be the generic point of* $\operatorname{Spec} A$. *Then,*

(1) *for any $k \geq 0$ and $m \geq 0$, we have*

$$\lim_{s \to s_0} \sum_j r_j(F^{k*}V_s) a_j(F^{k*}V_s)^m = \sum_i r_i(V_{s_0}) \mu_i(V_{s_0})^m.$$

(2) *Similarly*

$$\lim_{s \to s_0} \sum_j \widetilde{r}_j(V_s) \widetilde{a}_j(V_s)^m = \sum_i r_i(V_{s_0}) \mu_i(V_{s_0})^m,$$

where in both the limits, s runs over closed points of $\operatorname{Spec} A$.

Proof. To prove the proposition, one can replace $\operatorname{Spec} A$ by an affine open subset (after localizing A if necessary), so that

$$(A, X_A, V_A) \text{ is a spread of } (X_A \otimes k(s_0), V_A \otimes k(s_0))$$

as defined above. Moreover we can choose A such that, for any closed point $s \in \operatorname{Spec} A$, we have

$$\operatorname{char} k(s) > 4(\operatorname{genus} X_s - 1)(\operatorname{rank} V_s)^3 = 4(\operatorname{genus} X_{s_0} - 1)(\operatorname{rank} V_{s_0})^3.$$

Therefore, if we denote

$$M = 8(\operatorname{genus}(X_{s_0})) r(V_{s_0}) (\max\{2, 2|\mu_1(V_{s_0})|, \ldots, |\mu_{l+1}(V_{s_0})|\}^{m-1}),$$

where $r(V_{s_0}) = \mathrm{rank}(V_{s_0})$, then, by Proposition 1.16, we have

$$\begin{aligned}\sum_j r_j(F^{k*}V_s)a_j(F^{k*}V_s)^m &= \sum_i r_i(V_s)\left(\mu_i(V_s)^m + \frac{C_i}{p}\right), \quad \text{where } |C_i| \leq M \\ &= \sum_i r_i(V_{s_0})\mu_i(V_{s_0})^m + \frac{C_{s_k}}{p},\end{aligned}$$

where $|C_{s_k}| \leq r(V_{s_0})M$. In particular, for every closed point $s \in \operatorname{Spec} A$, we have

$$\sum_j \widetilde{r}_j(V_s)\widetilde{a}_j(V_s)^m = \sum_i r_i(V_{s_0})\mu_i(V_{s_0})^m + \frac{C_s}{p},$$

where $|C_s| \leq r(V_{s_0})M$. Now the proposition follows easily. ∎

Corollary 2.3. *Along with Notation* 2.1*, if we denote* (*as defined in* [B2])*, for* $\operatorname{char} k > 0$*,* $\mu_{HK}(V) = \sum_i \widetilde{r}_i(V)\widetilde{a}_i(V)^2$*, and for* $\operatorname{char} k = 0$*,* $\mu_{HK}(V) = \sum_j r_j(V)\mu_j(V)^2$*, then*

$$\lim_{s \to s_0} \mu_{HK}(V_s) = \mu_{HK}(V_{s_0}).$$

Proof. The corollary follows by substituting $m = 2$ in the second statement of Proposition 2.2. ∎

Here recall similar notion of spread for the pair (R, I), where R is a finitely generated $\mathbb{N}$-graded two dimensional domain over an algebraically closed field k of characteristic 0 and $I \subset R$ is a homogeneous ideal of finite colength. For such a pair, there exists a finitely generated $\mathbb{Z}$-algebra $A \subseteq k$, a finitely generated $\mathbb{N}$-graded algebra R_A over A and a homogeneous ideal $I_A \subset R_A$ such that $R_A \otimes_A k = R$ and for any closed point $s \in \operatorname{Spec} A$ (*i.e.* maximal ideal of A) the ring $R_s = R_A \otimes_A k(s)$ is a finitely generated $\mathbb{N}$-graded 2-dimensional domain (which is a normal domain if R is normal) over $k(s)$ and the ideal $I_s = \operatorname{Im}(I_A \otimes_A k(s)) \subset R_s$ is a homogeneous ideal of finite colength. We call (A, R_A, I_A) a spread of the pair (R, I).

Moreover, if, for the pair (R, I), we have a *spread* (A, R_A, I_A) as above and $A \subset A' \subset k$, for some finitely generated $\mathbb{Z}$-algebra A' then $(A', R_{A'}, I_{A'})$ satisfy the same properties as (A, R_A, I_A). Hence we may always assume that the spread (A, R_A, I_A) as above is chosen such that A contains a given finitely generated algebra $A_0 \subseteq k$.

THEOREM 2.4. *Let R be a standard graded two dimensional domain over an algebraically closed field k of characteristic* 0. *Let $I \subset R$ be a homogeneous ideal of finite colength. Let (A, R_A, I_A) be a spread as given above. Then*

$$\lim_{s \to s_0} e_{HK}(R_s, I_s)$$

exists and is a rational number, where $s_0 = \operatorname{Spec} Q(A)$ is the generic point of $\operatorname{Spec} A$, *and the limit is taken over closed points $s \in \operatorname{Spec} A$.*

Proof. Let $R \to S$ be the normalization of R. Then $R \to S$ is a finite graded map of degree 0, and $Q(R) = Q(S)$, such that S is a finitely generated $\mathbb{N}$-graded 2-dimensional normal domain over k. Now, for pairs (R, I), (S, IS), we choose spreads (A, R_A, I_A) and (A, S_A, IS_A) such that for every closed point $s \in \operatorname{Spec} A$, the natural map $R_s = R_A \otimes k(s) \to S_s = S_A \otimes k(s)$ is a finite graded map of degree 0. Therefore we have the following commutative diagrams of horizontal finite maps

$$\begin{array}{ccc} \operatorname{Proj} R & \longleftarrow & \operatorname{Proj} S \\ \big\downarrow & & \big\downarrow \\ \operatorname{Proj} R_A & \longleftarrow & \operatorname{Proj} S_A. \end{array}$$

It follows that, for every $s \in \operatorname{Spec} A$, the corresponding map of curves

$$\operatorname{Proj} S_A \otimes_A k(s) \longrightarrow \operatorname{Proj} R_A \otimes_A k(s)$$

is a finite map, where the curve $\operatorname{Proj} S_A \otimes_A k(s)$ is nonsingular. Since $R_s \to S_s$ is a finite map such that S_s is a module of rank 1 over R_s, by Lemma 1.3 in [M], Theorem 2.7 in [WY] and [BCP], we have

$$e_{HK}(R_s, I_s) = e_{HK}(S_s, IS_s), \text{ for every closed point } s \in \operatorname{Spec} A.$$

Therefore it is enough to prove the following

CLAIM. $\lim_{s \to s_0} e_{HK}(S_s, IS_s)$ *exists.*

Proof of the claim. Let I and IS_A be generated by the set $\{f_1, \dots, f_k\}$, where $\deg f_i = d_i$. We have a short exact sequence of $\mathcal{O}_{X_A}$-sheaves (see [B1] and [T1]):

$$0 \longrightarrow V_A \longrightarrow \bigoplus_{i=1}^{k} \mathcal{O}_{X_A}(1 - d_i) \longrightarrow \mathcal{O}_{X_A}(1) \longrightarrow 0 \tag{2.1}$$

where $\mathcal{O}_{X_A}(1-d_i) \to \mathcal{O}_{X_A}(1)$ is multiplication by f_i. Restricting (2.1) to the fiber X_s, we get

$$0 \longrightarrow V_s \longrightarrow \bigoplus_{i=1}^{k} \mathcal{O}_{X_s}(1-d_i) \longrightarrow \mathcal{O}_{X_s}(1) \longrightarrow 0.$$

Note that (see [B1] and [T1]),

$$e_{HK}(S_s, IS_s) = \frac{\deg \operatorname{Proj} S_s}{2}\left(\sum_i \widetilde{r}_i(V_s)\widetilde{a}_i(V_s)^2 - \sum_{i=1}^{k} d_i^2\right).$$

Therefore

$$\lim_{s\to s_0} e_{HK}(S_s, IS_s) = \frac{\deg \operatorname{Proj} S}{2}\left(\lim_{s\to s_0}\sum_i \widetilde{r}_i(V_s)\widetilde{a}_i(V_s)^2 - \sum_{i=1}^{k} d_i^2\right).$$

Hence, by Proposition 2.2,

$$\lim_{s\to s_0} e_{HK}(S_s, IS_s) = \frac{\deg \operatorname{Proj} S}{2}\left(\sum_i r_i(V_{s_0})\mu_i(V_{s_0})^2 - \sum_{i=1}^{k} d_i^2\right).$$

In particular $\lim_{s\to s_0} e_{HK}(S_s, IS_s)$ exists and is a rational number. This proves the theorem. ∎

Remark 2.5. Let R be a standard graded 2 dimensional domain over a field of characteristic 0. Let $I \subset R$ be a homogeneous ideal of finite colength. Then for the pair (R, I) we choose a spread (A, X_A, I_A) as described earlier and define

$$e_{HK}(R, I) = \lim_{s\to s_o} e_{HK}(R_s, I_s). \tag{2.2}$$

This is, inherently, a well defined notion (*i.e.*, irrespective of a choice of generators of I), since in positive characteristic $e_{HK}(R_s, I_s)$ is independent of a choice of generators of I_s. We extend this definition to a standard graded 2-dimensional ring R, over a field k of characteristic 0, and a homogeneous ideal $I \subset R$ of finite colength as

$$e_{HK}(R, I) = \sum_{\mathbf{p}\in \operatorname{Spec} R,\, \dim R/\mathbf{p}=2} \ell_{R_\mathbf{p}}(R_\mathbf{p}) e_{HK}(R/\mathbf{p}, IR/\mathbf{p}),$$

This is a always a rational number, by Theorem 2.4.

Note that a notion of $e_{HK}(R,I)$, when R is also a normal domain (*i.e.*, $\mathrm{Proj}\,R$ is a smooth curve) over a field of characteristic 0, is given in [B2] as

$$e_{HK}(R,I) = \frac{\deg \mathrm{Proj}\,R}{2}\left(\mu_{HK}(V) - \sum_{i=1}^{k} d_i^2\right), \tag{2.3}$$

where V is the vector bundle given by

$$0 \longrightarrow V \longrightarrow \bigoplus_i \mathcal{O}_X(1-d_i) \longrightarrow \mathcal{O}_X(1) \longrightarrow 0.$$

By Corollary 2.3, these two definitions (2.3) and (2.2) coincide, in this case.

Remark 2.6. It follows from Remark 4.13 of [T1] that, for every closed point s in $\mathrm{Spec}\,A$, where (A, R_A, I_A) is a spread for the pair (R,I), we have

$$e_{HK}(R_s, I_s) \geq e_{HK}(R,I),$$

and $e_{HK}(R_s,I_s) = e_{HK}(R,I)$ if and only if HN filtration of V_s is the strongly semistable HN filtration, where $e_{HK}(R_s,I_s)$ is the HK multiplicity defined (as given in the introduction) over the residue field $k(s)$, of the point s, which is of positive characteristic and $e_{HK}(R,I)$ is defined (as in Remark 2.5), over the quotient field of A which is of characteristic 0. If V_s is semistable then

$$e_{HK}(R,I) = \frac{\deg \mathrm{Proj}\,R_s}{2}\left(\Big(\sum_i d_i\Big)^2/(t-1) - \sum_i d_i^2\right).$$

Remark 2.7. As observed in the above remark,

$$\{e_{HK}(R_s,I_s) - e_{HK}(R,I) \mid s \in \{\text{closed points of } \mathrm{Spec}\,A\}\}$$

is a sequence of non-negative rational numbers (indexed by the closed points of $\mathrm{Spec}\,A$), converging to 0. Examples show that it could be *oscillating*.

First we recall the following result of [T2]

Corollary. *Let $X_p = \mathrm{Proj}\,R_p$, where $R_p = k[x,y,z]/(f))$, be a non-singular plane curve of degree d over an algebraically closed field k of characteristic $p > 0$. Then*

$$e_{HK}(X_p, \mathcal{O}_{X_p}(1)) = e_{HK}(R_p, (x,y,z)R_p) = \frac{3d}{4} + \frac{l^2}{4dp^{2s}},$$

where $s \geq 1$ is a number such that $F^{(s-1)*}V_{X_p}$ is semistable and $F^{s*}V_{X_p}$ is not semistable (if $F^{t*}V_{X_p}$ is semistable for all $t \geq 0$, we take $s = \infty$) and l is an integer congruent to pd (mod 2) with $0 \leq l \leq d(d-3)$. ∎

Monsky (around 1990) calculated the Hilbert Kunz function for plane curves $k[x,y,z]/(x^d+y^d+z^d)$, this result was later generalized in [H] and [HM] to diagonal hypersurfaces $k[x_1,x_2,\ldots]/(\sum_i x_i^d)$. In particular, arguing as in the examples of [HM] we have the following

Let

$$R_p = k[X,Y,Z]/(x^4+y^4+z^4), \quad \text{where } \operatorname{char} k = p.$$

Then

$$\begin{aligned} e_{HK}(R_p,(x,y,z)R_p) &= 3 + \frac{1}{p^2}, \quad \text{if } p \equiv \pm 3(8) \\ &= 3, \qquad\quad \text{if } p \equiv \pm 1(8). \end{aligned}$$

Now, let $X_p = \operatorname{Proj} R_p$. Consider the short exact sequence

$$0 \longrightarrow V_{X_p} \longrightarrow \mathcal{O}_{X_p} \oplus \mathcal{O}_{X_p} \oplus \mathcal{O}_{X_p} \longrightarrow \mathcal{O}_{X_p}(1) \longrightarrow 0,$$

where the second map is given by $(f_1,f_2,f_3) \to xf_1 + yf_2 + zf_3$.

By the above Corollary, we have

(1) if $p \equiv \pm -3\,(8)$ and $p \gg 0$ then $l = 4$ and $s = 1$, *i.e.* V_{X_p} is semistable, and $F^*(V_{X_p})$ is not semistable and has strongly semistable HN filtration and

$$a_1(V_{X_p}) = \mu(V_{X_p}) + \frac{2}{p} \quad \text{and} \quad a_2(V_{X_p}) = \mu(V_{X_p}) - \frac{2}{p}$$

In particular $\mu_{HK}(V_{X_p}) = 2\mu(V_{X_p})^2 + \frac{8}{p^2}$.

(2) if $p \equiv \pm -1\,(8)$ then $l = 0$, *i.e.* V_{X_p} is strongly semistable, and

$$a_1(V_{X_p}) = \mu(V_{X_p})$$

In particular $\mu_{HK}(V_{X_p}) = 2\mu(V_{X_p})^2$.

In particular, for $p \gg 0$ the numbers $a_1(V_{X_p})$ $a_2(V_{X_p})$ do not eventually become constant or a well defined function of p, but keep oscillating and converge to $\mu(V_X)$.

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