T. Tanisaki and N. Xi
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KAZHDAN-LUSZTIG BASIS AND A GEOMETRIC FILTRATION OF AN AFFINE HECKE ALGEBRA

TOSHIYUKI TANISAKI AND NANHUA XI

Dedicated to Professor George Lusztig on his sixtieth birthday

Abstract. According to Kazhdan-Lusztig and Ginzburg, the Hecke algebra of an affine Weyl group is identified with the equivariant K-group of Steinberg's triple variety. The K-group is equipped with a filtration indexed by closed G-stable subvarieties of the nilpotent variety, where G is the corresponding reductive algebraic group over $\mathbb{C}$. In this paper we will show in the case of type A that the filtration is compatible with the Kazhdan-Lusztig basis of the Hecke algebra.

§0. Introduction

Let G be a connected reductive algebraic group over the complex number field $\mathbb{C}$ with simply-connected derived group. Let W and P be its Weyl group and weight lattice respectively. The semidirect product $\tilde{W}_a = WP$ with respect to the action of W on P is called an (extended) affine Weyl group. Let $H(\tilde{W}_a)$ be the associated Hecke algebra. According to Kazhdan-Lusztig and Ginzburg ([6], [3]) we have a geometric realization of $H(\tilde{W}_a)$ in terms of equivariant K-theory. Namely, we have an isomorphism

$$\Phi : H(\tilde{W}_a) \longrightarrow K^{G\times\mathbb{C}^*}(Z)$$

of $\mathbb{Z}[q^{1/2}, q^{-1/2}]$-algebras, where $K^{G\times\mathbb{C}^*}(Z)$ denotes the equivariant K-group of Steinberg's triple variety Z with respect to the natural action of $G\times\mathbb{C}^*$. Let $\mathcal{N}$ be the nilpotent variety of the Lie algebra $\mathfrak{g}$ of G. For each G-stable closed subset V of $\mathcal{N}$ there corresponds a $G\times\mathbb{C}^*$-stable closed subvariety Z_V of Z, and the associated equivariant K-group $K^{G\times\mathbb{C}^*}(Z_V)$ is identified with a two-sided ideal of $K^{G\times\mathbb{C}^*}(Z)$. Moreover, we have $K^{G\times\mathbb{C}^*}(Z_{V_1}) \subset K^{G\times\mathbb{C}^*}(Z_{V_2})$ if $V_1 \subset V_2$.

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Recall that $H(\tilde{W}_a)$ is equipped with the Kazhdan-Lusztig basis $\{C_w \mid w \in \tilde{W}_a\}$ ([5]). It plays very important roles in various aspects of the representation theory of reductive algebraic groups. It should be an interesting problem to give a geometric description of $\Phi(C_w)$ for $w \in \tilde{W}_a$. An answer in the case $w \in W$ is given in [18]. Moreover, the answer for certain elements corresponding to dominant elements in P is given in [13]. Related to this problem, it is conjectured that $K^{G\times\mathbb{C}^*}(Z_V)$ is spanned by a subset of $\{\Phi(C_w) \mid w \in \tilde{W}_a\}$ for any G-stable closed subset V of $\mathcal{N}$. In particular, any $H(\tilde{W}_a)$-bimodule associated to a two-sided cell of $\tilde{W}_a$ should be identified with $K^{G\times\mathbb{C}^*}(Z_{\overline{O}})/K^{G\times\mathbb{C}^*}(Z_{\overline{O}\setminus O})$ for a nilpotent orbit O.

The aim of this paper is to prove this conjecture in the case G is of type A. A key to this result is the fact that the $H(\tilde{W}_a)$-bimodule corresponding to a two-sided cell of $\tilde{W}_a$ is generated by a single element (see Theorem 4.3 below).

The contents of this paper are as follows. In Section 1 and Section 2 we will recall some fundamental facts on (affine) Hecke algebras. A precise formulation of the above stated conjecture in view of the bijection between the set of nilpotent orbits and that of two-sided cells will be given in Section 3. In Section 4 we will give a proof of the conjecture in the case $G = GL_n(\mathbb{C})$. The arguments works for $SL_n(\mathbb{C})$ as well. In Appendix A we will collect well-known facts on equivariant K-theory, and in Appendix B we will give a description of the product on the quotient $K^{G\times\mathbb{C}^*}(Z_{\overline{O}})/K^{G\times\mathbb{C}^*}(Z_{\overline{O}\setminus O})$ for any G in terms of the Springer fiber and Slodowy's variety, where O is a nilpotent orbit.

§1. Hecke algebras

Let (W, S) be a Coxeter system with the length function $\ell : W \to \mathbb{Z}_{\geq 0}$ and the standard partial order $\geq$. Assume that we are given a group Ω and a group homomorphism $\Omega \to \mathrm{Aut}(W, S)$, where $\mathrm{Aut}(W, S)$ denotes the automorphism group of (W, S). We denote by $\tilde{W}$ the semidirect product ΩW with respect to the action of Ω on W. The length function ℓ and the standard partial order $\geq$ for W are naturally extended to $\tilde{W}$ by

$$\ell(\omega w) = \ell(w) \quad (\omega \in \Omega,\, w \in W),$$
$$\omega_1 w_1 \geq \omega_2 w_2 \Longleftrightarrow \omega_1 = \omega_2,\, w_1 \geq w_2 \quad (\omega_1, \omega_2 \in \Omega,\, w_1, w_2 \in W).$$

For w in $\tilde{W}$ we set

$$L(w) = \{s \in S \mid sw \leq w\}, \quad R(w) = \{s \in S \mid ws \leq w\}.$$

We denote by $H(\tilde{W})$ the Hecke algebra associated to $\tilde{W}$. It is an associative algebra over the Laurent polynomial ring $\mathbb{Z}[q^{1/2}, q^{-1/2}]$. As a $\mathbb{Z}[q^{1/2}, q^{-1/2}]$-module it has a free basis $\{T_w \mid w \in \tilde{W}\}$, and the multiplication is determined by

$$\begin{aligned} &T_y T_w = T_{yw} \quad (y, w \in \tilde{W},\, \ell(y) + \ell(w) = \ell(yw)), \\ &(T_s + 1)(T_s - q) = 0 \quad (s \in S). \end{aligned}$$

There is a unique ring automorphism $h \mapsto \overline{h}$ of $H(\tilde{W})$ determined by

$$\overline{q^{1/2}} = q^{-1/2}, \quad \overline{T}_w = T_{w^{-1}}^{-1} \quad (w \in \tilde{W}).$$

According to Kazhdan-Lusztig [5], for each $w \in \tilde{W}$ there exists uniquely an element

$$C_w = \sum_{y \le w} P_{y,w}(q) T_y$$

of $H(\tilde{W})$ satisfying

(a) $P_{w,w}(q) = 1$,
(b) for $y < w$ we have $P_{y,w}(q) \in \mathbb{Z}[q]$, and $\deg(P_{y,w}(q)) \le (\ell(w) - \ell(y) - 1)/2$,
(c) $\overline{C}_w = q^{-\ell(w)} C_w$.

The basis $\{C_w \mid w \in \tilde{W}\}$ of $H(\tilde{W})$ is called the Kazhdan-Lusztig basis. We will also use

$$C'_w = q^{-\ell(w)/2} C_w \quad (w \in \tilde{W}).$$

For $w \in \tilde{W}$ let $\mathcal{I}_w$ (resp. $\mathcal{I}_w^L$, $\mathcal{I}_w^R$) denote the set of two-sided (resp. left, right) ideals I of $H(\tilde{W})$ subject to the conditions

(a) $C_w \in I$,
(b) I is spanned over $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ by a subset of $\{C_y \mid y \in \tilde{W}\}$.

It contains the unique minimal element $I_w = \bigcap_{I \in \mathcal{I}_w} I$ (resp. $I_w^L = \bigcap_{I \in \mathcal{I}_w^L} I$, $I_w^R = \bigcap_{I \in \mathcal{I}_w^R} I$). We define a preorder $\underset{LR}{\le}$ (resp. $\underset{L}{\le}$, $\underset{R}{\le}$) and an equivalence relation $\underset{LR}{\sim}$ (resp. $\underset{L}{\sim}$, $\underset{R}{\sim}$) on $\tilde{W}$ by

$$\begin{aligned}
&y \underset{LR}{\leq} w \Longleftrightarrow I_y \subset I_w,\\
&(\text{resp. } \; y \underset{L}{\leq} w \Longleftrightarrow I_y^L \subset I_w^L, \; y \underset{R}{\leq} w \Longleftrightarrow I_y^R \subset I_w^R)\\
&y \underset{LR}{\sim} w \Longleftrightarrow I_y = I_w,\\
&(\text{resp. } \; y \underset{L}{\sim} w \Longleftrightarrow I_y^L = I_w^L, \; y \underset{R}{\sim} w \Longleftrightarrow I_y^R = I_w^R).
\end{aligned}$$

Equivalence classes with respect to $\underset{LR}{\sim}$ (resp. $\underset{L}{\sim}$, $\underset{R}{\sim}$) are called two-sided (resp. left, right) cells of $\tilde{W}$. The preorder $\underset{LR}{\leq}$ on $\tilde{W}$ induces a partial order on the set of two-sided cells which is also denoted by $\underset{LR}{\leq}$. For a two-sided cell $\mathcal{C}$ of $\tilde{W}$ with $w \in \mathcal{C}$ we define two-sided ideals $H(\tilde{W})_{\underset{LR}{\leq}\mathcal{C}}$ and $H(\tilde{W})_{\underset{LR}{<}\mathcal{C}}$ of $H(\tilde{W})$ by

$$H(\tilde{W})_{\underset{LR}{\leq}\mathcal{C}} = I_w, \quad H(\tilde{W})_{\underset{LR}{<}\mathcal{C}} = \sum_{y \underset{LR}{\leq} w,\, y \notin \mathcal{C}} I_y.$$

The $H(\tilde{W})$-bimodule

$$H(\tilde{W})_{\mathcal{C}} = H(\tilde{W})_{\underset{LR}{\leq}\mathcal{C}} / H(\tilde{W})_{\underset{LR}{<}\mathcal{C}}$$

has a canonical $\mathbb{Z}[q^{1/2}, q^{-1/2}]$-basis parametrized by $\mathcal{C}$. The multiplication of $H(\tilde{W})$ induces a multiplication of $H(\tilde{W})_{\mathcal{C}}$ which is associative; however, $H(\tilde{W})_{\mathcal{C}}$ does not contain the identity element in general.

LEMMA 1.1. (Kazhdan-Lusztig [5]) *If $y \underset{L}{\leq} w$ (resp. $y \underset{R}{\leq} w$), then $R(w) \subset R(y)$ (resp. $L(w) \subset L(y)$). In particular, if $y \underset{L}{\sim} w$ (resp. $y \underset{R}{\sim} w$), then $R(w) = R(y)$ (resp. $L(w) = L(y)$).*

For a subset T of S such that $\langle T \rangle$ is a finite subgroup of W we denote the longest element of $\langle T \rangle$ by w_T. We call $w \in \tilde{W}$ a parabolic element if there exists some $T \subset S$ such that $|\langle T \rangle| < \infty$ and $w = w_T$.

We will need the following simple assertion later.

LEMMA 1.2. *Let $x, y \in \tilde{W}$ and let w be a parabolic element of $\tilde{W}$. Assume that $x \underset{L}{\leq} w$ and $y \underset{R}{\leq} w$. Then $C'_x = hC'_w$ and $C'_y = C'_w h'$ for some $h, h' \in H(\tilde{W})$.*

Proof. By Lemma 1.1 we have $R(w) \subset R(x)$ and $L(w) \subset L(y)$. Since w is a parabolic element, there are x_1 and y_1 in $\tilde{W}$ such that $x = x_1 w$, $y = wy_1$ and $\ell(x) = \ell(x_1) + \ell(w)$, $\ell(y) = \ell(w) + \ell(y_1)$. Now using induction on the length of x_1 and of y_1 we see the assertion is true (see [5, (2.3.a), (2.3.b)]). ∎

In the analysis of two-sided cells the star operations defined in Kazhdan-Lusztig [5] and the a-function defined in Lusztig [10] play important roles.

Let s and t be in S such that st has order 3, i.e. $sts = tst$. Define

$$D_L(s,t) = \{w \in \tilde{W} \mid L(w) \cap \{s,t\} \text{ has exactly one element}\},$$
$$D_R(s,t) = \{w \in \tilde{W} \mid R(w) \cap \{s,t\} \text{ has exactly one element}\}.$$

If w is in $D_L(s,t)$, then $\{sw, tw\}$ contains exactly one element in $D_L(s,t)$, denoted by ${}^*w$, here $* = \{s,t\}$. The map

$$D_L(s,t) \ni w \longmapsto {}^*w \in D_L(s,t)$$

is an involution and is called a left star operation. Similarly we can define the right star operation $D_R(s,t) \ni w \mapsto w^* \in D_R(s,t)$ by $\{w^*\} = \{ws, wt\} \cap D_R(s,t)$.

Proposition 1.3. (Kazhdan-Lusztig [5]) *Let s and t be in S such that st has order 3, and set $* = \{s,t\}$.*

(i) *For $w \in D_L(s,t)$ (resp. $D_R(s,t)$) we have ${}^*w \underset{L}{\sim} w$ (resp. $w^* \underset{R}{\sim} w$).*

(ii) *For $y, w \in D_L(s,t)$ (resp. $D_R(s,t)$) with $y \underset{R}{\sim} w$ (resp. $y \underset{L}{\sim} w$) we have ${}^*y \underset{R}{\sim} {}^*w$ (resp. $y^* \underset{L}{\sim} w^*$).*

Given w, u in $\tilde{W}$, we write

$$C'_w C'_u = \sum_{v \in \tilde{W}_a} h_{w,u,v} C'_v \quad (h_{w,u,v} \in \mathbb{Z}[q^{1/2}, q^{-1/2}]).$$

The a-function

$$a : \tilde{W} \longrightarrow \mathbb{Z}_{\geq 0} \sqcup \{\infty\}$$

is defined as follows. Let $v \in \tilde{W}$. If for any $i \in \mathbb{Z}_{\geq 0}$ there exist some $w, u \in \tilde{W}$ such that $q^{-i/2} h_{w,u,v} \notin \mathbb{Z}[q^{-1/2}]$, then we set $a(v) = \infty$. Otherwise we set

$$a(v) = \min\{i \in \mathbb{Z}_{\geq 0} \mid q^{-i/2} h_{w,u,v} \in \mathbb{Z}[q^{-1/2}] \text{ for all } w, u \in \tilde{W}\}.$$

Proposition 1.4. (Lusztig [10]) *Assume that* (W,S) *is crystallographic. Then for any* $w, y \in \tilde{W}$ *with* $y \underset{LR}{\leq} w$ *we have* $a(y) \geq a(w)$*. In particular, the function* a *is constant on each two-sided cell of* $\tilde{W}$*.*

§2. Affine Hecke algebras

Let G be a connected reductive algebraic group over $\mathbb{C}$ with simply-connected derived group. Let B and T be a Borel subgroup and a maximal torus of G respectively such that $B \supset T$. We denote the Lie algebras of G, B, T by $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{t}$ respectively. Let $\Delta \subset \mathfrak{t}^*$ be the root system. For $\alpha \in \Delta$ we denote the corresponding root subspace by $\mathfrak{g}_\alpha$. We choose a system Δ^+ of positive roots as the weights of $\mathfrak{g}/\mathfrak{b}$, and denote the corresponding set of simple roots by Π. Let $P \subset \mathfrak{t}^*$ denote the weight lattice and let Q be its sublattice spanned by Δ. We denote the subset of P consisting of dominant weights by P^+.

In the rest of this paper we denote by W the Weyl group of G. It is a Coxeter group with canonical generator system $S = \{s_\alpha \mid \alpha \in \Pi\}$. Here, the reflection with respect to $\alpha \in \Delta$ is denoted by s_α.

Let $W_a = WQ$ (resp. $\tilde{W}_a = WP$) denote the semidirect product with respect to the action W on Q (resp. P). W_a and $\tilde{W}_a$ are called the affine Weyl group and the extended affine Weyl group respectively. The element of W_a (resp. $\tilde{W}_a$) corresponding to $\lambda \in Q$ (resp. $\lambda \in P$) is denoted by t_λ. Let Δ_c denote the set of roots β such that the corresponding coroots $\beta^\vee$ are the highest coroots of irreducible components of the coroot system, and set

$$S_a = S \sqcup \{t_\beta s_\beta \mid \beta \in \Delta_c\}.$$

Then (W_a, S_a) is a Coxeter system. Set

$$\Omega = \{\omega \in \tilde{W}_a \mid \omega S_a = S_a \omega\}.$$

Then $\tilde{W}_a$ is canonically isomorphic to the semidirect product ΩW_a with respect to the conjugation action of Ω on W_a. Especially, we have the Hecke algebra $H(\tilde{W}_a)$ of $\tilde{W}_a$. We identify the Hecke algebra $H(W)$ of W with a subalgebra of $H(\tilde{W}_a)$ by the canonical embedding $T_w \mapsto T_w$ $(w \in W)$.

We have the following properties on the a-function on $\tilde{W}_a$.

Proposition 2.1. (Lusztig [10])

(i) $a(w) = \ell(w)$ *if* w *is a parabolic element of* $\tilde{W}_a$*.*

(ii) $a(w) \leq a(w_S)\,(= \ell(w_S))$ *for any* $w \in \tilde{W}_a$.

Note also that the function a is constant on each two-sided cell of $\tilde{W}_a$ by Proposition 1.4. For $w, u, v \in \tilde{W}$ we define $\gamma_{w,u,v} \in \mathbb{Z}$ by

$$h_{w,u,v} = \gamma_{w,u,v} q^{a(v)/2} + \text{lower degree terms.}$$

Now we present a property of $\gamma_{w,u,v}$ related to the star operations.

By a similar argument to that for Theorem 1.4.5 in [22], we can see the following result.

Proposition 2.2. *Let* s, t *be in* S_a *such that* st *has order* 3. *Set* $* = \{s,t\}$. *Assume* $w, v \in D_L(s,t)$. *Then we have*

$$\gamma_{w,u,v} = \gamma_{{}^*w,u,{}^*v}.$$

For $\lambda \in P$ we define $\theta_\lambda \in H(\tilde{W}_a)$ as follows. Take $\lambda_1, \lambda_2 \in P^+$ such that $\lambda = \lambda_1 - \lambda_2$ and set

$$\theta_\lambda = q^{(-\ell(t_{\lambda_1})+\ell(t_{\lambda_2}))/2} T_{t_{\lambda_1}} T_{t_{\lambda_2}}^{-1}.$$

It does not depend on the choice of λ_1, λ_2. Moreover, we have

$$\begin{aligned}
&\theta_0 = 1,\\
&\theta_\lambda\theta_\mu = \theta_{\lambda+\mu} \quad (\lambda, \mu \in P),\\
&T_{s_\alpha}\theta_\lambda = \theta_{s\lambda}T_{s_\alpha} + (q-1)\frac{\theta_\alpha(\theta_\lambda - \theta_{s\lambda})}{\theta_\alpha - 1} \quad (\lambda \in P,\ \alpha \in \Pi).
\end{aligned}$$

This presentation in terms of the $\mathbb{Z}[q^{1/2}, q^{-1/2}]$-basis $\{T_w\theta_\lambda \mid w \in W,\ \lambda \in P\}$ of $H(\tilde{W}_a)$ is due to Bernstein-Zelevinski (see [8]).

§3. Affine Hecke algebras and equivariant K-groups

For an algebraic variety Y over $\mathbb{C}$ we denote its structure sheaf by $\mathcal{O}_Y$. If Y is smooth, then its canonical sheaf is denoted by Ω_Y.

We denote the flag variety G/B of G by $\mathcal{B}$. As a set $\mathcal{B}$ is identified with the set of Borel subalgebras of $\mathfrak{g}$ by the correspondence $gB \mapsto \mathrm{Ad}(g)(\mathfrak{b})$ $(g \in G)$. For $x \in \mathcal{B}$ we denote by $\mathfrak{b}_x$ the corresponding Borel subalgebra of $\mathfrak{g}$, and set $\mathfrak{n}_x = [\mathfrak{b}_x, \mathfrak{b}_x]$. For $w \in W$ set

$$Y_w = G(B, wB) \subset \mathcal{B} \times \mathcal{B}.$$

Then we have $\mathcal{B}\times\mathcal{B}=\bigsqcup_{w\in W}Y_w$, and $\overline{Y}_w=\bigsqcup_{y\leq w}Y_y$. We denote by

$$i_w:\overline{Y}_w\longrightarrow\mathcal{B}\times\mathcal{B}$$

the embedding.

Set

$$\begin{aligned}\Lambda&=\{(a,x)\in\mathfrak{g}\times\mathcal{B}\mid a\in\mathfrak{n}_x\},\\ Z&=\{(a,x,y)\in\mathfrak{g}\times\mathcal{B}\times\mathcal{B}\mid a\in\mathfrak{n}_x\cap\mathfrak{n}_y\}.\end{aligned}$$

Let $\pi:\Lambda\to\mathcal{B}$ be the projection. The algebraic group $G\times\mathbb{C}^*$ acts on the variety Λ by

$$(g,z):(a,x)\longmapsto(z^{-2}\operatorname{Ad}(g)(a),gx)\in\Lambda.$$

We sometimes identify Z with a $G\times\mathbb{C}^*$-stable closed subvariety of $\Lambda\times\Lambda$ by the embedding

$$Z\longrightarrow\Lambda\times\Lambda\quad((a,x,y)\longmapsto((a,x),(a,y))).$$

In particular, Z is a $G\times\mathbb{C}^*$-variety. For $w\in W$ set

$$Z_w=\{(a,x,y)\in Z\mid(x,y)\in Y_w\}.$$

We denote by

$$r_w:\overline{Z}_w\longrightarrow Z,\quad\pi_w:\overline{Z}_w\longrightarrow\overline{Y}_w$$

the embedding and the projection respectively.

Let us consider the equivariant K-group $K^{G\times\mathbb{C}^*}(Z)=K^{G\times\mathbb{C}^*}(\Lambda\times\Lambda;Z)$ (see Section A for the equivariant K-groups and notation concerning them). It is a module over the representation ring $R^{G\times\mathbb{C}^*}=R^G\otimes_{\mathbb{Z}}R^{\mathbb{C}^*}$ of $G\times\mathbb{C}^*$. We will identify $R^{\mathbb{C}^*}$ with $\mathbb{Z}[q^{1/2},q^{-1/2}]$ by associating the $\mathbb{C}^*$-module given by $z\mapsto z^n$ to $q^{n/2}$. In particular, $K^{G\times\mathbb{C}^*}(Z)$ is a $\mathbb{Z}[q^{1/2},q^{-1/2}]$-module.

For $(i,j)=(1,2),(2,3),(1,3)$ we denote by $p_{ij}:\Lambda\times\Lambda\times\Lambda\to\Lambda\times\Lambda$ the projections onto (i,j)-factors. Note that $p_{13}(p_{12}^{-1}Z\cap p_{23}^{-1}Z)\subset Z$. Since the morphism $p_{12}^{-1}Z\cap p_{23}^{-1}Z\to Z$ induced by p_{13} is proper, we can define an $R^{G\times\mathbb{C}^*}$-bilinear map

$$\begin{aligned}&\bigstar:K^{G\times\mathbb{C}^*}(Z)\times K^{G\times\mathbb{C}^*}(Z)\longrightarrow K^{G\times\mathbb{C}^*}(Z)\\ &\qquad\big((m,n)\longmapsto m\bigstar n=p_{13*}(p_{12}^*m\otimes_{\mathcal{O}_{\Lambda\times\Lambda\times\Lambda}}p_{23}^*n)\big).\end{aligned}$$

Then it is easily seen that the convolution product $\bigstar$ endows with $K^{G\times\mathbb{C}^*}(Z)$ a structure of associative algebra over $R^{G\times\mathbb{C}^*}$ with the identity element $[r_{1*}\mathcal{O}_{Z_1}]$. For $\lambda\in P$ we denote by $\mathcal{O}_{\mathcal{B}}(\lambda)$ the G-equivariant invertible $\mathcal{O}_{\mathcal{B}}$-module whose fiber at B is the B-module corresponding to λ.

Theorem 3.1. (Ginzburg [3], Kazhdan-Lusztig [6]) *There exists an isomorphism*

$$\Phi : H(\tilde{W}_a) \longrightarrow K^{G\times\mathbb{C}^*}(Z)$$

of $\mathbb{Z}[q^{1/2},q^{-1/2}]$*-algebras satisfying*

$$\Phi(\theta_\lambda) = [r_{1*}\pi_1^*\mathcal{O}_{\mathcal{B}}(-\lambda)] \quad (\lambda\in P),$$
$$\Phi(T_s+1) = -[r_{s*}\pi_s^*(\Omega_{\overline{Y}_s}\otimes i_s^*(\mathcal{O}_{\mathcal{B}}\boxtimes\Omega_{\mathcal{B}}^{\otimes -1}))] \quad (s\in S).$$

Here, we have identified $\overline{Y}_1\,(=Y_1)$ *with* $\mathcal{B}$.

Remark 3.2. Note that $\Phi(T_s+1)$ is not symmetric with respect to the the symmetry of $(\Lambda\times\Lambda, Z)$ given by $\Lambda\times\Lambda\ni(x,y)\mapsto(y,x)\in\Lambda\times\Lambda$. This can be resolved if we use the twisted product

$$K^{G\times\mathbb{C}^*}(Z)\times K^{G\times\mathbb{C}^*}(Z)\longrightarrow K^{G\times\mathbb{C}^*}(Z)$$
$$\big((m,n)\longmapsto p_{13*}(p_{12}^*m\otimes_{\mathcal{O}_{\Lambda\times\Lambda\times\Lambda}}p_{23}^*n\otimes_{\mathcal{O}_{\Lambda\times\Lambda\times\Lambda}}p_2^*\pi^*\Omega_{\mathcal{B}})\big)$$

as in [18], where $p_2:\Lambda\times\Lambda\times\Lambda\to\Lambda$ is the projection onto the second factor. There is another way to recover the symmetry by modifying the definition of Φ without changing the product (see Lusztig [13]).

Let $\mathcal{N}$ denote the closed subvariety of $\mathfrak{g}$ consisting of nilpotent elements. For a locally closed G-stable subvariety V of $\mathcal{N}$ we set

$$Z_V = \{(a,x,y)\in Z \mid a\in V\}.$$

Proposition 3.3. (Ginzburg [3], Kazhdan-Lusztig [6]) *Let* V *be a locally closed* G*-stable subvariety of* $\mathcal{N}$*. Then we have an exact sequence*

$$0\longrightarrow K^{G\times\mathbb{C}^*}(Z_{\overline{V}\setminus V})\longrightarrow K^{G\times\mathbb{C}^*}(Z_{\overline{V}})\longrightarrow K^{G\times\mathbb{C}^*}(Z_V)\longrightarrow 0.$$

Here $K^{G\times\mathbb{C}^*}(Z_{\overline{V}\setminus V})\to K^{G\times\mathbb{C}^*}(Z_{\overline{V}})$ *is given by the direct image with respect to the inclusion* $Z_{\overline{V}\setminus V}\to Z_{\overline{V}}$*, and* $K^{G\times\mathbb{C}^*}(Z_{\overline{V}})\to K^{G\times\mathbb{C}^*}(Z_V)$ *is given by the inverse image with respect to the inclusion* $Z_V\to Z_{\overline{V}}$*.*

In particular, if V is closed, then the homomorphism $K^{G\times\mathbb{C}^*}(Z_V) \to K^{G\times\mathbb{C}^*}(Z)$ given by the direct image with respect to the closed embedding $Z_V \to Z$ is injective. By this we will identify $K^{G\times\mathbb{C}^*}(Z_V)$ for a closed G-stable subvariety V of $\mathcal{N}$ with a two-sided ideal of $K^{G\times\mathbb{C}^*}(Z)$.

The following remarkable fact conjectured in Lusztig [7] was proved by Lusztig himself [12] using the theory of character sheaves among other things.

Theorem 3.4. *There exists a natural one-to-one correspondence between the set of two-sided cells of $\tilde{W}_a$ and that of nilpotent orbits of $\mathfrak{g}$.*

For a nilpotent orbit O we denote by $\mathcal{C}_O$ the corresponding two-sided cell.

In view of Theorem 3.1, it is natural to expect the following (see [4], [19], [14]).

Conjecture 3.5. *Let O be a nilpotent orbit. Then we have*

$$\Phi\big(H(\tilde{W}_a)_{\underset{LR}{\leq}\mathcal{C}_O}\big) = K^{G\times\mathbb{C}^*}(Z_{\overline{O}}).$$

Remark 3.6. This conjecture is known to be true when $O = \{0\}$ (see [21]). In [1] Bezrukavnikov established a closely related result, which involves affine flag manifolds, derived categories and the Springer resolution (see Theorem 4(a) there).

Let $w \in W$. In [18] a $G\times\mathbb{C}^*$-equivariant coherent sheaf M_w on $\Lambda\times\Lambda$ such that $\mathrm{Supp}(M_w) \subset Z$ and $\Phi(C_w) = (-1)^{\ell(w)}[M_w]$ is associated using the theory of Hodge modules. This together with a deep result related to the associated varieties of primitive ideals of the enveloping algebra $U(\mathfrak{g})$ implies

$$\Phi(C_w) \in K^{G\times\mathbb{C}^*}(Z_{\overline{O}}) \setminus K^{G\times\mathbb{C}^*}(Z_{\overline{O}\setminus O}),$$

where O is the nilpotent orbit satisfying $w \in \mathcal{C}_O$. In Section 4 we will need the following weaker result which is much easier.

Proposition 3.7. *Let Π_1 be a subset of Π. Set $w = w_T \in W$ with $T = \{s_\alpha \mid \alpha \in \Pi_1\} \subset S$. Let O be the nilpotent orbit satisfying*

$$\overline{O} = \mathrm{Ad}(G)\Big(\sum_{\alpha\in\Delta^+\setminus\Delta_1} \mathfrak{g}_\alpha\Big),$$

where $\Delta_1 = \Delta \cap (\sum_{\alpha\in\Pi_1} \mathbb{Z}\alpha)$. *Then we have*

$$\Phi(C_w) \in K^{G\times\mathbb{C}^*}(Z_{\overline{O}}).$$

Proof. Note that $\overline{Y}_w$ is smooth. Hence by [18] we have $\Phi(C_w) = (-1)^{\ell(w)}[M_w]$ with

$$M_w = \mathrm{gr}(\mathbb{Q}^H_{\overline{Y}_w}[\dim \overline{Y}_w]) \otimes_{\mathcal{O}_{\Lambda\times\Lambda}} (\pi\times\pi)^*(\mathcal{O}_{\mathcal{B}} \boxtimes \Omega_{\mathcal{B}}^{\otimes -1}),$$

where $\mathbb{Q}^H_{\overline{Y}_w}[\dim \overline{Y}_w]$ denotes the canonical irreducible G-equivariant Hodge module whose underlying perverse sheaf is $\mathbb{Q}_{\overline{Y}_w}[\dim \overline{Y}_w]$. By

$$\mathrm{gr}\big(\mathbb{Q}^H_{\overline{Y}_w}[\dim \overline{Y}_w]\big) = r_{w*}\pi_w^*(\Omega_{\overline{Y}_w})$$

we obtain

$$M_w = r_{w*}\pi_w^*(\Omega_{\overline{Y}_w}) \otimes_{\mathcal{O}_{\Lambda\times\Lambda}} (\pi\times\pi)^*(\mathcal{O}_{\mathcal{B}} \boxtimes \Omega_{\mathcal{B}}^{\otimes -1}). \tag{3.1}$$

It follows that $\mathrm{Supp}(M_w) = \overline{Z}_w \subset Z_{\overline{O}}$. □

Remark 3.8. We can prove (3.1) directly without appealing to the theory of Hodge modules. Details are omitted.

§4. The case $G = GL_n(\mathbb{C})$

The main result of this paper is the following.

THEOREM 4.1. *Conjecture* 3.5 *holds for* $G = GL_n(\mathbb{C})$.

In the rest of this section we assume that $G = GL_n(\mathbb{C})$. In this case the extended affine Weyl group $\tilde{W}_a$ is identified with the group of all permutations σ of $\mathbb{Z}$ satisfying $\sigma(i+n) = \sigma(i)+n$ $(i\in\mathbb{Z})$ and $\sum_{i=1}^n(\sigma(i)-i) \in n\mathbb{Z}$. Define $\omega, s_k \in \tilde{W}_a$ $(0 \le k \le n-1)$ by

$$\omega(i) = i+1 \quad (i\in\mathbb{Z}),$$
$$s_k(i) = \begin{cases} i+1 & (i\in n\mathbb{Z}+k),\\ i-1 & (i \in n\mathbb{Z}+k+1),\\ i & \text{(otherwise)}.\end{cases}$$

Then we have

$$S = \{s_i \mid 1\le i\le n-1\}, \quad S_a = S \sqcup \{s_0\}, \quad \Omega = \langle\omega\rangle,$$

and W is identified with the symmetric group $\mathfrak{S}_n$.

Let $\mathcal{P}(n)$ denote the set of partitions of n, that is,

$$\mathcal{P}(n) = \left\{ \rho = (\rho_1, \rho_2, \dots, \rho_n) \in \mathbb{Z}^n_{\geq 0} \;\middle|\; \rho_i \geq \rho_{i+1}, \sum_{i=1}^{n} \rho_i = n \right\}.$$

For $\rho \in \mathcal{P}(n)$ we set

$$N_j(\rho) = \sharp\{i \mid \rho_i = j\}.$$

We denote by $\rho \mapsto \rho^*$ the duality operation on $\mathcal{P}(n)$ induced by the transpose of the corresponding Young diagram, that is, $\rho^*_i = \sum_{k=i}^{n} N_k(\rho)$.

The set of nilpotent orbits in $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ is parametrized by $\mathcal{P}(n)$. The nilpotent orbit O_ρ corresponding to $\rho \in \mathcal{P}(n)$ is the one containing the Jordan normal form with exactly $N_i(\rho^*)$ Jordan blocks of size i (with eigenvalue 0) for each i. In particular, $O_{(n,0,\dots,0)} = \{0\}$ and $O_{(1,\dots,1)}$ is the regular nilpotent orbit.

By Theorem 3.4 the set of two-sided cells of $\tilde{W}_a$ is also parametrized by $\mathcal{P}(n)$ (in our case $G = GL_n(\mathbb{C})$ this is due to Lusztig [9] and Shi [15]). We denote by $\mathcal{C}_\rho$ the two-sided cell of $\tilde{W}_a$ corresponding to O_ρ.

Let T be a proper subset of S_a such that $\langle T \rangle$ is of type $A_{k_1} \times \cdots \times A_{k_r}$. Then the corresponding parabolic element w_T belongs to $\mathcal{C}_\rho$ if and only if

$$\sharp\{j \mid k_j + 1 = i\} = N_i(\rho)$$

for any i.

For $\rho \in \mathcal{P}(n)$ set $\mathcal{C}^W_\rho = W \cap \mathcal{C}_\rho$. It is known that $\mathcal{C}^W_\rho$ is a two-sided cell of W. In particular, the set of two-sided cells of W is also parametrized by $\mathcal{P}(n)$ (see Kazhdan-Lusztig [5]).

Proposition 4.2. (Shi [16]) *The following conditions on $\rho, \xi \in \mathcal{P}(n)$ are equivalent.*

(a) $\mathcal{C}_\xi \underset{LR}{\leq} \mathcal{C}_\rho$.

(b) $\mathcal{C}^W_\xi \underset{LR}{\leq} \mathcal{C}^W_\rho$.

(c) $O_\xi \subset \overline{O}_\rho$.

Hence we have $H(W)_{\underset{LR}{\leq} \mathcal{C}^W_\rho} = H(\tilde{W}_a)_{\underset{LR}{\leq} \mathcal{C}_\rho} \cap H(W)$, and $H(W)_{\mathcal{C}^W_\rho}$ is identified with an $(H(W), H(W))$-submodule of $H(\tilde{W}_a)_{\mathcal{C}_\rho}$.

The following is crucial for the proof of Theorem 4.1.

THEOREM 4.3. *Let v be a parabolic element in $\mathcal{C}_\rho$. Then the $H(\tilde{W}_a)$-bimodule $H(\tilde{W}_a)_{\mathcal{C}_\rho}$ is generated by the image of C_v.*

We first show the following corresponding statement for $H(W)$.

PROPOSITION 4.4. *Let v be a parabolic element in $\mathcal{C}_\rho^W$. Then the $H(W)$-bimodule $H(W)_{\mathcal{C}_\rho^W}$ is generated by the image of C_v.*

Proof. Let $u \in \mathcal{C}_\rho^W$. Let $\mathcal{L}$ be the left cell of W containing u and $\mathcal{R}$ the right cell of W containing u. Then $\mathcal{L}$ contains a unique element y such that $y \underset{R}{\sim} v$, and $\mathcal{R}$ contains a unique element x such that $x \underset{L}{\sim} v$ (see the proof of Theorem 1.4 in [5, §5]). By Lemma 1.2 we have $C'_x = hC'_v$ and $C'_y = C'_v h'$ for some h, h' in $H(W)$.

Let $\pi : H(W)_{\leq \mathcal{C}_\rho^W} \to H(W)_{\mathcal{C}_\rho^W}$ be the canonical projection and let V_1 and V_2 be the left $H(W)$-submodules of $H(W)_{\mathcal{C}_\rho^W}$ generated by $\pi(C'_v)$ and $\pi(C'_y)$ respectively. Then $H(W)_{\mathcal{C}_\rho^W} \ni k \mapsto kh' \in H(W)_{\mathcal{C}_\rho^W}$ is a homomorphism of left $H(W)$-modules satisfying $\pi(C'_v) \mapsto \pi(C'_y)$. Hence we obtain a homomorphism $f : V_1 \to V_2$ given by $f(k) = kh'$.

On the other hand by the proof of Theorem 1.4 in [5, §5] there exists an isomorphism $g : V_1 \to V_2$ of left $H(W)$-modules such that $g(\pi(C'_v)) = \pi(C'_y)$ and $g(\pi(C'_x)) = \pi(C'_u)$. By $f(\pi(C'_v)) = g(\pi(C'_v))$ we have $f = g$. Hence

$$\pi(C'_u) = g(\pi(C'_x)) = f(\pi(C'_x)) = hf(\pi(C'_v)) = h\pi(C'_v)h'.$$

The proof is complete. ∎

Now we give a proof of Theorem 4.3. Let $\pi : H(\tilde{W}_a)_{\leq \mathcal{C}_\rho} \to H(\tilde{W}_a)_{\mathcal{C}_\rho}$ be the canonical homomorphism. According to [15, Lemma 18.3.2] one has a parabolic element $w \in \mathcal{C}_\rho^W$ such that for any $u \in \mathcal{C}_\rho$ there exists a sequence of left star operations $\phi_1, \phi_2, \ldots, \phi_r$ and an integer m satisfying

$$w \underset{R}{\sim} \omega^m \phi_r \phi_{r-1} \cdots \phi_1(u). \tag{4.1}$$

We first show the statement for this special parabolic element w.

Let $u \in \mathcal{C}_\rho$. Take left star operations $\phi_1, \phi_2, \ldots, \phi_r$ and an integer m satisfying (4.1), and set $y = \omega^m \phi_r \phi_{r-1} \cdots \phi_1(u)$, $x = \phi_1 \phi_2 \cdots \phi_r \omega^{-m}(w)$. Note that x is well-defined and $x \underset{L}{\sim} w$ by definition and Proposition 1.3. Since w is a parabolic element, there exist $h, h' \in H(\tilde{W}_a)$ such that $C'_x =$

hC'_w and $C'_y = C'_w h'$ by Lemma 1.2. Note that $C'_w C'_w = \eta C'_w$ where $\eta \in \mathbb{Z}[q^{1/2}, q^{-1/2}]$ satisfies $\overline{\eta} = \eta$ and $\eta = q^{\ell(w)/2} +$ (lower degree terms). Hence

$$\eta h\pi(C'_w)h' = \pi(C'_x C'_y) = \sum_{z\in\mathcal{C}_\rho} h_{x,y,z}\pi(C'_z),$$

where $h_{x,y,z} \in \mathbb{Z}[q^{1/2}, q^{-1/2}]$ satisfies $\overline{h}_{x,y,z} = h_{x,y,z}$ and $h_{x,y,z} = \gamma_{x,y,z}q^{a(z)/2}$ + (lower degree terms). For any $z \in \mathcal{C}_\rho$ we have $a(z) = a(w) = \ell(w)$, and hence we obtain

$$h\pi(C'_w)h' = \sum_{z\in\mathcal{C}_\rho} \gamma_{x,y,z}\pi(C'_z).$$

Note that $\gamma_{\omega^m w_1,w_2,w_3} = \gamma_{w_1,w_2,\omega^{-m}w_3}$ for any w_1, w_2, w_3 in $\tilde{W}_a$. Hence we have $\gamma_{x,y,z} = \gamma_{w,\,y,\,\omega^m\phi_r\cdots\phi_1(z)}$ by Proposition 2.2. Since w is a distinguished involution, we have $\gamma_{x,y,z} \neq 0$ if and only if $\omega^m\phi_r\phi_{r-1}\cdots\phi_1(z) = y$ and in this case $\gamma_{x,y,z} = 1$ (see Lusztig [11]). Thus $\gamma_{x,y,z} \neq 0$ if and only $z = u$ and in this case $\gamma_{x,y,u} = 1$. Therefore we have $h\pi(C'_w)h' = \pi(C'_u)$.

Now let v be any parabolic element in $\mathcal{C}_\rho$. Then there exists an integer k such that $\omega^k v\omega^{-k}$ is in W. By Proposition 4.4 we have $H(W)\pi(C'_{\omega^k v\omega^{-k}})H(W) = H(W)\pi(C'_w)H(W)$ and hence

$$H(\tilde{W}_a)\pi(C'_v)H(\tilde{W}_a) = H(\tilde{W}_a)\pi(C'_w)H(\tilde{W}_a) = H(\tilde{W}_a)_{\mathcal{C}_\rho}.$$

The proof of Theorem 4.3 is complete.

Remark 4.5. (a) The assertion for W_a similar to that in Theorem 4.3 does not hold in general.

(b) Let v be as in Theorem 4.3. It is not difficult to prove that for any $w \underset{LR}{\leq} v$, there exists a polynomial f_w in $q^{1/2} + q^{-1/2}$ such that $f_w C_w$ is in the two-sided ideal of $H(\tilde{W}_a)$ generated by C_v. However, in general it is not true that C_w is in the two-sided ideal of $H(\tilde{W}_a)$ generated by C_v. Example: $n = 4$ and let $\mathcal{C}_\rho$ be the two-sided cell containing $v = s_1s_3$. Then $(q^{1/2} + q^{-1/2})C_{s_1s_2s_1}$ is in $H(\tilde{W}_a)C_vH(\tilde{W}_a)$, but $C_{s_1s_2s_1}$ is not in $H(\tilde{W}_a)C_vH(\tilde{W}_a)$.

Let $\mathbb{F}$ be an algebraic closure of $\mathbb{C}(q^{1/2})$, and set $H^{\mathbb{F}} = \mathbb{F} \otimes H(\tilde{W}_a)$, $G_{\mathbb{F}} = GL_n(\mathbb{F})$, $\mathfrak{g}_{\mathbb{F}} = \mathfrak{gl}_n(\mathbb{F})$. Then $H^{\mathbb{F}}$ is an $\mathbb{F}$-algebra and $G_{\mathbb{F}}$ is an algebraic group over $\mathbb{F}$ with Lie algebra $\mathfrak{g}_{\mathbb{F}}$.

Let $\mathcal{Q}$ denote the $G_{\mathbb{F}}$-conjugacy classes of the pairs $(s, e) \in G_{\mathbb{F}} \times \mathfrak{g}_{\mathbb{F}}$ where s is semisimple, e is nilpotent, and $\mathrm{Ad}(s)(e) = qe$. For such a pair

(s,e) Kazhdan-Lusztig [6] and Ginzburg [3] constructed a finite-dimensional $H^{\mathbb{F}}$-module $M_{(s,e)}$. Moreover, we have a unique irreducible quotient $L_{(s,e)}$ of $M_{(s,e)}$, and the set of irreducible $H^{\mathbb{F}}$-modules is parametrized by $\mathcal{Q}$ via $(s,e) \mapsto L_{(s,e)}$ (note that $\mathbb{F}$ is isomorphic to $\mathbb{C}$ as an abstract field). In particular, we can associate to each irreducible $H^{\mathbb{F}}$-module L a nilpotent orbit $O(L)$ in $\mathfrak{g}$ by $\mathrm{Ad}(G_{\mathbb{F}})(O(L_{(s,e)})) \ni e$ (note that the set of $G_{\mathbb{F}}$-conjugacy classes of nilpotent elements in $\mathfrak{g}_{\mathbb{F}}$ is in one-to-one correspondence with that of G-conjugacy classes of nilpotent elements in $\mathfrak{g}$).

We need the following deep result of Lusztig [12].

Proposition 4.6. *For any irreducible subquotient L of the (left) $H^{\mathbb{F}}$-module $\mathbb{F} \otimes_{\mathbb{Z}[q^{1/2},q^{-1/2}]} H(\tilde{W}_a)_{\mathcal{C}_\rho}$ we have $\overline{O(L)} \supset O_\rho$.*

Proposition 4.7. *Let O be a nilpotent orbit. Then for any irreducible quotient L of the (left) $H^{\mathbb{F}}$-module $\mathbb{F} \otimes_{\mathbb{Z}[q^{1/2},q^{-1/2}]} K^{G\times\mathbb{C}^*}(Z_O)$ we have $O(L) = O$.*

Proof. By [6, Corollary 5.9] we see that L is a quotient of $M_{(s,e)}$ for $(s,e) \in \mathcal{Q}$ with $e \in O$. Since $L_{(s,e)}$ is the unique irreducible quotient of $M_{(s,e)}$, we have $L = L_{(s,e)}$ and hence $O(L) = O$. □

Now we are ready to give a proof of Theorem 4.1. We show

$$\Phi\big(H(\tilde{W}_a)_{\underset{LR}{\le}\mathcal{C}_\xi}\big) = K^{G\times\mathbb{C}^*}(Z_{\overline{O}_\xi}) \tag{4.2}$$

for any $\xi \in \mathcal{P}(n)$ by induction on $\dim O_\xi$. Let $\rho \in \mathcal{P}(n)$ and assume that (4.2) is true for any $\xi \in \mathcal{P}(n)$ with $\dim O_\xi < \dim O_\rho$.

For any $\tau \in \mathcal{P}(n)$ with $\mathcal{C}_\tau \underset{LR}{\le} \mathcal{C}_\rho$ any parabolic element $v \in \mathcal{C}_\tau^W$ satisfies $\Phi(C_v) \in K^{G\times\mathbb{C}^*}(Z_{\overline{O}_\rho})$ by Proposition 3.7. Hence we see by Theorem 4.3 that $\Phi(H(\tilde{W}_a)_{\underset{LR}{\le}\mathcal{C}_\rho}) \subset K^{G\times\mathbb{C}^*}(Z_{\overline{O}_\rho})$. Moreover, the hypothesis of induction together with Proposition 4.2 implies $\Phi(H(\tilde{W}_a)_{\underset{LR}{<}\mathcal{C}_\rho}) = K^{G\times\mathbb{C}^*}(Z_{\overline{O}_\rho\setminus O_\rho})$. Hence it is sufficient to show that the induced injection $\overline{\Phi} : H(\tilde{W}_a)_{\mathcal{C}_\rho} \to K^{G\times\mathbb{C}^*}(Z_{O_\rho})$ is surjective. Assume that $\mathrm{Coker}(\overline{\Phi}) \neq 0$. Since $H(\tilde{W}_a)_{\underset{LR}{\le}\mathcal{C}_\rho}$ is a direct summand of the $\mathbb{Z}[q^{1/2},q^{-1/2}]$-module $H(\tilde{W}_a)$ and $\Phi(H(\tilde{W}_a)) = K^{G\times C^*}(Z)$, we see that the cokernel of the injective homomorphism

$$\overline{\Phi}^{\mathbb{F}} : \mathbb{F} \otimes_{\mathbb{Z}[q^{1/2},q^{-1/2}]} H(\tilde{W}_a)_{\mathcal{C}_\rho} \longrightarrow \mathbb{F} \otimes_{\mathbb{Z}[q^{1/2},q^{-1/2}]} K^{G\times\mathbb{C}^*}(Z_{O_\rho})$$

is also non-trivial. Take an irreducible quotient L of the $H^{\mathbb{F}}$-module $\mathrm{Coker}(\overline{\Phi}^{\mathbb{F}})$. Since L is an irreducible quotient of $\mathbb{F}\otimes_{\mathbb{Z}[q^{1/2},q^{-1/2}]}K^{G\times\mathbb{C}^*}(Z_{O_\rho})$, we have $O(L)=O_\rho$ by Proposition 4.7. On the other hand since L is an irreducible subquotient of the $H^{\mathbb{F}}$-module

$$\mathbb{F}\otimes_{\mathbb{Z}[q^{1/2},q^{-1/2}]} H(\tilde{W}_a)/\mathbb{F}\otimes_{\mathbb{Z}[q^{1/2},q^{-1/2}]} H(\tilde{W}_a)_{\underset{LR}{\leq}\mathcal{C}_\rho},$$

there exists a nilpotent orbit O such that $O\not\subset\overline{O_\rho}$ and $O\subset\overline{O(L)}$ by Proposition 4.6. This is a contradiction. Hence $\overline{\Phi}$ is surjective. The proof of Theorem 4.1 is complete.

Appendix A. Equivariant K-theory

In this section we recall basic notions concerning equivariant K-groups (see Thomason [20]). All algebraic varieties are assumed to be quasi-projective over $\mathbb{C}$ and all algebraic groups are assumed to be affine over $\mathbb{C}$. The structure sheaf of an algebraic variety X is denoted by $\mathcal{O}_X$. When we consider an action of an algebraic group A on an algebraic variety X, we always assume the existence of a closed A-equivariant embedding $X\to X'$ where X' is a smooth variety with an action of A. In this case we say that X is an A-variety.

Let A be an algebraic group. For a pair (Y,X) such that Y is an A-variety and X is its A-stable closed subvariety, we denote by $\mathrm{Coh}^A(Y;X)$ the abelian category of A-equivariant coherent sheaves on Y whose supports are contained in X. Its Grothendieck group $K^A(Y;X)$ is called the equivariant K-group. Note that the direct image functor $i_*:\mathrm{Coh}^A(X;X)\to\mathrm{Coh}^A(Y;X)$ with respect to the embedding $i:X\to Y$ induces an isomorphism $K^A(X;X)\cong K^A(Y;X)$. It means that $K^A(Y;X)$ depends only on the A-variety X, and hence we sometimes denote it by $K^A(X)$. However, we will need to specify the ambient space Y in defining some operations on equivariant K-groups. Note that $K^A(X)$ is a module over the representation ring

$$R^A=K^A(\mathrm{pt}) \tag{A.1}$$

of A. Here pt denotes the variety consisting of a single point.

Assume that we are given an A-equivariant morphism $f:Y\to Y'$ of A-varieties and A-stable closed subvarieties X and X' of Y and Y' respectively such that $f(X) \subset X'$ and the restriction $X \to X'$ of f is a proper morphism. Then the derived functors

$$R^n f_* : \mathrm{Coh}^A(Y;X) \longrightarrow \mathrm{Coh}^A(Y';X') \quad (n \in \mathbb{Z})$$

of the direct image functor f_* induce a homomorphism

$$\text{(A.2)} \quad f_* : K^A(Y;X) \longrightarrow K^A(Y';X') \quad \Big([M] \longmapsto \sum_n (-1)^n [R^n f_*(M)]\Big)$$

of R^A-modules. We note that (A.2) does not depend on the choice of the ambient spaces Y and Y'.

Lemma A.1. *Let $f : X \to X'$ and $g : X' \to X''$ be A-equivariant proper morphisms of A-varieties. Then we have*

$$(g \circ f)_* = g_* \circ f_* : K^A(X) \longrightarrow K^A(X'').$$

Assume that we are given an A-equivariant morphism $f : Y \to Y'$ of A-varieties and an A-stable closed subvariety X' of Y'. Set $X = f^{-1}(X')$. If f is smooth or if Y' is a smooth variety, then the derived functors

$$L^n f^* : \mathrm{Coh}^A(Y';X') \longrightarrow \mathrm{Coh}^A(Y;X) \quad (n \in \mathbb{Z})$$

of the inverse image functor

$$f^* : \mathrm{Coh}^A(Y';X') \longrightarrow \mathrm{Coh}^A(Y;X) \quad (M \longmapsto \mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_{Y'}} f^{-1}M)$$

are zero except for finitely many n's, and they induce a homomorphism

$$\text{(A.3)} \quad f^* : K^A(Y';X') \longrightarrow K^A(Y;X) \quad \Big([M] \longmapsto \sum_n (-1)^n [L^n f^*(M)]\Big)$$

of R^A-modules. If f is smooth, we have $L^n f^* = 0$ for $n \neq 0$.

Lemma A.2. *Let $f : Y \to Y'$ and $g : Y' \to Y''$ be A-equivariant morphisms of A-varieties. Let X'' be a closed subvariety of Y'', and set $X = (g \circ f)^{-1}(X'')$, $X' = f^{-1}(X'')$. Assume that $f^* : K^A(Y';X') \to K^A(Y;X)$ and $g^* : K^A(Y'';X'') \to K^A(Y';X')$ are defined. Then we have*

$$(g \circ f)^* = f^* \circ g^* : K^A(Y'';X'') \longrightarrow K^A(Y;X).$$

Assume that we are given a smooth A-variety Y and its A-stable closed subvarieties X_1 and X_2. The derived functors

$$\operatorname{Tor}_n^{\mathcal{O}_Y}(\ ,\ ) : \operatorname{Coh}^A(Y;X_1) \times \operatorname{Coh}^A(Y;X_2) \longrightarrow \operatorname{Coh}^A(Y;X_1 \cap X_2)$$
$$\big((M_1, M_2) \longmapsto \operatorname{Tor}_n^{\mathcal{O}_Y}(M_1, M_2) = H^{-n}(M_1 \otimes^{\mathbb{L}}_{\mathcal{O}_Y} M_2)\big)$$

of the tensor product functor $\otimes_{\mathcal{O}_Y}$ are zero except for finitely many n's, and induce a bilinear map

$$\text{(A.4)}\quad \otimes_{\mathcal{O}_Y} : K^A(Y;X_1) \times K^A(Y;X_2) \longrightarrow K^A(Y;X_1 \cap X_2)$$
$$\Big(([M_1],[M_2]) \longmapsto [M_1] \otimes_{\mathcal{O}_Y} [M_2] = \sum_n (-1)^n \operatorname{Tor}_n^{\mathcal{O}_Y}(M_1, M_2)\Big)$$

of R^A-modules. Note that $\otimes_{\mathcal{O}_Y}$ does depend on the choice of the ambient space Y.

Lemma A.3. *Let $f : Y \to Y'$ be an A-equivariant smooth morphism of smooth A-varieties. Let X_1', X_2' be closed subvarieties of Y', and set $X_1 = f^{-1}(X_1')$, $X_2 = f^{-1}(X_2')$. Then we have*

$$f^*(m_1) \otimes_{\mathcal{O}_Y} f^*(m_2) = f^*(m_1 \otimes_{\mathcal{O}_{Y'}} m_2) \in K^A(Y;X_1 \cap X_2)$$

for any $m_1 \in K^A(Y';X_1')$, $m_2 \in K^A(Y';X_2')$.

Lemma A.4. (Projection formula) *Let $f : Y \to Y'$ be an A-equivariant morphism of smooth A-varieties. Let X_1' be an A-stable closed subvariety of Y' and set $X_1 = f^{-1}(X_1')$. Let X_2 and X_2' be closed subvarieties of Y and Y' respectively such that $f(X_2) = X_2'$ and $X_2 \to X_2'$ is proper. Then we have*

$$f_*(f^*(m) \otimes_{\mathcal{O}_Y} n) = m \otimes_{\mathcal{O}_{Y'}} f_*n \in K^A(Y';X_1' \cap X_2')$$

for any $m \in K^A(Y';X_1')$, $n \in K^A(Y;X_2)$.

Lemma A.5. (Base change theorem 1) *Let $f : Y' \to Y$ and $g : Y'' \to Y$ be A-equivariant morphism of A-varieties. We assume that g is smooth. Set $Y''' = Y' \times_Y Y''$ and let $f' : Y''' \to Y''$ and $g' : Y''' \to Y'$ be canonical morphisms. Let X, X' be closed A-stable closed subvarieties of Y, Y' respectively such that $f(X') \subset X$ and $X' \to X$ is proper. Then we have*

$$g^* \circ f_* = f'_* \circ g'^* : K^A(Y';X') \longrightarrow K^A(Y'';g^{-1}(X)).$$

LEMMA A.6. (Base change theorem 2) *Let Y be a smooth A-variety and let Y_1, Y_2 be A-stable smooth closed subvarieties of Y. Set $Y_3 = Y_1 \cap Y_2$. We assume that Y_3 is smooth and that*

$$T_yY = T_yY_1 + T_yY_2, \quad T_yY_3 = T_yY_1 \cap T_yY_2$$

for any $y \in Y_3$. Here, T_yY denotes the tangent space of Y at y. Let $i : Y_1 \to Y$, $j : Y_2 \to Y$, $i' : Y_3 \to Y_2$, $j' : Y_3 \to Y_1$ be the inclusions. Let X_1 be an A-stable closed subvariety of Y_1. Then we have

$$j^* \circ i_* = i'_* \circ j'^* : K^A(Y_1; X_1) \longrightarrow K^A(Y_2; X_1 \cap Y_2).$$

Appendix B. Convolution product

In this section G is as in Section 2. In particular, G is not necessarily of type A. We fix a nilpotent orbit O of $\mathfrak{g}$ in the following.

According to Conjecture 3.5 the quotient

$$H(\tilde{W}_a)_{\mathcal{C}_O} = H(\tilde{W}_a)_{\underset{LR}{\leq} \mathcal{C}_O} / H(\tilde{W}_a)_{\underset{LR}{<} \mathcal{C}_O}$$

should be identified with

$$K^{G\times\mathbb{C}^*}(Z_O) \cong K^{G\times\mathbb{C}^*}(Z_{\overline{O}}) / K^{G\times\mathbb{C}^*}(Z_{\overline{O}\setminus O}).$$

For $e \in O$ set

$$\mathcal{B}_e = \{x \in \mathcal{B} \mid e \in \mathfrak{n}_x\}.$$

Since Z_O is a $G \times \mathbb{C}^*$-equivariant fiber bundle on O whose fiber at $e \in O$ is canonically isomorphic to $\mathcal{B}_e \times \mathcal{B}_e$, we have

$$K^{G\times\mathbb{C}^*}(Z_O) \cong K^{M(e)}(\mathcal{B}_e \times \mathcal{B}_e), \tag{B.1}$$

where

$$M(e) = \{(g, z) \in G \times \mathbb{C}^* \mid \mathrm{Ad}(g)(e) = z^2 e\}.$$

The aim of this section is to give a description of the product on $K^{M(e)}(\mathcal{B}_e \times \mathcal{B}_e)$ induced from the convolution product $\bigstar$ on $K^{G\times\mathbb{C}^*}(Z)$.

We say that a triple $(h, e, f) \in \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ is an $\mathfrak{sl}_2$-triple if $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$. Then e and f are nilpotent elements belonging to the same conjugacy class. Moreover, the map $(h, e, f) \mapsto e$ induces a bijection between the set of G-conjugacy classes of $\mathfrak{sl}_2$-triples and that of nilpotent orbits. Set

$$\hat{O} = \{(e, f) \in \mathfrak{g} \times \mathfrak{g} \mid e \in O,\ ([e, f], e, f) \text{ is an } \mathfrak{sl}_2\text{-triple}\}.$$

The group $G \times \mathbb{C}^*$ acts transitively on $\hat{O}$ by

$$(g, z) : (e, f) \longrightarrow (z^{-2}\operatorname{Ad}(g)(e), z^2 \operatorname{Ad}(g)(f)).$$

In particular, $\hat{O}$ is a smooth variety. For $(e, f) \in \hat{O}$, Slodowy's variety $\Lambda_{(e,f)}$ is defined by

$$\Lambda_{(e,f)} = \{(a, x) \in \Lambda \mid a \in e + \mathfrak{z}_{\mathfrak{g}}(f)\},$$

where

$$\mathfrak{z}_{\mathfrak{g}}(f) = \{a \in \mathfrak{g} \mid [a, f] = 0\}.$$

PROPOSITION B.1. (Slodowy [17])

(i) $\Lambda_{(e,f)}$ *is a smooth variety with* $\dim \Lambda_{(e,f)} = 2 \dim \mathcal{B}_e$.

(ii) $\operatorname{Ad}(G)(\mathcal{N} \cap (e + \mathfrak{z}_{\mathfrak{g}}(f)) \subset \mathcal{N} \setminus (\overline{O} \setminus O)$.

(iii) $O \cap (e + \mathfrak{z}_{\mathfrak{g}}(f)) = \{e\}$.

We identify $\mathcal{B}_e$ with a closed subvariety of $\Lambda_{(e,f)}$ via the embedding $x \mapsto (e, x)$. Set

$$M(e, f) = \{(g, z) \in G \times \mathbb{C}^* \mid \operatorname{Ad}(g)(e) = z^2 e,\ \operatorname{Ad}(g)(f) = z^{-2} f\}.$$

Then $M(e, f)$ is a subgroup of $M(e)$ acting naturally on $\Lambda_{(e,f)}$. Moreover, $M(e, f)$ and $M(e)$ contain a common maximal reductive subgroup (see [6]). Hence we have the identification

$$K^{M(e)}(\mathcal{B}_e \times \mathcal{B}_e) = K^{M(e,f)}(\Lambda_{(e,f)} \times \Lambda_{(e,f)}; \mathcal{B}_e \times \mathcal{B}_e). \tag{B.2}$$

For $(i, j) = (1, 2), (2, 3), (1, 3)$ we denote by $\pi_{ij} : \Lambda_{(e,f)} \times \Lambda_{(e,f)} \times \Lambda_{(e,f)} \to \Lambda_{(e,f)} \times \Lambda_{(e,f)}$ the projections onto (i, j)-factors.

THEOREM B.2. *The product on* $K^{M(e,f)}(\Lambda_{(e,f)} \times \Lambda_{(e,f)}; \mathcal{B}_e \times \mathcal{B}_e)$ *induced from the convolution product* $\bigstar$ *on* $K^{G \times \mathbb{C}^*}(Z)$ *is given by*

$$(m, n) \longmapsto \pi_{13*}(\pi_{12}^* m \otimes_{\mathcal{O}_{\Lambda_{(e,f)} \times \Lambda_{(e,f)} \times \Lambda_{(e,f)}}} \pi_{23}^* n).$$

The rest of this section is devoted to proving Theorem B.2.

Set

$$\tilde{\Lambda} = \{(a, x) \in \Lambda \mid a \notin \overline{O} \setminus O\}.$$

Then $\tilde{\Lambda}$ is an open subset of Λ, and Z_O is a closed subset of $\tilde{\Lambda} \times \tilde{\Lambda}$. We denote by

$$k : Z_O \longrightarrow \tilde{\Lambda} \times \tilde{\Lambda}$$

the closed embedding. For $(i,j) = (1,2),(2,3),(1,3)$ we denote by $\tilde{p}_{ij} : \tilde{\Lambda} \times \tilde{\Lambda} \times \tilde{\Lambda} \to \tilde{\Lambda} \times \tilde{\Lambda}$ the projections onto (i,j)-factors. We see easily the following.

LEMMA B.3. *The product on $K^{G\times\mathbb{C}^*}(Z_O) = K^{G\times\mathbb{C}^*}(\tilde{\Lambda} \times \tilde{\Lambda}; Z_O)$ induced from the convolution product $\bigstar$ on $K^{G\times\mathbb{C}^*}(Z)$ is given by*

$$(m,n) \longmapsto m \bigstar n = \tilde{p}_{13*}(\tilde{p}_{12}^* m \otimes_{\mathcal{O}_{\tilde{\Lambda}\times\tilde{\Lambda}\times\tilde{\Lambda}}} \tilde{p}_{23}^* n).$$

Set

$$\begin{aligned}\tilde{\Lambda}_O &= \{(e,x) \in \Lambda \mid e \in O\},\\ Y_O &= \tilde{\Lambda}_O \times_O \hat{O} = \{(e,f,x) \mid (e,f) \in \hat{O},\ (e,x) \in \Lambda\},\\ Y &= \{(e,f,a,x) \mid (e,f) \in \hat{O},\ (a,x) \in \Lambda_{(e,f)}\}.\end{aligned}$$

We identify Y_O with a closed subvariety of Y by the embedding

$$i : Y_O \longrightarrow Y \quad ((e,f,x) \longmapsto (e,f,e,x)).$$

Then Y is a $G\times\mathbb{C}^*$-equivariant fiber bundle on $\hat{O}$ whose fiber at $(e,f) \in \hat{O}$ is $\Lambda_{(e,f)}$, and Y_O is its subbundle whose fiber at $(e,f) \in \hat{O}$ is $\mathcal{B}_e$. In particular, Y is a smooth variety and the projection $Y \to \hat{O}$ is a smooth morphism.

We set

$$\begin{aligned}Y^{(2)} &= Y \times_{\hat{O}} Y\\ &= \{(e,f,a,x,b,y) \mid (e,f) \in \hat{O},\ (a,x),(b,y) \in \Lambda_{(e,f)}\},\\ Y_O^{(2)} &= Y_O \times_{\hat{O}} Y_O = Z_O \times_O \hat{O}\\ &= \{(e,f,x,y) \mid (e,f) \in \hat{O},\ x,y \in \mathcal{B}_e\}.\end{aligned}$$

We regard $Y_O^{(2)}$ as a closed subvariety of $Y^{(2)}$ by the embedding

$$i^{(2)} = i \times_{\hat{O}} i : Y_O^{(2)} \longrightarrow Y^{(2)}.$$

Define

$$\varphi : Y_O^{(2)} \longrightarrow Z_O$$

by $\varphi(e,f,x,y) = (e,x,y)$. It is a smooth surjective morphism. Since $Y_O^{(2)}$ is a $G\times\mathbb{C}^*$-equivariant fiber bundle whose fiber at $(e,f) \in \hat{O}$ is $\Lambda_{(e,f)} \times \Lambda_{(e,f)}$, we have a commutative diagram

$$\begin{array}{ccc} K^{G\times\mathbb{C}^*}(Z_O) & \xrightarrow{\varphi^*} & K^{G\times\mathbb{C}^*}(Y_O^{(2)}) \\ \| & & \| \\ K^{M(e)}(\mathcal{B}_e\times\mathcal{B}_e) & \longrightarrow & K^{M(e,f)}(\mathcal{B}_e\times\mathcal{B}_e). \end{array}$$

Hence we see by (B.2) that

$$\varphi^* : K^{G\times\mathbb{C}^*}(Z_O) \longrightarrow K^{G\times\mathbb{C}^*}(Y_O^{(2)}) \tag{B.3}$$

is an isomorphism of $R^{G\times\mathbb{C}^*}$-modules. Set

$$Y^{(3)} = Y \times_{\hat{O}} Y \times_{\hat{O}} Y,$$
$$Y_O^{(3)} = Y_O \times_{\hat{O}} Y \times_{\hat{O}} Y,$$

and regard $Y_O^{(3)}$ as a subvariety of $Y^{(3)}$ by

$$i^{(3)} = i \times_{\hat{O}} i \times_{\hat{O}} i : Y_O^{(3)} \longrightarrow Y^{(3)}.$$

For $(i,j) = (1,2),(2,3),(1,3)$ we denote by $q_{ij} : Y^{(3)} \to Y^{(2)}$ the projections onto (i,j)-factors. Note that q_{ij} is a morphism of $G\times\mathbb{C}^*$-equivariant fiber bundles on $\hat{O}$ whose fiber at $(e,f)\in\hat{O}$ is given by $\pi_{ij} : \Lambda_{(e,f)} \times \Lambda_{(e,f)} \times \Lambda_{(e,f)} \to \Lambda_{(e,f)} \times \Lambda_{(e,f)}$. Therefore, Theorem B.2 is equivalent to the following.

Proposition B.4. *The product on $K^{G\times\mathbb{C}^*}(Y^{(2)};Y_O^{(2)})$ induced from the convolution product $\bigstar$ on $K^{G\times\mathbb{C}^*}(Z_O)$ via φ^* is given by*

$$(m,n) \longmapsto q_{13*}(q_{12}^* m \otimes_{\mathcal{O}_{Y^{(3)}}} q_{23}^* n).$$

By Proposition B.1 we have a morphism

$$\theta : Y \longrightarrow \tilde{\Lambda} \quad ((e,f,a,x) \longmapsto (a,x)).$$

We define

$$\tau : Y_O \longrightarrow \tilde{\Lambda}_O$$

as the restriction of θ.

Lemma B.5. (i) *The commutative diagram*

$$\begin{array}{ccc} Y_O & \xrightarrow{\tau} & \tilde{\Lambda}_O \\ \downarrow & & \downarrow \\ Y & \xrightarrow[\theta]{} & \tilde{\Lambda} \end{array}$$

is cartesian.

(ii) *θ is a smooth morphism.*

Proof. The statement (i) follows from Proposition B.1 (iii). By a result of Slodowy [17] we see that the composition of the smooth surjective morphism

$$G \times \Lambda_{(e,f)} \longrightarrow Y \quad ((g,(a,x)) \longmapsto (\mathrm{Ad}(g)(e), \mathrm{Ad}(g)(f), \mathrm{Ad}(g)(a), gx))$$

with $\theta : Y \to \tilde{\Lambda}$ is smooth (see the proof of Proposition 11.10 in Lusztig [13]). Hence θ is also smooth. ∎

Consider the following diagrams

$$\text{(B.4)} \qquad \begin{array}{ccccc} Y_O^{(2)} \times_{\hat{O}} Y & \xrightarrow{i^{(2)} \times_{\hat{O}} 1} & Y^{(3)} & \xrightarrow{k_{23}} & \tilde{\Lambda} \times Y^{(2)} \\ \beta_{12} \downarrow & & \downarrow k_{12} & & \downarrow \ell_{23} \\ Y_O^{(2)} \times \tilde{\Lambda} & \xrightarrow[i^{(2)} \times 1]{} & Y^{(2)} \times \tilde{\Lambda} & \xrightarrow[\ell_{12}]{} & \tilde{\Lambda} \times Y \times \tilde{\Lambda} \\ \alpha_{12} \downarrow & & \gamma_{12} \downarrow & & \\ Y_O^{(2)} & \xrightarrow[i^{(2)}]{} & Y^{(2)}, & & \end{array}$$

$$\text{(B.5)} \qquad \begin{array}{ccc} Y_O^{(2)} \times \tilde{\Lambda} & \xrightarrow{\ell_{12} \circ (i^{(2)} \times 1)} & \tilde{\Lambda} \times Y \times \tilde{\Lambda} \\ \varphi \circ \alpha_{12} \downarrow & & \downarrow \tilde{p}_{12} \circ (1 \times \theta \times 1) \\ Z_O & \xrightarrow[k]{} & \tilde{\Lambda} \times \tilde{\Lambda} \end{array}$$

where α_{12}, γ_{12} are the projections, and β_{12}, k_{12}, k_{23}, ℓ_{12}, ℓ_{23} are the closed embeddings induced by $\theta : Y \to \tilde{\Lambda}$. We can check the commutativity easily.

Moreover, we see easily that all of the squares in the diagrams are cartesian. We set

$$\psi = \ell_{12} \circ k_{12} = \ell_{23} \circ k_{23} : Y^{(3)} \longrightarrow \tilde{\Lambda} \times Y \times \tilde{\Lambda}.$$

Let $m, n \in K^{G\times\mathbb{C}^*}(Z_O; Z_O)$. Then the corresponding elements in $K^{G\times\mathbb{C}^*}(Y^{(2)}; Y_O^{(2)})$ are given by $\tilde{m} = i_*^{(2)}\varphi^* m$, $\tilde{n} = i_*^{(2)}\varphi^* n$ respectively. By $q_{12} = \gamma_{12} \circ k_{12}$ we see from (B.4) that

$$q_{12}^*\tilde{m} = k_{12}^*\gamma_{12}^* i_*^{(2)}\varphi^* m = k_{12}^*(i^{(2)} \times 1)_*\alpha_{12}^*\varphi^* m.$$

Similarly, we have $q_{23}^*\tilde{n} = k_{23}^*(1 \times i^{(2)})_*\alpha_{23}^*\varphi^* n$, where $\alpha_{23} : \tilde{\Lambda} \times Y_O^{(2)} \to Y_O^{(2)}$ is the projection. Hence we have

$$\begin{aligned} \psi_*(q_{12}^*\tilde{m} \otimes q_{23}^*\tilde{n}) &= \ell_{12*}k_{12*}\big(k_{12}^*(i^{(2)} \times 1)_*\alpha_{12}^*\varphi^* m \otimes k_{23}^*(1 \times i^{(2)})_*\alpha_{23}^*\varphi^* n\big) \\ &= \ell_{12*}\big((i^{(2)} \times 1)_*\alpha_{12}^*\varphi^* m \otimes k_{12*}k_{23}^*(1 \times i^{(2)})_*\alpha_{23}^*\varphi^* n\big) \\ &= \ell_{12*}\big((i^{(2)} \times 1)_*\alpha_{12}^*\varphi^* m \otimes \ell_{12}^*\ell_{23*}(1 \times i^{(2)})_*\alpha_{23}^*\varphi^* n\big) \\ &= \ell_{12*}(i^{(2)} \times 1)_*\alpha_{12}^*\varphi^* m \otimes \ell_{23*}(1 \times i^{(2)})_*\alpha_{23}^*\varphi^* n. \end{aligned}$$

Here we have used Lemma A.4 for the second and the fourth identities and Lemma A.6 for the third identity. By Lemma A.5, Lemma B.5 and (B.5) we have

$$\ell_{12*}(i^{(2)} \times 1)_*\alpha_{12}^*\varphi^* m = (1 \times \theta \times 1)^*\tilde{p}_{12}^* k_* m.$$

Similarly we have

$$\ell_{23_*}(1 \times i^{(2)})_*\alpha_{23}^*\varphi^* n = (1 \times \theta \times 1)^*\tilde{p}_{23}^* k_* n.$$

Therefore, we obtain

$$\psi_*(q_{12}^*\tilde{m} \otimes q_{23}^*\tilde{n}) = (1 \times \theta \times 1)^*(\tilde{p}_{12}^* k_* m \otimes \tilde{p}_{23}^* k_* n)$$

by Lemma A.3.

Set

$$\tilde{\Lambda}_O^{(3)} = \tilde{\Lambda}_O \times_O \tilde{\Lambda}_O \times_O \tilde{\Lambda}_O = \tilde{p}_{12}^{-1} Z_O \cap \tilde{p}_{23}^{-1} Z_O,$$

and consider the commutative diagram

$$\text{(B.6)} \qquad \begin{array}{ccc} \tilde{\Lambda} \times \tilde{\Lambda} \times \tilde{\Lambda} & \xleftarrow{1\times\theta\times 1} & \tilde{\Lambda} \times Y \times \tilde{\Lambda} \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle \psi\circ i^{(3)}} \\ \tilde{\Lambda}_O^{(3)} & \xleftarrow{\tilde{\varphi}} & Y_O^{(3)} \\ {\scriptstyle \overline{p}_{13}}\downarrow & & \downarrow{\scriptstyle \overline{q}_{13}} \\ Z_O & \xleftarrow[\varphi]{} & Y_O^{(2)}. \end{array}$$

Here, f is the natural inclusion, $\tilde{\varphi}$ is the canonical morphism, and $\overline{p}_{13}$, $\overline{q}_{13}$ are the restrictions of $\tilde{p}_{13}$, q_{13} respectively. We see easily that both of the squares in (B.6) are cartesian.

Define $u \in K^{G\times\mathbb{C}^*}(\tilde{\Lambda}_O^{(3)}; \tilde{\Lambda}_O^{(3)})$ by $f_*u = \tilde{p}_{12}^* k_* m \otimes \tilde{p}_{23}^* k_* n$. Then we have

$$\psi_*(q_{12}^*\tilde{m} \otimes q_{23}^*\tilde{n}) = (1 \times \theta \times 1)^* f_* u = \psi_*(i_*^{(3)} \tilde{\varphi}^* u)$$

and hence $q_{12}^*\tilde{m} \otimes q_{23}^*\tilde{n} = i_*^{(3)}\tilde{\varphi}^* u$. It follows that

$$q_{13*}(q_{12}^*\tilde{m} \otimes q_{23}^*\tilde{n}) = q_{13*} i_*^{(3)} \tilde{\varphi}^* u = i_*^{(2)} \overline{q}_{13*} \tilde{\varphi}^* u = i_*^{(2)} \varphi^*(\overline{p}_{13*} u).$$

By

$$k_*(\overline{p}_{13*} u) = \tilde{p}_{13*} f_* u = \tilde{p}_{13*}(\tilde{p}_{12}^* k_* m \otimes \tilde{p}_{23}^* k_* n)$$

we conclude that the element of $K^{G\times\mathbb{C}^*}(Y^{(2)}; Y_O^{(2)})$ corresponding to $m \bigstar n \in K^{G\times\mathbb{C}^*}(Z_O; Z_O)$ is given by $q_{13*}(q_{12}^*\tilde{m} \otimes q_{23}^*\tilde{n})$. Proposition B.4 is verified. This completes the proof of Theorem B.2.

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Toshiyuki Tanisaki
Department of Mathematics
Osaka City University
3-3-138 Sugimoto
Sumiyoshi-ku
Osaka, 558-8585
Japan
tanisaki@sci.osaka-cu.ac.jp

Nanhua Xi
Institute of Mathematics
Chinese Academy of Sciences
Beijing, 100080
China
nanhua@math.ac.cn