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# TWO-DIMENSIONAL SYMMETRIC STABLE DISTRIBUTIONS AND THEIR PROJECTIONS

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**Abstract.** We study the problem whether a given 2-dimensional symmetric stable distribution with index $\alpha$ $(0 < \alpha \leq 1)$ is determined by its 1-dimensional projections in some specified directions. We give some conditions for the affirmative answer and for the negative answer.

## §1. Introduction and preliminaries

An $\mathbf{R}^d$-valued random variable $X = (X_1, X_2, \ldots, X_d)$ is said to be *stable* if for any $A, B > 0$, there exist $C > 0$ and $D \in \mathbf{R}$ such that

$$AX^{(1)} + BX^{(2)} \stackrel{d}{=} CX + D \quad (\stackrel{d}{=} \text{ means equality in distribution}) \tag{1.1}$$

where $X^{(1)}$ and $X^{(2)}$ are independent copies of $X$. If $X$ is stable and non-constant, then there exists a constant $\alpha$ $(0 < \alpha \leq 2)$ such that $C = (A^\alpha + B^\alpha)^{1/\alpha}$ and therefore $X$ is called $\alpha$-stable ($\alpha$ is called the index of stability of $X$). $X$ is called *strictly stable* if (1.1) holds with $D = 0$ for any $A, B > 0$. $X$ is called *symmetric stable* if $X$ is stable and satisfies $-X \stackrel{d}{=} X$. An $\mathbf{R}$-valued random variable $X$ is symmetric $\alpha$-stable $(0 < \alpha \leq 2)$ if and only if $E \exp(izX) = \exp(-c|z|^\alpha)$, $z \in \mathbf{R}$, for some $c \geq 0$. Especially, when $\alpha = 2$, $X$ is Gaussian with mean 0. An $\mathbf{R}^d$-valued random variable $X$ is $d$-dimensional symmetric $\alpha$-stable $(0 < \alpha < 2)$ if and only if

$$E \exp\left(i \sum_{j=1}^{d} z_j X_j\right) = \exp\left(-\int_{\xi=(\xi_1,\xi_2,\ldots,\xi_d)\in S^{d-1}} \left|\sum_{j=1}^{d} z_j \xi_j\right|^\alpha \Gamma(d\xi)\right),$$
$$z = (z_1, z_2, \ldots, z_d) \in \mathbf{R}^d,$$

for some symmetric finite measure $\Gamma$ on the $(d-1)$-dimensional unit sphere $S^{d-1}$. This $\Gamma$ is uniquely determined by the distribution of $X$ and is called the *spectral measure* of $X$.

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The problem whether an $\mathbf{R}^d$-valued random variable $X = (X_1, X_2, \ldots, X_d)$ is stable if all 1-dimensional projections $\sum_{j=1}^d z_j X_j$, $z_j \in \mathbf{R}$, are stable is studied variously. An $\mathbf{R}^d$-valued random variable is symmetric stable (respectively, strictly stable) if and only if all 1-dimensional projections are symmetric stable (respectively, strictly stable) (see Theorem 2.1.5 in G. Samorodnitsky and S. Taqqu [2]). If $1 \le \alpha \le 2$, an $\mathbf{R}^d$-valued random variable is $\alpha$-stable if and only if all 1-dimensional projections are $\alpha$-stable. However, if $0 < \alpha < 1$, there exists a non-stable $\mathbf{R}^2$-valued random variable such that all 1-dimensional projections are $\alpha$-stable (D. J. Marcus [1]).

Two $\mathbf{R}^d$-valued random variables $X = (X_1, X_2, \ldots, X_d)$ and $\tilde{X} = (\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_d)$ are identically distributed if $\sum_{j=1}^d z_j X_j \overset{d}{=} \sum_{j=1}^d z_j \tilde{X}_j$ for all $z_j \in \mathbf{R}$. In the case where two 2-dimensional random variables $X = (X_1, X_2)$ and $\tilde{X} = (\tilde{X}_1, \tilde{X}_2)$ are Gaussian with mean 0, they are identically distributed if $(\cos\theta_k)X_1 + (\sin\theta_k)X_2 \overset{d}{=} (\cos\theta_k)\tilde{X}_1 + (\sin\theta_k)\tilde{X}_2$, $k = 1, 2, 3$, for some $\theta_1$, $\theta_2$ and $\theta_3$ $(0 \le \theta_1 < \theta_2 < \theta_3 < \pi)$. In this paper we study the problem whether a given 2-dimensional symmetric $\alpha$-stable distribution $(0 < \alpha \le 1)$ is determined by its 1-dimensional projections in some specified directions. In Section 2, we see that for any 2-dimensional symmetric $\alpha$-stable random variable $X = (X_1, X_2)$ $(0 < \alpha \le 1)$, there exists a 2-dimensional symmetric $\alpha$-stable random variable $\tilde{X} = (\tilde{X}_1, \tilde{X}_2)$ such that $z_1\tilde{X}_1 + z_2\tilde{X}_2 \overset{d}{=} z_1X_1 + z_2X_2$ for uncountably many pairs $(z_1, z_2)$ with $z_1^2 + z_2^2 = 1$ although $\tilde{X} \overset{d}{\neq} X$. In Section 3, we see that for a certain 2-dimensional symmetric $\alpha$-stable random variable $X = (X_1, X_2)$ $(0 < \alpha < 1)$, there does not exist a 2-dimensional symmetric $\alpha$-stable random variable $\tilde{X} = (\tilde{X}_1, \tilde{X}_2)$ such that $\tilde{X} \overset{d}{\neq} X$ and $z_1\tilde{X}_1 + z_2\tilde{X}_2 \overset{d}{=} z_1X_1 + z_2X_2$ for some specified vectors $(z_1, z_2) \in \mathbf{R}^2$.

## §2. The existence of distribution with common projections in uncountably many directions

Henceforth we identify the unit circle $S^1$ with $[0, 2\pi)$ and denote $\xi_1 = \cos\xi$, $\xi_2 = \sin\xi$ for $\xi \in [0, 2\pi)$.

THEOREM 2.1. *Let* $X = (X_1, X_2)$ *be a 2-dimensional symmetric* $\alpha$*-stable random variable* $(0 < \alpha \le 1)$*. If the spectral measure* $\Gamma$ *of* $X$ *satisfies* $\Gamma((0, \pi/2)) > 0$*, then there exists a 2-dimensional symmetric* $\alpha$*-stable ran-*

*dom variable* $\tilde{X} = (\tilde{X}_1, \tilde{X}_2)$ *satisfying* $\tilde{X} \overset{d}{\neq} X$ *such that*

$$z_1\tilde{X}_1 + z_2\tilde{X}_2 \overset{d}{=} z_1X_1 + z_2X_2 \quad \text{if } z_1z_2 \geq 0. \tag{C1}$$

*Proof.* First we consider the case $\Gamma([\pi/2, \pi]) = 0$. If there exists an $\mathbf{R}^2$-valued random variable $\tilde{X} = (\tilde{X}_1, \tilde{X}_2)$ such that

$$\begin{aligned} &E\exp\big(i(z_1\tilde{X}_1 + z_2\tilde{X}_2)\big) \\ &\quad = \begin{cases} \exp\Big(-\int_{[0,2\pi)} |z_1\xi_1 + z_2\xi_2|^\alpha\Gamma(d\xi)\Big) & \text{if } z_1z_2 \geq 0, \\ \exp\Big(-\int_{[0,2\pi)} |z_1\xi_1 - z_2\xi_2|^\alpha\Gamma(d\xi)\Big) & \text{if } z_1z_2 < 0, \end{cases} \end{aligned} \tag{2.1}$$

then $\tilde{X}$ is symmetric $\alpha$-stable and satisfies the condition (C1). Further, we have $\tilde{X} \overset{d}{\neq} X$ since $\Gamma((0, \pi/2)) > 0$ and $|\xi_1+\xi_2|^\alpha > |\xi_1-\xi_2|^\alpha$ for $\xi \in (0, \pi/2)$. Therefore we have only to show the existence of $\tilde{X}$.

Let $\tilde{\varphi}(z_1, z_2)$ denote the right hand side of (2.1). Let $\epsilon \in (0, \pi/4)$ be a constant with $\Gamma((\epsilon, \pi/2 - \epsilon)) > 0$ and we have

$$\begin{aligned} c &= \int_{\mathbf{R}^2} \tilde{\varphi}(z_1, z_2)\, dz_1dz_2 \\ &= 2\int_{z_1>0,\, z_2>0} \tilde{\varphi}(z_1, z_2)\, dz_1dz_2 + 2\int_{z_1>0,\, z_2<0} \tilde{\varphi}(z_1, z_2)\, dz_1dz_2 \\ &= 4\int_{z_1>0,\, z_2>0} \exp\Big(-2\int_{[0,\pi)} |z_1\xi_1 + z_2\xi_2|^\alpha\Gamma(d\xi)\Big) dz_1dz_2 \\ &\leq 4\int_{z_1>0,\, z_2>0} \exp\Big(-2\int_{(\epsilon,\pi/2-\epsilon)} |z_1\xi_1 + z_2\xi_2|^\alpha\Gamma(d\xi)\Big) dz_1dz_2 \\ &\leq 4\int_{z_1>0,\, z_2>0} \exp(-c_1|z_1|^\alpha - c_2|z_2|^\alpha)\, dz_1dz_2 < \infty \end{aligned}$$

for some $c_1, c_2 > 0$. Since $\tilde{\varphi}(z_1, z_2)$ is positive, $\tilde{\varphi}(z_1, z_2)/c$ is a density function of a distribution on $\mathbf{R}^2$. Let $f(x_1, x_2)$ be

$$f(x_1, x_2) = \frac{1}{(2\pi)^2}\int_{\mathbf{R}^2} \exp(-i(x_1z_1 + x_2z_2))\tilde{\varphi}(z_1, z_2)\, dz_1dz_2.$$

Then we have

$$\frac{1}{c}\int_{\mathbf{R}^2} \exp(i(x_1z_1 + x_2z_2))\tilde{\varphi}(z_1, z_2)\, dz_1dz_2 = \frac{(2\pi)^2}{c} f(x_1, x_2),$$

using $\tilde{\varphi}(-z_1,-z_2) = \tilde{\varphi}(z_1,z_2)$. Suppose that we have shown that $\int_{\mathbf{R}^2} |f(x_1,x_2)|\,dx_1dx_2 < \infty$. Then we have

$$\frac{1}{c}\tilde{\varphi}(z_1,z_2) = \frac{1}{(2\pi)^2}\int_{\mathbf{R}^2} \exp(-i(x_1z_1+x_2z_2))\frac{(2\pi)^2}{c}f(x_1,x_2)\,dx_1dx_2$$

by inverse Fourier transform and therefore

$$\tilde{\varphi}(z_1,z_2) = \int_{\mathbf{R}^2} \exp(i(x_1z_1+x_2z_2))f(x_1,x_2)\,dx_1dx_2,$$

using $f(-x_1,-x_2) = f(x_1,x_2)$. If we further show that $f(x_1,x_2)$ is non-negative, we find that $\tilde{\varphi}(z_1,z_2)$ is the characteristic function of a distribution with density $f(x_1,x_2)$. That is, $\tilde{X}$ exists.

Let us show that $f(x_1,x_2) > 0$. The function $f(x_1,x_2)$ can be written as

$$f(x_1,x_2) = \frac{1}{\pi^2}\int_0^\infty \cos(x_2z_2)\int_0^\infty \cos(x_1z_1) \times \exp\biggl(-2\int_{(0,\pi/2)}(z_1\xi_1+z_2\xi_2)^\alpha\Gamma(d\xi)\biggr)dz_1dz_2.$$

We note that if a function $g : (0,\infty) \to \mathbf{R}$ is summable and $g'' > 0$ on $(0,\infty)$, then $\int_0^\infty g(x)\cos kx\,dx > 0$ for any $k \in \mathbf{R}$. Let us define

$$I_{x_1}(z_2) = \int_0^\infty \cos(x_1z_1)\exp\biggl(-2\int_{(0,\pi/2)}(z_1\xi_1+z_2\xi_2)^\alpha\Gamma(d\xi)\biggr)dz_1.$$

Then

$$\frac{d^2}{d{z_2}^2}I_{x_1}(z_2) = \int_0^\infty \cos(x_1z_1)\,\frac{\partial^2}{\partial {z_2}^2}\exp\biggl(-2\int_{(0,\pi/2)}(z_1\xi_1+z_2\xi_2)^\alpha\Gamma(d\xi)\biggr)dz_1.$$

We find that

$$\begin{aligned}
&\frac{\partial^2}{\partial {z_2}^2}\exp\biggl(-2\int_{(0,\pi/2)}(z_1\xi_1+z_2\xi_2)^\alpha\Gamma(d\xi)\biggr)\\
&\quad= \exp\biggl(-2\int_{(0,\pi/2)}(z_1\xi_1+z_2\xi_2)^\alpha\Gamma(d\xi)\biggr)\\
&\qquad\times\biggl\{\biggl(2\alpha\int_{(0,\pi/2)}(z_1\xi_1+z_2\xi_2)^{\alpha-1}\xi_2\Gamma(d\xi)\biggr)^2\\
&\qquad\qquad- 2\alpha(\alpha-1)\int_{(0,\pi/2)}(z_1\xi_1+z_2\xi_2)^{\alpha-2}\xi_2^2\Gamma(d\xi)\biggr\}
\end{aligned}$$

is summable on $(0,\infty)$ with respect to $z_1$ for any fixed $z_2>0$ because

$$\begin{aligned}&\left|\frac{\partial^2}{\partial {z_2}^2}\exp\Big(-2\int_{(0,\pi/2)}(z_1\xi_1+z_2\xi_2)^\alpha\Gamma(d\xi)\Big)\right|\\&\quad\le\exp\Big(-2\int_{(0,\pi/2)}(z_1\xi_1+z_2\xi_2)^\alpha\Gamma(d\xi)\Big)\\&\qquad\times\left\{\Big(2\alpha\int_{(0,\pi/2)}(z_2\xi_2)^{\alpha-1}\xi_2\Gamma(d\xi)\Big)^2\right.\\&\qquad\qquad\left.+2\alpha(1-\alpha)\int_{(0,\pi/2)}(z_2\xi_2)^{\alpha-2}\xi_2^2\Gamma(d\xi)\right\}.\end{aligned}$$

And we obtain that $\frac{\partial^2}{\partial {z_1}^2}\frac{\partial^2}{\partial {z_2}^2}\exp\big(-2\int_{(0,\pi/2)}(z_1\xi_1+z_2\xi_2)^\alpha\Gamma(d\xi)\big)>0$ by direct calculations, so that $\frac{d^2}{d{z_2}^2}I_{x_1}(z_2)$ is positive for any $x_1$. Moreover, $I_{x_1}(z_2)$ is summable for any $x_1$ and therefore we find $f(x_1,x_2)>0$.

We write $I=\int_{\mathbf{R}^2}f(x_1,x_2)\,dx_1dx_2$. Although we need only to show that $I<\infty$, let us show that $I=1$. We have

$$\begin{aligned}I&=\lim_{k_1,k_2\to\infty}\int_{-k_2}^{k_2}\int_{-k_1}^{k_1}f(x_1,x_2)\,dx_1dx_2\\&=\frac{1}{\pi^2}\lim_{k_1,k_2\to\infty}\int_0^\infty\int_0^\infty\exp\Big(-2\int_{(0,\pi/2)}(z_1\xi_1+z_2\xi_2)^\alpha\Gamma(d\xi)\Big)dz_1dz_2\\&\qquad\times\int_{-k_2}^{k_2}\int_{-k_1}^{k_1}\cos(x_1z_1)\cos(x_2z_2)\,dx_1dx_2\\&=\frac{4}{\pi^2}\lim_{k_1,k_2\to\infty}\int_0^\infty\int_0^\infty\frac{\sin(k_1z_1)}{z_1}\frac{\sin(k_2z_2)}{z_2}\\&\qquad\times\exp\Big(-2\int_{(0,\pi/2)}(z_1\xi_1+z_2\xi_2)^\alpha\Gamma(d\xi)\Big)dz_1dz_2\\&=\frac{4}{\pi^2}\lim_{k_1,k_2\to\infty}\int_0^\infty\int_0^\infty\frac{\sin z_1}{z_1}\frac{\sin z_2}{z_2}\\&\qquad\times\exp\Big(-2\int_{(0,\pi/2)}\Big(\frac{z_1}{k_1}\xi_1+\frac{z_2}{k_2}\xi_2\Big)^\alpha\Gamma(d\xi)\Big)dz_1dz_2\\&=\frac{4}{\pi^2}\lim_{k_1,k_2\to\infty}\sum_{m,n=0}^\infty\int_{2n\pi}^{2(n+1)\pi}\int_{2m\pi}^{2(m+1)\pi}\sin z_1\sin z_2\\&\qquad\times\frac{1}{z_1z_2}\exp\Big(-2\int_{(0,\pi/2)}\Big(\frac{z_1}{k_1}\xi_1+\frac{z_2}{k_2}\xi_2\Big)^\alpha\Gamma(d\xi)\Big)dz_1dz_2.\end{aligned}$$

Here we define $q_{k_1,k_2}(z_1,z_2) = \frac{1}{z_1z_2}\exp\bigl(-2\int_{(0,\pi/2)}\bigl(\frac{z_1}{k_1}\xi_1 + \frac{z_2}{k_2}\xi_2\bigr)^\alpha\Gamma(d\xi)\bigr)$ and $\tilde{Q}(k_1,k_2,z_1,z_2,w_1,w_2) = q_{k_1,k_2}(z_1,z_2) - q_{k_1,k_2}(w_1,z_2) - q_{k_1,k_2}(z_1,w_2) + q_{k_1,k_2}(w_1,w_2)$. For $w_1 > z_1$ and $w_2 > z_2$, we find $\tilde{Q}(k_1,k_2,z_1,z_2,w_1,w_2) > 0$ because

$$\begin{aligned}
&\frac{\partial}{\partial w_1}\tilde{Q}(k_1,k_2,z_1,z_2,w_1,w_2)\\
&\quad= \frac{1}{w_1^2}\biggl\{\frac{1}{z_2}\exp\biggl(-2\int_{(0,\pi/2)}\Bigl(\frac{w_1}{k_1}\xi_1 + \frac{z_2}{k_2}\xi_2\Bigr)^\alpha\Gamma(d\xi)\biggr)\\
&\qquad\qquad - \frac{1}{w_2}\exp\biggl(-2\int_{(0,\pi/2)}\Bigl(\frac{w_1}{k_1}\xi_1 + \frac{w_2}{k_2}\xi_2\Bigr)^\alpha\Gamma(d\xi)\biggr)\biggr\}\\
&\qquad + \frac{2\alpha}{w_1}\biggl\{\frac{1}{z_2}\exp\biggl(-2\int_{(0,\pi/2)}\Bigl(\frac{w_1}{k_1}\xi_1 + \frac{z_2}{k_2}\xi_2\Bigr)^\alpha\Gamma(d\xi)\biggr)\\
&\qquad\qquad\qquad \times\biggl(\int_{(0,\pi/2)}\Bigl(\frac{w_1}{k_1}\xi_1 + \frac{z_2}{k_2}\xi_2\Bigr)^{\alpha-1}\frac{\xi_1}{k_1}\Gamma(d\xi)\biggr)\\
&\qquad\qquad - \frac{1}{w_2}\exp\biggl(-2\int_{(0,\pi/2)}\Bigl(\frac{w_1}{k_1}\xi_1 + \frac{w_2}{k_2}\xi_2\Bigr)^\alpha\Gamma(d\xi)\biggr)\\
&\qquad\qquad\qquad \times\biggl(\int_{(0,\pi/2)}\Bigl(\frac{w_1}{k_1}\xi_1 + \frac{w_2}{k_2}\xi_2\Bigr)^{\alpha-1}\frac{\xi_1}{k_1}\Gamma(d\xi)\biggr)\biggr\}\\
&\quad> 0
\end{aligned}$$

and $\frac{\partial}{\partial w_2}\tilde{Q}(k_1,k_2,z_1,z_2,w_1,w_2) > 0$. We define $Q_{k_1,k_2}(z_1,z_2) = q_{k_1,k_2}(z_1,z_2) - q_{k_1,k_2}(z_1+\pi,z_2) - q_{k_1,k_2}(z_1,z_2+\pi) + q_{k_1,k_2}(z_1+\pi,z_2+\pi)$, then we have

$$I = \frac{4}{\pi^2}\lim_{k_1,k_2\to\infty}\sum_{m,n=0}^{\infty}\int_{2n\pi}^{(2n+1)\pi}\int_{2m\pi}^{(2m+1)\pi}\sin z_1\sin z_2\,Q_{k_1,k_2}(z_1,z_2)\,dz_1dz_2.$$

By $Q_{k_1,k_2}(z_1,z_2) = \tilde{Q}(k_1,k_2,z_1,z_2,z_1+\pi,z_2+\pi) > 0$, we obtain $I_{m,n,k_1,k_2} = \int_{2n\pi}^{(2n+1)\pi}\int_{2m\pi}^{(2m+1)\pi}\sin z_1\sin z_2\,Q_{k_1,k_2}(z_1,z_2)\,dz_1dz_2 > 0$. And we have

$$\begin{aligned}
&\frac{\partial}{\partial k_1}\tilde{Q}(k_1,k_2,z_1,z_2,w_1,w_2)\\
&\quad= \frac{2\alpha}{k_1^2}\biggl\{\frac{1}{z_2}\biggl(\exp\biggl(-2\int_{(0,\pi/2)}\Bigl(\frac{z_1}{k_1}\xi_1 + \frac{z_2}{k_2}\xi_2\Bigr)^\alpha\Gamma(d\xi)\biggr)\\
&\qquad\qquad\qquad \times\biggl(\int_{(0,\pi/2)}\Bigl(\frac{z_1}{k_1}\xi_1 + \frac{z_2}{k_2}\xi_2\Bigr)^{\alpha-1}\xi_1\Gamma(d\xi)\biggr)
\end{aligned}$$

$$
\begin{aligned}
&- \exp\Big(-2\int_{(0,\pi/2)}\Big(\frac{w_1}{k_1}\xi_1+\frac{z_2}{k_2}\xi_2\Big)^{\alpha}\Gamma(d\xi)\Big)\\
&\qquad\times\Big(\int_{(0,\pi/2)}\Big(\frac{w_1}{k_1}\xi_1+\frac{z_2}{k_2}\xi_2\Big)^{\alpha-1}\xi_1\Gamma(d\xi)\Big)\Big)\\
&-\frac{1}{w_2}\Big(\exp\Big(-2\int_{(0,\pi/2)}\Big(\frac{z_1}{k_1}\xi_1+\frac{w_2}{k_2}\xi_2\Big)^{\alpha}\Gamma(d\xi)\Big)\\
&\qquad\times\Big(\int_{(0,\pi/2)}\Big(\frac{z_1}{k_1}\xi_1+\frac{w_2}{k_2}\xi_2\Big)^{\alpha-1}\xi_1\Gamma(d\xi)\Big)\\
&- \exp\Big(-2\int_{(0,\pi/2)}\Big(\frac{w_1}{k_1}\xi_1+\frac{w_2}{k_2}\xi_2\Big)^{\alpha}\Gamma(d\xi)\Big)\\
&\qquad\times\Big(\int_{(0,\pi/2)}\Big(\frac{w_1}{k_1}\xi_1+\frac{w_2}{k_2}\xi_2\Big)^{\alpha-1}\xi_1\Gamma(d\xi)\Big)\Big)\Big\}.
\end{aligned}
$$

We find that

$$
\begin{aligned}
&\exp\Big(-2\int_{(0,\pi/2)}\Big(\frac{z_1}{k_1}\xi_1+\frac{z_2}{k_2}\xi_2\Big)^{\alpha}\Gamma(d\xi)\Big)\\
&\qquad- \exp\Big(-2\int_{(0,\pi/2)}\Big(\frac{w_1}{k_1}\xi_1+\frac{z_2}{k_2}\xi_2\Big)^{\alpha}\Gamma(d\xi)\Big)\\
&> \exp\Big(-2\int_{(0,\pi/2)}\Big(\frac{z_1}{k_1}\xi_1+\frac{w_2}{k_2}\xi_2\Big)^{\alpha}\Gamma(d\xi)\Big)\\
&\qquad- \exp\Big(-2\int_{(0,\pi/2)}\Big(\frac{w_1}{k_1}\xi_1+\frac{w_2}{k_2}\xi_2\Big)^{\alpha}\Gamma(d\xi)\Big),
\end{aligned}
$$

$$
\begin{aligned}
&\int_{(0,\pi/2)}\Big(\frac{z_1}{k_1}\xi_1+\frac{z_2}{k_2}\xi_2\Big)^{\alpha-1}\xi_1\Gamma(d\xi)\\
&\qquad-\int_{(0,\pi/2)}\Big(\frac{w_1}{k_1}\xi_1+\frac{z_2}{k_2}\xi_2\Big)^{\alpha-1}\xi_1\Gamma(d\xi)\\
&> \int_{(0,\pi/2)}\Big(\frac{z_1}{k_1}\xi_1+\frac{w_2}{k_2}\xi_2\Big)^{\alpha-1}\xi_1\Gamma(d\xi)\\
&\qquad-\int_{(0,\pi/2)}\Big(\frac{w_1}{k_1}\xi_1+\frac{w_2}{k_2}\xi_2\Big)^{\alpha-1}\xi_1\Gamma(d\xi),
\end{aligned}
$$

using that $x^{\alpha}$ $(0<\alpha\le 1)$ is concave and $x^{\alpha-1}$ $(0<\alpha<1)$ and $e^{-x}$ are convex for $x>0$. We note that if $A_1$, $A_2$, $A_3$, $A_4$, $B_1$, $B_2$, $B_3$ and $B_4$ are positive numbers such that $A_1>A_3$, $B_2>B_4$, $A_1-A_2>A_3-A_4>0$

and $B_1 - B_2 > B_3 - B_4 > 0$, then $A_1B_1 - A_2B_2 = A_1(B_1 - B_2) + (A_1 - A_2)B_2 > A_3(B_3 - B_4) + (A_3 - A_4)B_4 = A_3B_3 - A_4B_4$. Therefore we find $\frac{\partial}{\partial k_1}\tilde{Q}(k_1, k_2, z_1, z_2, w_1, w_2)$, $\frac{\partial}{\partial k_2}\tilde{Q}(k_1, k_2, z_1, z_2, w_1, w_2) > 0$, so that $I_{m,n,k_1,k_2}$ monotonously increases in $k_1$, $k_2$. Hence we obtain

$$\begin{aligned}
I &= \frac{4}{\pi^2}\sum_{m,n=0}^{\infty}\lim_{k_1,k_2\to\infty}\int_{2n\pi}^{(2n+1)\pi}\int_{2m\pi}^{(2m+1)\pi}\sin z_1 \sin z_2\, Q_{k_1,k_2}(z_1,z_2)\,dz_1dz_2\\
&= \frac{4}{\pi^2}\sum_{m,n=0}^{\infty}\int_{2n\pi}^{2(n+1)\pi}\int_{2m\pi}^{2(m+1)\pi}\frac{\sin z_1}{z_1}\frac{\sin z_2}{z_2}\\
&\qquad\qquad \times \lim_{k_1,k_2\to\infty}\exp\Big(-2\int_{(0,\pi/2)}\Big(\frac{z_1}{k_1}\xi_1 + \frac{z_2}{k_2}\xi_2\Big)^{\alpha}\Gamma(d\xi)\Big)dz_1dz_2\\
&= \frac{4}{\pi^2}\int_0^{\infty}\frac{\sin z_1}{z_1}\,dz_1\int_0^{\infty}\frac{\sin z_2}{z_2}\,dz_2\\
&= 1.
\end{aligned}$$

Next we consider the case $\Gamma([\pi/2,\pi]) > 0$. Let $\Gamma_1$, $\Gamma_2$ be

$$\begin{aligned}
&\Gamma_1(B) = \Gamma(B\cap((0,\pi/2)\cup(\pi,3\pi/2))),\\
&\Gamma_2(B) = \Gamma(B\cap([\pi/2,\pi]\cup[3\pi/2,2\pi])) \text{ for a Borel set } B\subset[0,2\pi).
\end{aligned}$$

Then $\Gamma = \Gamma_1 + \Gamma_2$. Let us define $\tilde{X} = (\tilde{X}_1, \tilde{X}_2)$ such that

$$\begin{aligned}
&E\exp(i(z_1\tilde{X}_1 + z_2\tilde{X}_2))\\
&= \begin{cases}
\exp\Big(-\int_{S^1}|z_1\xi_1 + z_2\xi_2|^{\alpha}\Gamma_1(d\xi)\Big)\exp\Big(-\int_{S^1}|z_1\xi_1 + z_2\xi_2|^{\alpha}\Gamma_2(d\xi)\Big) & \text{if } z_1z_2 \ge 0,\\
\exp\Big(-\int_{S^1}|z_1\xi_1 - z_2\xi_2|^{\alpha}\Gamma_1(d\xi)\Big)\exp\Big(-\int_{S^1}|z_1\xi_1 + z_2\xi_2|^{\alpha}\Gamma_2(d\xi)\Big) & \text{if } z_1z_2 < 0.
\end{cases}
\end{aligned}$$

We can easily see that there exists a 2-dimensional symmetric stable random variable whose characteristic function is $\exp\big(-\int_{S^1}|z_1\xi_1 + z_2\xi_2|^{\alpha}\Gamma_1(d\xi)\big)$ if $z_1z_2 \ge 0$, and $\exp\big(-\int_{S^1}|z_1\xi_1 - z_2\xi_2|^{\alpha}\Gamma_1(d\xi)\big)$ if $z_1z_2 < 0$. Its spectral measure is denoted by $\tilde{\Gamma}_1$ and we have $E\exp(i(z_1\tilde{X}_1 + z_2\tilde{X}_2)) = \exp\big(-\int_{S^1}|z_1\xi_1 + z_2\xi_2|^{\alpha}(\tilde{\Gamma}_1 + \Gamma_2)(d\xi)\big)$. Obviously, $\tilde{\Gamma}_1 + \Gamma_2$ satisfies the properties of spectral measures and we find that $\tilde{X}$ exists and satisfies the condition (C1). ∎

We note that, if $\alpha > 1$, any symmetric $\alpha$-stable random variable has a differentiable characteristic function. Therefore, if $\alpha > 1$, $\Gamma((0, \pi/2)) > 0$ and $\Gamma([\pi/2, \pi]) = 0$, then the right hand side of (2.1), $\tilde{\varphi}(z_1, z_2)$, cannot be the characteristic function of a symmetric $\alpha$-stable random variable, because $\tilde{\varphi}(z_1, z_2)$ is non-differentiable on $z_1 = 0$ and $z_2 = 0$.

Theorem 2.1 can be extended to the following proposition. Here, if $(z_1, z_2) \in \mathbf{R}^2 \backslash \{(0,0)\}$ is written as $z_1 = r\cos\theta$, $z_2 = r\sin\theta$ $(r > 0)$, then $\theta$ is denoted by $\arg(z_1, z_2)$.

PROPOSITION 2.2. *Let $X = (X_1, X_2)$ be a 2-dimensional symmetric $\alpha$-stable random variable* $(0 < \alpha \leq 1)$ *whose spectral measure $\Gamma$ satisfies* $\Gamma((\theta_1, \theta_2)) > 0$ $(0 \leq \theta_1 < \theta_2 < \pi)$. *Then there exists a 2-dimensional symmetric $\alpha$-stable random variable $\tilde{X} = (\tilde{X}_1, \tilde{X}_2)$ satisfying $\tilde{X} \overset{d}{\neq} X$ such that*

$$\text{(C2)} \quad z_1\tilde{X}_1 + z_2\tilde{X}_2 \overset{d}{=} z_1 X_1 + z_2 X_2 \quad \textit{if } \arg(z_1, z_2) \in [\theta_2 - \pi/2, \theta_1 + \pi/2].$$

*Proof.* Let $Y = (Y_1, Y_2)$ be a 2-dimensional symmetric stable random variable such that

$$\begin{cases} Y_1 = X_1 \sin\theta_2 - X_2\cos\theta_2 \\ Y_2 = -X_1\sin\theta_1 + X_2\cos\theta_1. \end{cases}$$

Then

$$\begin{aligned}
&E\exp(i(z_1Y_1 + z_2Y_2)) \\
&\quad = E\exp(i(z_1(X_1\sin\theta_2 - X_2\cos\theta_2) + z_2(-X_1\sin\theta_1 + X_2\cos\theta_1))) \\
&\quad = E\exp(i((z_1\sin\theta_2 - z_2\sin\theta_1)X_1 - (z_1\cos\theta_2 - z_2\cos\theta_1)X_2)) \\
&\quad = \exp\bigg(-\int_{[0,2\pi)} |(z_1\sin\theta_2 - z_2\sin\theta_1)\cos\xi \\
&\qquad\qquad\qquad - (z_1\cos\theta_2 - z_2\cos\theta_1)\sin\xi|^\alpha \Gamma(d\xi)\bigg) \\
&\quad = \exp\bigg(-\int_{[0,2\pi)} |z_1(\sin\theta_2\cos\xi - \cos\theta_2\sin\xi) \\
&\qquad\qquad\qquad + z_2(\sin\xi\cos\theta_1 - \cos\xi\sin\theta_1)|^\alpha \Gamma(d\xi)\bigg) \\
&\quad = \exp\bigg(-\int_{[0,2\pi)} |z_1\sin(\theta_2 - \xi) + z_2\sin(\xi - \theta_1)|^\alpha \Gamma(d\xi)\bigg).
\end{aligned}$$

Here let us define the function $\sigma : \xi \in [0, 2\pi) \to \eta \in [0, 2\pi)$ such that

$$\begin{cases} \cos\eta = \sin(\theta_2 - \xi)/\sqrt{\sin^2(\theta_2 - \xi) + \sin^2(\xi - \theta_1)} \\ \sin\eta = \sin(\xi - \theta_1)/\sqrt{\sin^2(\theta_2 - \xi) + \sin^2(\xi - \theta_1)}. \end{cases}$$

We note that the function $\sigma$ is one-to-one correspondence. Therefore we have

$$E\exp(i(z_1Y_1 + z_2Y_2)) = \exp\left(-\int_{[0,2\pi)} |z_1\cos\eta + z_2\sin\eta|^\alpha \Gamma_Y(d\eta)\right),$$

where $\Gamma_Y(d\eta)$ is the spectral measure of $Y$ and satisfies $\Gamma_Y(d\eta) = \big(\sin^2(\theta_2 - \sigma^{-1}(\eta)) + \sin^2(\sigma^{-1}(\eta) - \theta_1)\big)^{\alpha/2}\Gamma(d\sigma^{-1}(\eta))$. We see that $\eta = \sigma(\xi)$ changes from 0 to $\pi/2$ as $\xi$ changes from $\theta_1$ to $\theta_2$ and therefore $\Gamma_Y((0, \pi/2)) > 0$. By Theorem 2.1, there exists a 2-dimensional symmetric stable random variable $\tilde{Y} = (\tilde{Y}_1, \tilde{Y}_2)$ such that $z_1\tilde{Y}_1 + z_2\tilde{Y}_2 \overset{d}{=} z_1Y_1 + z_2Y_2$ for $z_1z_2 \geq 0$ although $\tilde{Y} \overset{d}{\neq} Y$.

Let us define a 2-dimensional symmetric stable random variable $\tilde{X} = (\tilde{X}_1, \tilde{X}_2)$ such that

$$\begin{cases} \tilde{Y}_1 = \tilde{X}_1\sin\theta_2 - \tilde{X}_2\cos\theta_2 \\ \tilde{Y}_2 = -\tilde{X}_1\sin\theta_1 + \tilde{X}_2\cos\theta_1. \end{cases}$$

Let $w = (w_1, w_2)$ be $w_1 = z_1\sin\theta_2 - z_2\sin\theta_1$ and $w_2 = -z_1\cos\theta_2 + z_2\cos\theta_1$. Then we can easily see that $w_1\tilde{X}_1 + w_2\tilde{X}_2 \overset{d}{=} w_1X_1 + w_2X_2$ for $z_1z_2 \geq 0$. When $z_2 = 0$ and $z_1 > 0$, we have $w_1 = z_1\sin\theta_2$ and $w_2 = -z_1\cos\theta_2$ so that $\arg w = \theta_2 - \pi/2$. When $z_1 = 0$ and $z_2 > 0$, we have $w_1 = -z_2\sin\theta_1$ and $w_2 = z_2\cos\theta_1$ so that $\arg w = \theta_1 + \pi/2$. Since the linear transformation from $(z_1, z_2)$ to $(w_1, w_2)$ has a positive determinant, we obtain that $\arg w$ changes from $\theta_2 - \pi/2$ to $\theta_1 + \pi/2$ for $z_1, z_2 \geq 0$. Hence we find that $w_1\tilde{X}_1 + w_2\tilde{X}_2 \overset{d}{=} w_1X_1 + w_2X_2$ for $\arg w \in [\theta_2 - \pi/2, \theta_1 + \pi/2]$. Obviously $\tilde{X} \overset{d}{\neq} X$. □

Proposition 2.2 implies an interesting corollary as follows.

COROLLARY 2.3. *For any non-zero* 2*-dimensional symmetric* $\alpha$*-stable random variable* $X = (X_1, X_2)$ $(0 < \alpha \leq 1)$*, there exists a* 2*-dimensional symmetric* $\alpha$*-stable random variable* $\tilde{X} = (\tilde{X}_1, \tilde{X}_2)$ *such that* $z_1\tilde{X}_1 + z_2\tilde{X}_2 \overset{d}{=} z_1X_1 + z_2X_2$ *for uncountably many pairs* $(z_1, z_2)$ *with* $z_1^2 + z_2^2 = 1$ *although* $\tilde{X} \overset{d}{\neq} X$.

## §3. The distribution determined by its projections in some specified directions

As for Theorem 2.1, in the case $\Gamma((0,\pi/2)) = 0$, we have an example with the opposite property.

PROPOSITION 3.1. *Let $X = (X_1, X_2)$ be a 2-dimensional symmetric $\alpha$-stable random variable* $(0 < \alpha < 1)$ *whose spectral measure $\Gamma$ satisfies that $\Gamma|_{[0,\pi)}$ is concentrated on at most three points on $[\pi/2, \pi)$. Suppose that a 2-dimensional symmetric $\alpha$-stable random variable $\tilde{X} = (\tilde{X}_1, \tilde{X}_2)$ satisfies the condition* (C1)*. Then $\tilde{X} \overset{d}{=} X$.*

*Proof.* We handle the case where $\Gamma|_{[0,\pi)}$ is supported on exactly three points in $[\pi/2, \pi)$. (In the case where it is supported on one or two points, the proof is similar.) Namely we assume that $\Gamma(\{\theta_j\}) = p_j > 0$ $(j = 1, 2, 3)$ and $\Gamma([0,\pi)\backslash\{\theta_1, \theta_2, \theta_3\}) = 0$ for $\pi/2 \le \theta_1 < \theta_2 < \theta_3 < \pi$. Calculating the characteristic function of 1-dimensional projection of $X$, we have

$$\begin{aligned} &-\frac{1}{2}\log E\exp(i((\cos\theta)X_1 + (\sin\theta)X_2)) \\ &\quad= \int_{[0,\pi)} |\cos\theta\cos\xi + \sin\theta\sin\xi|^\alpha \Gamma(d\xi) = \int_{[0,\pi)} |\cos(\theta - \xi)|^\alpha \Gamma(d\xi). \end{aligned}$$

Suppose that $\tilde{X}$ is a 2-dimensional symmetric $\alpha$-stable random variable which satisfies the condition (C1) and let $\tilde{\Gamma}$ be the spectral measure of $\tilde{X}$. If $\theta = \theta_j - \pi/2$ $(j = 1, 2, 3)$, then we have $(\cos\theta)\tilde{X}_1 + (\sin\theta)\tilde{X}_2 \overset{d}{=} (\cos\theta)X_1 + (\sin\theta)X_2$. Therefore we have

$$\begin{aligned} (3.1)\qquad \int_{[0,\pi)} |\sin(\theta_j - \xi)|^\alpha \tilde{\Gamma}(d\xi) &= |\sin(\theta_j - \theta_1)|^\alpha p_1 \\ &+ |\sin(\theta_j - \theta_2)|^\alpha p_2 + |\sin(\theta_j - \theta_3)|^\alpha p_3, \quad j = 1, 2, 3. \end{aligned}$$

If $\theta = \theta_j - \pi/2 + \epsilon$ $(j = 1, 2, 3)$ for sufficiently small $\epsilon > 0$, then we also have $(\cos\theta)\tilde{X}_1 + (\sin\theta)\tilde{X}_2 \overset{d}{=} (\cos\theta)X_1 + (\sin\theta)X_2$. Therefore we have

$$\begin{aligned} (3.2)\qquad \int_{[0,\pi)} |\sin(\theta_j + \epsilon - \xi)|^\alpha \tilde{\Gamma}(d\xi) &= |\sin(\theta_j - \theta_1 + \epsilon)|^\alpha p_1 \\ &+ |\sin(\theta_j - \theta_2 + \epsilon)|^\alpha p_2 + |\sin(\theta_j - \theta_3 + \epsilon)|^\alpha p_3, \quad j = 1, 2, 3, \end{aligned}$$

for small $\epsilon > 0$. Adding the equations (3.1) and (3.2) respectively, we obtain

$$\int_{[0,\pi)} \varphi(\xi)\tilde{\Gamma}(d\xi) = \sum_{j=1}^{3}\sum_{k=1}^{3} |\sin(\theta_j - \theta_k)|^\alpha p_k \tag{3.3}$$

and

$$\int_{[0,\pi)} \varphi_\epsilon(\xi)\tilde{\Gamma}(d\xi) = \sum_{j=1}^{3}\sum_{k=1}^{3} a_j |\sin(\theta_j - \theta_k + \epsilon)|^\alpha p_k \tag{3.4}$$

for any $a_1$, $a_2$, $a_3$, where $\varphi(\xi) = |\sin(\theta_1 - \xi)|^\alpha + |\sin(\theta_2 - \xi)|^\alpha + |\sin(\theta_3 - \xi)|^\alpha$ and $\varphi_\epsilon(\xi) = a_1|\sin(\theta_1 + \epsilon - \xi)|^\alpha + a_2|\sin(\theta_2 + \epsilon - \xi)|^\alpha + a_3|\sin(\theta_3 + \epsilon - \xi)|^\alpha$. Now let us determine the coefficients $a_1$, $a_2$, $a_3$ so that $\varphi(\xi) = \varphi_\epsilon(\xi)$ at $\xi = \theta_1, \theta_2, \theta_3$. That is, determine $a_1$, $a_2$, $a_3$ satisfying the equation

$$\begin{pmatrix} |\sin\epsilon|^\alpha & |\sin(\theta_2 - \theta_1 + \epsilon)|^\alpha & |\sin(\theta_3 - \theta_1 + \epsilon)|^\alpha \\ |\sin(\theta_1 - \theta_2 + \epsilon)|^\alpha & |\sin\epsilon|^\alpha & |\sin(\theta_3 - \theta_2 + \epsilon)|^\alpha \\ |\sin(\theta_1 - \theta_3 + \epsilon)|^\alpha & |\sin(\theta_2 - \theta_3 + \epsilon)|^\alpha & |\sin\epsilon|^\alpha \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$$= \begin{pmatrix} |\sin(\theta_2 - \theta_1)|^\alpha + |\sin(\theta_3 - \theta_1)|^\alpha \\ |\sin(\theta_1 - \theta_2)|^\alpha + |\sin(\theta_3 - \theta_2)|^\alpha \\ |\sin(\theta_1 - \theta_3)|^\alpha + |\sin(\theta_2 - \theta_3)|^\alpha \end{pmatrix}.$$

Let $A_\epsilon$ ($\epsilon \geq 0$) be the determinant of the coefficient matrix in the left hand side. Then we see that $A_\epsilon$ is continuous in $\epsilon$ and $A_0 = 2|\sin(\theta_1 - \theta_2)|^\alpha|\sin(\theta_2 - \theta_3)|^\alpha|\sin(\theta_3 - \theta_1)|^\alpha > 0$, so that we have $A_\epsilon > 0$ for sufficiently small $\epsilon > 0$. Thus $a_1$, $a_2$, $a_3$ exist uniquely for any fixed small $\epsilon > 0$. For these $a_1$, $a_2$, $a_3$, we notice that the right hand sides of (3.3) and (3.4) are equal to each other and therefore

$$\int_{[0,\pi)} \varphi(\xi)\tilde{\Gamma}(d\xi) = \int_{[0,\pi)} \varphi_\epsilon(\xi)\tilde{\Gamma}(d\xi). \tag{3.5}$$

Next let us show that

$$a_j = 1 - k_j\epsilon^\alpha + O(\epsilon^{2\alpha\wedge 1}) \quad (j = 1, 2, 3) \tag{3.6}$$

for some $k_j > 0$ as $\epsilon \to 0$. We show it only for $a_1$, as $a_2$ and $a_3$ are treated similarly. By Cramer's rule, we have

$$a_1 = \frac{1}{A_\epsilon} \begin{vmatrix} |\sin(\theta_2-\theta_1)|^\alpha+|\sin(\theta_3-\theta_1)|^\alpha & |\sin(\theta_2-\theta_1+\epsilon)|^\alpha & |\sin(\theta_3-\theta_1+\epsilon)|^\alpha \\ |\sin(\theta_1-\theta_2)|^\alpha+|\sin(\theta_3-\theta_2)|^\alpha & |\sin\epsilon|^\alpha & |\sin(\theta_3-\theta_2+\epsilon)|^\alpha \\ |\sin(\theta_1-\theta_3)|^\alpha+|\sin(\theta_2-\theta_3)|^\alpha & |\sin(\theta_2-\theta_3+\epsilon)|^\alpha & |\sin\epsilon|^\alpha \end{vmatrix}.$$

Therefore we obtain

$$\begin{aligned} a_1 &= \frac{A_0 + c_1\epsilon^\alpha + O(\epsilon^{2\alpha\wedge 1})}{A_0 + b\epsilon^\alpha + O(\epsilon^{2\alpha\wedge 1})} = \frac{1 + \frac{c_1}{A_0}\epsilon^\alpha + O(\epsilon^{2\alpha\wedge 1})}{1 + \frac{b}{A_0}\epsilon^\alpha + O(\epsilon^{2\alpha\wedge 1})} \\ &= \Big(1 + \frac{c_1}{A_0}\epsilon^\alpha + O(\epsilon^{2\alpha\wedge 1})\Big)\Big(1 - \frac{b}{A_0}\epsilon^\alpha - O(\epsilon^{2\alpha\wedge 1})\Big) \\ &= 1 - \frac{b - c_1}{A_0}\epsilon^\alpha + O(\epsilon^{2\alpha\wedge 1}), \end{aligned}$$

where

$$\begin{aligned} b &= -\big\{(|\sin(\theta_2 - \theta_1)|^\alpha)^2 + (|\sin(\theta_3 - \theta_1)|^\alpha)^2 + (|\sin(\theta_3 - \theta_2)|^\alpha)^2\big\}, \\ c_1 &= -|\sin(\theta_2 - \theta_1)|^\alpha(|\sin(\theta_1 - \theta_2)|^\alpha + |\sin(\theta_3 - \theta_2)|^\alpha) \\ &\quad - |\sin(\theta_3 - \theta_1)|^\alpha(|\sin(\theta_1 - \theta_3)|^\alpha + |\sin(\theta_2 - \theta_3)|^\alpha). \end{aligned}$$

We see that

$$b - c_1 = |\sin(\theta_3 - \theta_2)|^\alpha(|\sin(\theta_2 - \theta_1)|^\alpha + |\sin(\theta_3 - \theta_1)|^\alpha - |\sin(\theta_3 - \theta_2)|^\alpha) > 0$$

and hence we obtain the formula (3.6) by setting $k_1 = (b - c_1)/A_0$.

Now let us show that $\varphi_\epsilon(\xi) < \varphi(\xi)$ at $\xi \neq \theta_1, \theta_2, \theta_3$ for sufficiently small $\epsilon$. We show it on $(\theta_1, \theta_2)$. (The proof is similar on $(\theta_2, \theta_3)$ or $(\theta_3, \pi) \cup [0, \theta_1)$.) We note that $\frac{d}{d\xi}|\sin(\theta_j - \xi)|^\alpha \to -\infty$ as $\xi \to \theta_j - 0$, $\frac{d}{d\xi}|\sin(\theta_j - \xi)|^\alpha \to \infty$ as $\xi \to \theta_j + 0$ and $\frac{d^2}{d\xi^2}|\sin(\theta_j - \xi)|^\alpha < 0$ at $\xi \neq \theta_j$ for $j = 1, 2, 3$. Therefore we have $\varphi_\epsilon(\xi) < \varphi(\xi)$ on $(\theta_1, \theta_1 + \epsilon]$ for sufficiently small $\epsilon$ by $\varphi_\epsilon(\theta_1) = \varphi(\theta_1)$. And we have

$$\begin{aligned} a_1|\sin(\theta_1 + \epsilon - \xi)|^\alpha &< |\sin(\theta_1 + \epsilon - \xi)|^\alpha < |\sin(\theta_1 - \xi)|^\alpha, \\ a_3|\sin(\theta_3 + \epsilon - \xi)|^\alpha &= (1 - k_3\epsilon^\alpha + O(\epsilon^{2\alpha\wedge 1}))(|\sin(\theta_3 - \xi)|^\alpha + O(\epsilon)) \\ &< |\sin(\theta_3 - \xi)|^\alpha \end{aligned}$$

on $(\theta_1 + \epsilon, \theta_2)$ for sufficiently small $\epsilon$. Let $\delta$ be $\delta = 2\alpha\epsilon^{1-\alpha}/k_2$. (We note that $\delta \to 0$ as $\epsilon \to 0$.) Then we obtain

$$|\sin(\theta_2 + \epsilon - \xi)|^\alpha < |\sin(\theta_2 - \xi)|^\alpha + \epsilon\alpha|\sin(\theta_2 - \xi)|^{\alpha-1}\cos(\theta_2 - \xi)$$

for $\theta_1+\epsilon<\xi\le\theta_2-\delta$ by mean value theorem and we have

$$\begin{aligned}
&a_2|\sin(\theta_2+\epsilon-\xi)|^\alpha\\
&\quad<(1-k_2\epsilon^\alpha+O(\epsilon^{2\alpha\wedge 1}))(|\sin(\theta_2-\xi)|^\alpha+\epsilon\alpha|\sin(\theta_2-\xi)|^{\alpha-1})\\
&\quad=|\sin(\theta_2-\xi)|^\alpha+\epsilon\alpha|\sin(\theta_2-\xi)|^{\alpha-1}-k_2\epsilon^\alpha|\sin(\theta_2-\xi)|^\alpha+O(\epsilon^{2\alpha\wedge 1})\\
&\quad\le|\sin(\theta_2-\xi)|^\alpha+\epsilon\alpha\delta^{\alpha-1}-k_2\epsilon^\alpha\delta^\alpha+O(\epsilon^{2\alpha\wedge 1})\\
&\quad=|\sin(\theta_2-\xi)|^\alpha-2^{\alpha-1}\alpha^\alpha k_2^{1-\alpha}\epsilon^{-\alpha^2+2\alpha}+O(\epsilon^{2\alpha\wedge 1})\\
&\quad<|\sin(\theta_2-\xi)|^\alpha
\end{aligned}$$

on $(\theta_1+\epsilon,\theta_2-\delta]$ for sufficiently small $\epsilon$. Therefore we find $\varphi_\epsilon(\xi)<\varphi(\xi)$ on $(\theta_1+\epsilon,\theta_2-\delta]$. And we see $\varphi_\epsilon(\xi)<\varphi(\xi)$ on $(\theta_2-\delta,\theta_2)$ for sufficiently small $\epsilon$ by $\varphi_\epsilon(\theta_2)=\varphi(\theta_2)$. Therefore we have $\varphi_\epsilon(\xi)<\varphi(\xi)$ on $(\theta_1,\theta_2)$.

Hence, by (3.5), we conclude that $\tilde\Gamma|_{[0,\pi)}$ is concentrated on $\{\theta_1,\theta_2,\theta_3\}$. We can easily verify that $\tilde\Gamma(\{\theta_j\})=p_j$ $(j=1,2,3)$. ∎

*Remark* 3.2. This proposition is also true if we replace $[\pi/2,\pi)$ in the statement with $[\pi/2,\pi)\cup\{0\}$.

*Remark* 3.3. We note that Proposition 3.1 fails when $\alpha=1$ and $\Gamma|_{[0,\pi)}$ is concentrated on $\{0,\pi/2\}$. For example, let us define $X=(X_1,X_2)$ and $\tilde X=(\tilde X_1,\tilde X_2)$ such that

$$\begin{aligned}
&E\exp(i(z_1X_1+z_2X_2))=\exp(-(|z_1|+|z_2|)),\\
&E\exp(i(z_1\tilde X_1+z_2\tilde X_2))=\exp\bigl(-(\tfrac13|z_1+2z_2|+\tfrac13|2z_1+z_2|)\bigr).
\end{aligned}$$

Then $z_1\tilde X_1+z_2\tilde X_2\overset{d}{=}z_1X_1+z_2X_2$ for $z_1z_2\ge 0$ although $\tilde X\overset{d}{\ne}X$.

CONJECTURE 3.4. *It seems that Proposition* 3.1 *remains true if we replace "at most three points" by "a finite number of points".*

Similarly to the extension of Theorem 2.1 to Proposition 2.2, we have the following proposition.

PROPOSITION 3.5. *Let $X=(X_1,X_2)$ be a 2-dimensional symmetric $\alpha$-stable random variable $(0<\alpha<1)$ whose spectral measure $\Gamma$ satisfies that $\Gamma|_{[0,\pi)}$ is concentrated on at most three points on $[\theta_1,\theta_2]$ $(0\le\theta_1<\theta_2<\pi)$. Suppose that a 2-dimensional symmetric $\alpha$-stable random variable $\tilde X=(\tilde X_1,\tilde X_2)$ satisfies the condition that*

(C3) $\quad z_1\tilde X_1+z_2\tilde X_2\overset{d}{=}z_1X_1+z_2X_2\quad$ *if* $\arg(z_1,z_2)\in[\theta_1+\pi/2,\theta_2+\pi/2]$.

*Then* $\tilde{X} \stackrel{d}{=} X$.

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