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# ENTIRE SOLUTIONS OF $(u_{z_1})^m + (u_{z_2})^n = e^g$

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**Abstract.** The paper is concerned with description of entire solutions of the partial differential equations $u_{z_1}^m + u_{z_2}^n = e^g$, where $m \geq 2$, $n \geq 2$ are integers and $g$ is a polynomial or an entire function in $\mathbf{C}^2$. Descriptions are given and complemented by various examples.

## §1. Introduction

This paper is concerned with description of entire solutions of the partial differential equations:

$$u_{z_1}^m + u_{z_2}^n = e^g \tag{1.1}$$

in $\mathbf{C}^2$, where $m \geq 2$, $n \geq 2$ are integers and $g$ is a polynomial or, more generally, an entire function in $\mathbf{C}^2$.

The partial differential equations (1.1) in real variable case arise in geometrical optics and wave propagation. For example, when $m = n = 2$, it is one of the main equations in geometric optics and describes the wave fronts of light in an inhomogeneous medium with a variable index of refraction $e^g$ (see e.g. [CH] and [G]). The partial differential equations (1.1) are clearly related to the functional equations

$$f_1^m + f_2^n = e^g.$$

The study of these equations goes back to Montel ([Mo]) and Cartan ([Ca]), who showed that entire solutions $f_1$ and $f_2$ in the complex plane (and thus in $\mathbf{C}^n$) must be both constant for the equation $f_1^m + f_2^m = 1$ when $m \geq 3$ and for the more general equation $f_1^m + f_2^n = 1$ when $\frac{1}{m} + \frac{1}{n} < 1$, respectively. This also follows from the fact that when $\frac{1}{m} + \frac{1}{n} < 1$ the surface $(z_1)^m + (z_2)^n = 1$ is a Kobayashi hyperbolic manifold, which implies that there are no nonconstant entire holomorphic mappings $(f_1, f_2)$

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from $\mathbf{C}$ (and thus from $\mathbf{C}^n$) to the surface, i.e., entire solutions $f_1$ and $f_2$ of the equation $f_1^m + f_2^n = 1$ must be both constant when $\frac{1}{m} + \frac{1}{n} < 1$ ([Sh, p. 360]); and thus entire solutions of the partial differential equation $(u_{z_1})^m + (u_{z_2})^n = 1$ must be linear in this case. In the "critical" case that $\frac{1}{m} + \frac{1}{n} = 1$, i.e., $m = n = 2$, the functional equation $f_1^2 + f_2^2 = 1$ obviously has non-constant entire solutions: $f_1 = \cos h$ and $f_2 = \sin h$ for any non-constant entire function $h$; however, an entire solution of the partial differential equation $(u_{z_1})^2 + (u_{z_2})^2 = 1$ is still linear ([K], [Sa]).

The above result for the equations (1.1) with $g \equiv 0$ is however no longer true when $g$ is a general polynomial or entire function. For example, when $g = 2z_2$, which is a linear function, the transcendental entire function $u = e^{z_2} \sin z_1$ is a solution of $(u_{z_1})^2 + (u_{z_2})^2 = e^{g(z)}$.

The purpose of this paper is to describe entire solutions for the partial differential equations (1.1) when $g$ is a general polynomial or an entire function in $\mathbf{C}^2$. Unlike the equations $u_{z_1}^m + u_{z_2}^n = 1$, in which case $g \equiv 0$, entire solutions of (1.1) are in general non-linear, and the results for the cases $\frac{1}{m} + \frac{1}{n} < 1$ and the case $m = n = 2$ are no longer the same; when $m = n = 2$, whether $g$ is transcendental or not also makes situations different; and for some functions $g$ the equations (1.1) even do not have any entire solutions.

We will state the detailed results for the case $m = n = 2$ in Section 2 and for the case $\frac{1}{m} + \frac{1}{n} < 1$ in Section 3, respectively. The main theorems will be proved in Section 4. In the proofs, we will employ Nevanlinna theory; we will assume that the reader is familiar with basics of the theory, and also basics of one and several complex variables and partial differential equations (see e.g. [BG], [J], [Kr], [St]). Note that neither the problem nor the solution to the problem is based on Nevanlinna theory; it may be possible to give elementary proofs to the results in the paper without using Nevanlinna theory.

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## §2. The case $m = n = 2$

The following theorem is concerned with the partial differential equations (1.1) when $\frac{1}{m} + \frac{1}{n} = 1$, or $m = n = 2$.

THEOREM 2.1. *Let $g$ be a polynomial in $\mathbf{C}^2$. Then $u$ is an entire solution of the partial differential equation*

$$(u_{z_1})^2 + (u_{z_2})^2 = e^g \tag{2.1}$$

*in $\mathbf{C}^2$ if and only if*

(i) $u = f(c_1 z_1 + c_2 z_2)$; *or*

(ii) $u = \phi_1(z_1 + iz_2) + \phi_2(z_1 - iz_2)$,

*where $f$ is an entire function in $\mathbf{C}$ satisfying that*

$$f'(c_1 z_1 + c_2 z_2) = \pm e^{\frac{1}{2}g(z)}, \tag{2.2}$$

*$c_1$ and $c_2$ are two constants satisfying that $c_1^2 + c_2^2 = 1$, and $\phi_1$ and $\phi_2$ are entire functions in $\mathbf{C}$ satisfying that*

$$\phi_1'(z_1 + iz_2)\phi_2'(z_1 - iz_2) = \frac{1}{4}e^{g(z)}. \tag{2.3}$$

*Remark* 2.2. We can express $f$ in (2.2), and $\phi_1$ and $\phi_2$ in (2.3) in terms of $g$. Restricting (2.2) to the complex line $(z_1, z_2) = (c_1\zeta, c_2\zeta)$ and then integrating it with respect to $\zeta$, we obtain that for $\zeta \in \mathbf{C}$,

$$f(\zeta) = \int \pm e^{\frac{1}{2}g(c_1\zeta, c_2\zeta)} d\zeta.$$

Similarly, considering the complex lines $(z_1, z_2) = \left(\frac{\zeta}{2}, \pm i\frac{\zeta}{2}\right)$, we obtain from (2.3) that

$$\phi_1(\zeta) = A \int e^{g(\frac{\zeta}{2}, -i\frac{\zeta}{2})} d\zeta,$$

and

$$\phi_2(\zeta) = B \int e^{g(\frac{\zeta}{2}, i\frac{\zeta}{2})} d\zeta,$$

where $A$ $\left(= \frac{1}{4\phi_2'(0)}\right)$ and $B$ $\left(= \frac{1}{4\phi_1'(0)}\right)$ are constants satisfying that $AB = \frac{1}{4}e^{-g(0,0)}$ by (2.3).

As we mentioned in the introduction, when $g$ is identically zero, an entire solution of (2.1) is linear. This result is clearly a simple consequence of Theorem 2.1, since when $g = 0$, the functions $f$, $\phi_1$ and $\phi_2$ are all linear and thus $u$ is linear by (i) and (ii) of Theorem 2.1.

Note that not every polynomial $g$ can satisfy the conditions (2.2) or (2.3) of Theorem 2.1. Therefore, Theorem 2.1 also implies a necessary and sufficient condition for the equation (2.1) to admit an entire solution. In fact, we have the following

COROLLARY 2.3. *Let $g$ be a polynomial in $\mathbf{C}^2$. Then the partial differential equation* (2.1) *admits an entire solution in $\mathbf{C}^2$ if and only if*

(i) $g(z_1, z_2) \equiv g(c_1(c_1z_1 + c_2z_2), c_2(c_1z_1 + c_2z_2)) + 2k\pi$ *for an integer $k$ and some constants $c_1$, $c_2$ with $c_1^2 + c_2^2 = 1$; or*

(ii) $g(z_1, z_2)+g(0, 0) \equiv g\big(\frac{1}{2}(z_1+iz_2), -\frac{i}{2}(z_1+iz_2)\big)+g\big(\frac{1}{2}(z_1-iz_2), \frac{i}{2}(z_1-iz_2)\big) + 2k\pi$ *for an integer $k$.*

*Proof.* If (i) holds, then we set $u = f(c_1z_1 + c_2z_2)$, where $f(\zeta)$ is the entire function defined in Remark 2.2. Then it is easy to verify that (2.2) holds. Thus $u$ is an entire solution of the equation (2.1) by Theorem 2.1(i). If (ii) holds, then we set $u = \phi_1(z_1 + iz_2) + \phi_2(z_1 - iz_2)$, where $\phi_1(\zeta)$ and $\phi_2(\zeta)$ are the entire functions defined in Remark 2.2. Then it is easy to verify that (2.3) holds and thus $u$ is an entire solution of the equation (2.1) by Theorem 2.1(ii).

Conversely, if the equation (2.1) has an entire solution $u$, then one of (i) and (ii) of Theorem 2.1 holds. If Theorem 2.1(i) holds, then by Remark 2.2 we have that $f'(\zeta) = \pm e^{\frac{1}{2}g(c_1\zeta, c_2\zeta)}$, which clearly implies the conclusion (i) of the corollary by virtue of (2.2). If Theorem 2.1(ii) holds, then by Remark 2.2 we have that $\phi_1'(\zeta) = Ae^{g(\frac{\zeta}{2}, -\frac{i\zeta}{2})}$ and $\phi_2'(\zeta) = Be^{g(\frac{\zeta}{2}, \frac{i\zeta}{2})}$, where $AB = \frac{1}{4}e^{-g(0,0)}$, which clearly implies the conclusion (ii) of the corollary by virtue of (2.3). $\square$

EXAMPLE 2.4. By way of illustration, we give three explicit examples for the case that the equation (2.1) does not have any entire solutions, and for the cases that the equation (2.1) has entire solutions of the forms (i) and (ii) in Theorem 2.1.

(1) Let $g(z_1, z_2) = z_1^2 + z_2^2$. Then it is easy to verify that neither of Corollary 2.3(i) and Corollary 2.3(ii) can hold. Thus, by Corollary 2.3, the equation $(u_{z_1})^2 + (u_{z_2})^2 = e^g$ does not have any entire solutions.

(2) Let $g = 2c^d(z_1 + z_2)^d$, where $d \geq 0$ is an integer and $c = \frac{1}{\sqrt{2}}$. Then $u = f(cz_1 + cz_2)$, where $f(w) = \int_0^w e^{w^d} dw$, is an entire solution of $(u_{z_1})^2 + (u_{z_2})^2 = e^g$, which is of the form (i) of Theorem 2.1, and $f$ satisfies (2.2).

(3) Let $g = 2z_2$. Then $u = e^{z_2} \sin z_1 = \frac{e^{z_2+iz_1} - e^{z_2-iz_1}}{2i}$ is an entire solution of $(u_{z_1})^2 + (u_{z_2})^2 = e^g$, which is of the form (ii) of Theorem 2.1 with $\phi_1(w) = -\frac{1}{2i}e^{-iw}$ and $\phi_2(w) = \frac{1}{2i}e^{iw}$, which satisfy (2.3).

A relation between the equations (1.1) when $g$ is linear and Monge-Ampère type equations was noted in [Sa, p. 373]; and it was claimed there that entire solutions $u$ of (1.1) when $g$ is linear has the form $u = e^{\frac{g}{2}}(c_1 z_1 + c_2 z_2) + c$, where $c_1$, $c_2$, $c \in \mathbf{C}$. This is however incorrect, as we see from the example in Examples 2.4(3).

The function $g$ in Theorem 2.1 is assumed to be a polynomial in $\mathbf{C}^2$. A natural question is whether or not $g$ can be generalized to be a transcendental entire function in Theorem 2.1. The answer is negative, as shown by the following

PROPOSITION 2.5. *There exists a transcendental entire function $g$ in $\mathbf{C}^2$ such that the partial differential equation $(u_{z_1})^2 + (u_{z_2})^2 = e^g$ has an entire solution $u$ with the following properties*:

(i) *$u \not\equiv f(c_1 z_1 + c_2 z_2)$ for any entire function $f$ in $\mathbf{C}$ and any constants $c_1$ and $c_2$; and*

(ii) *$u \not\equiv \phi_1(z_1 + iz_2) + \phi_2(z_1 - iz_2)$ for any entire functions $\phi_1$ and $\phi_2$ in $\mathbf{C}$.*

*Proof.* Consider the partial differential equation $(u_{z_1})^2 + (u_{z_2})^2 = e^g$, where $g(z) = iz_1 + z_2 + G(z)$ and $G(z) = iz_1 - z_2 + e^{iz_1+z_2}$. Let us show that the solution $u := ie^{\frac{1}{2}G}$ has the properties (i) and (ii) in the proposition.

The conclusion (i) is obvious when $c_1$ and $c_2$ are all zero. When at least one of $c_1$ and $c_2$ is not zero, consider the restriction of $z$ to the complex plane $c_1 z_1 + c_2 z_2 = 0$. On this plane, $f(c_1 z_1 + c_2 z_2) = f(0)$ is a constant for any entire function $f$ in $\mathbf{C}$, while $u$ is a non-constant entire function of one complex variable. Thus, (i) follows.

To show (ii), suppose, to the contrary, that

$$u = \phi_1(z_1 + iz_2) + \phi_2(z_1 - iz_2) \tag{2.4}$$

for some entire functions $\phi_1$ and $\phi_2$ in $\mathbf{C}$. We may assume that one of $\phi_1$ and $\phi_2$, say $\phi_1$, is not a constant. We consider two cases: $\phi_2$ is a constant and $\phi_2$ is not a constant. If $\phi_2$ is a constant, then on the complex plane $z_1 + iz_2 = 0$, $u$ is a non-constant entire function of one complex variable, while $\phi_1(z_1 + iz_2) + \phi_2(z_1 - iz_2)$ is a constant, a contradiction to (2.4). If

$\phi_2$ is not a constant, we can take two points $a$ and $b$ in the complex plane such that $\phi_2(a) \neq \phi_2(b)$. Consider (2.4) on the complex plane $z_1 - iz_2 = a$. Since $u$ never vanishes, we have that $\phi_1(2iz_2 + a) + \phi_2(a) \neq 0$ for any $z_2 \in \mathbf{C}$, which clearly implies that $-\phi_2(a)$ is a Picard-value of the function $\phi_1$ in $\mathbf{C}$. Next, consider (2.4) on the complex plane $z_1 - iz_2 = b$. We have that $\phi_1(2iz_2 + b) + \phi_2(b) \neq 0$ for any $z_2 \in \mathbf{C}$, which implies that $-\phi_2(b)$ is also a Picard-value of $\phi_1$. Therefore, the entire function $\phi_1$ has two distinct Picard-values, which is of course absurd, since a non-constant entire function has at most one Picard-value. The conclusion (ii) is thus proved. $\square$

## §3. The case $\frac{1}{m} + \frac{1}{n} < 1$

The following theorem deals with entire solutions of the partial differential equations (1.1) when $\frac{1}{m} + \frac{1}{n} < 1$.

THEOREM 3.1. *Let $g$ be an entire function in $\mathbf{C}^2$, and $m$ and $n$ integers satisfying that $\frac{1}{m} + \frac{1}{n} < 1$. Then $u$ is an entire solution of the partial differential equation*

$$(u_{z_1})^m + (u_{z_2})^n = e^g \tag{3.1}$$

*in $\mathbf{C}^2$ if and only if*

(i) *$u = f(c_1 z_1 + c_2 z_2)$ when $m = n$, where $f$ is an entire function in $\mathbf{C}$ satisfying that*

$$f'(c_1 z_1 + c_2 z_2) = c e^{\frac{g(z)}{m}}, \tag{3.2}$$

*and $c_1$, $c_2$, $c$ are constants satisfying that $c_1^m + c_2^n = 1$ and $c^m = 1$; and*

(ii) *$u = c_1 z_1 + c_2 z_2 + c_3$, or $u = F(z_1)$, or $u = G(z_2)$, when $m \neq n$, where $c_1$, $c_2$, $c_3$ are constants satisfying that $c_1^m + c_2^n = e^g$, and $F$ and $G$ are entire functions in $\mathbf{C}$ satisfying that*

$$(F'(z_1))^m = e^g, \quad (G'(z_2))^n = e^g. \tag{3.3}$$

Similar to Remark 2.2, explicit expressions of $f$ in (3.2), and $F$ and $G$ in (3.3) can be given in terms of $g$. And we also have the following corollary, whose proof is similar to the one of Corollary 2.3.

Corollary 3.2. *Let $g$ be an entire function in $\mathbf{C}^2$. Then the partial differential equation* (3.1) *admits an entire solution in $\mathbf{C}^2$ if and only if*

(i) $g(z_1, z_2) \equiv g(c_1^{m-1}(c_1z_1+c_2z_2), c_2^{n-1}(c_1z_1+c_2z_2))+2k\pi$ *for an integer $k$ and some constants $c_1$, $c_2$ with $c_1^m + c_2^n = 1$, when $m = n$; or*

(ii) *$g$ is an one variable function in $z_1$ or $z_2$, when $m \neq n$.*

Example 3.3. (1) As in Examples 2.4(1), the partial differential equation (3.1) with $g(z) = z_1^2 + z_2^2$ does not have any entire solutions.

(2) Let $g = me^{c(z_1+z_2)} + mc(z_1 + c_2)$, a transcendental entire function in $\mathbf{C}^2$, where $c$ is a constant satisfying that $2c^m = 1$. Then $u = e^{e^{c(z_1+z_2)}}$ is an entire solution of $(u_{z_1})^m + (u_{z_2})^m = e^g$, which is of the form (i) of Theorem 3.1. Examples of entire solutions for the forms in (ii) of Theorem 3.1 can be trivially given.

## §4. Proofs of Theorems 2.1 and 3.1

We now give the proofs of Theorem 2.1 and Theorem 3.1.

*Proof of Theorem* 2.1. The sufficiency is obvious. To prove the necessity, let $u$ be an entire solution of (2.1). Then

$$\Big(\frac{u_{z_1} + iu_{z_2}}{e^{\frac{g}{2}}}\Big)\Big(\frac{u_{z_1} - iu_{z_2}}{e^{\frac{g}{2}}}\Big) = 1.$$

Therefore, there exists an entire function $h$ in $\mathbf{C}^2$ such that $\frac{u_{z_1}+iu_{z_2}}{e^{\frac{g}{2}}} = e^{ih}$ and then that $\frac{u_{z_1}-iu_{z_2}}{e^{\frac{g}{2}}} = e^{-ih}$, from which it follows that

$$u_{z_1} = e^{\frac{g}{2}}\frac{e^{ih} + e^{-ih}}{2} = e^{\frac{g}{2}}\cos h, \quad u_{z_2} = e^{\frac{g}{2}}\frac{e^{ih} - e^{-ih}}{2i} = e^{\frac{g}{2}}\sin h \tag{4.1}$$

(cf. [Ma], [GI]). Using the fact that $u_{z_1z_2} = u_{z_2z_1}$, we obtain that

$$\Big(\frac{1}{2}g_{z_1} + h_{z_2}\Big)\sin h = \Big(\frac{1}{2}g_{z_2} - h_{z_1}\Big)\cos h. \tag{4.2}$$

We consider two different cases in the following.

*Case* (I): $\frac{1}{2}g_{z_1} + h_{z_2} \not\equiv 0$. In this case, by (4.2), we have that $\cos h \not\equiv 0$. Thus, (4.2) can be written as

$$\tan h = \frac{\frac{1}{2}g_{z_2} - h_{z_1}}{\frac{1}{2}g_{z_1} + h_{z_2}}. \tag{4.3}$$

We will employ Nevanlinna theory to show that $h$ is a constant. To this end, we first assert that

$$T(r,h) = o\{T(r,\tan h)\} + O(1), \tag{4.4}$$

where $T(r,F)$ denotes the Nevanlinna characteristic function of a meromorphic function $F$ in $\mathbf{C}^2$. Recall the following basic facts: A meromorphic function $F$ is transcendental if and only if $\lim_{r\to\infty} \frac{T(r,F)}{\log r} = +\infty$; and if $F$ is a non-constant polynomial, then $T(r,F) = d\log r + O(1)$ for some $d \neq 0$. Thus, the equality (4.4) is obvious if $h$ is a non-constant polynomial, since $\tan h$ is transcendental. If $h$ is a constant, (4.4) is trivial. When $h$ is transcendental, (4.4) follows from the following theorem in our paper [CLY]: If $F$ is a transcendental meromorphic function in $\mathbf{C}$ and $G$ is a transcendental entire function in $\mathbf{C}^2$, then $\lim_{r\to\infty} \frac{T(r,F(G))}{T(r,G)} = +\infty$. Hence, (2.4) always holds. On the other hand, we have that $T(r,F_{z_j}) = O\{T(r,F)\}$ for any meromorphic function $F$ outside a set of finite Lebesgue measure (see e.g. [St], [V]). We thus deduce, from (4.3) and (4.4), that outside a set of finite Lebesgue measure,

$$T(r,h) = o\{T(r,\tan h)\} + O(1) = o\{T(r,g) + T(r,h)\} + O(1)$$

and so that

$$T(r,h) = o\{T(r,g)\} + O(1) = o\{\log r\}, \tag{4.5}$$

since $g$ is a polynomial. Then (4.5) implies that $h$ must be a constant by the basic facts on the Nevanlinna characteristic mentioned above. Therefore, we then obtain, by (4.1), that

$$u_{z_1} = c_1 e^{\frac{g}{2}}, \quad u_{z_2} = c_2 e^{\frac{g}{2}}, \tag{4.6}$$

where $c_1 = \cos h$ and $c_2 = \sin h$ are constants, which clearly satisfy that

$$c_1^2 + c_2^2 = 1. \tag{4.7}$$

From (4.6), we obtain that

$$c_2 u_{z_1} - c_1 u_{z_2} = 0. \tag{4.8}$$

Solving this equation yields that $u(z_1,z_2) = f(c_1 z_1 + c_2 z_2)$ for some entire function $f$, which is of the form (i) in Theorem 2.1.

Substituting $f(c_1z_1 + c_2z_2)$ for $u$ in the original equation (2.1) yields that $(f')^2(c_1z_1 + c_2z_2) = e^{g(z)}$, in view of (4.7). That is, $f$ satisfies (2.2). Thus, we have proved the theorem in Case (I).

*Case* (II): $\frac{1}{2}g_{z_1} + h_{z_2} \equiv 0$. In this case, by (4.2), we have that either $\cos h = 0$ or

$$\frac{1}{2}g_{z_2} - h_{z_1} = 0. \tag{4.9}$$

If $\cos h = 0$, then $h$ is a constant and thus (4.6) holds, in which case we have already known that $u$ is of the form (i), as shown in Case (I). Therefore, in the following, we assume that (4.9) holds. By (4.1), we have that

$$u_{z_1z_1} = \frac{1}{2}e^{\frac{g}{2}}g_{z_1}\cos h - e^{\frac{g}{2}}h_{z_1}\sin h,$$

and

$$u_{z_2z_2} = \frac{1}{2}e^{\frac{g}{2}}g_{z_2}\sin h + e^{\frac{g}{2}}h_{z_2}\cos h.$$

Thus, we have that $u_{z_1z_1} + u_{z_2z_2} = 0$. Make the transformation

$$z_1 = \frac{w_1 + w_2}{2}, \quad z_2 = \frac{w_1 - w_2}{2i},$$

or

$$w_1 = z_1 + iz_2, \quad w_2 = z_1 - iz_2.$$

By abuse of notation, we still use $u$ to denote the function in $w_1$ and $w_2$ after the transformation. Then we have that $u_{w_1} = \frac{1}{2}u_{z_1} + \frac{1}{2i}u_{z_2}$ and thus that

$$\begin{aligned} u_{w_2w_1} &= \frac{1}{2}\Big(\frac{1}{2}u_{z_1z_1} - \frac{1}{2i}u_{z_2z_1}\Big) + \frac{1}{2i}\Big(\frac{1}{2}u_{z_1z_2} - \frac{1}{2i}u_{z_2z_2}\Big) \\ &= \frac{1}{4}(u_{z_1z_1} + u_{z_2z_2}) = 0. \end{aligned}$$

Integrating the equality $u_{w_2w_1} = 0$ with respect to $w_1$ and then $w_2$, we obtain that $u_{w_2} = \Phi(w_2)$ and then that

$$\begin{aligned} u &= \phi_1(w_1) + \int \Phi(w_2)\,dw_2 \\ &:= \phi_1(w_1) + \phi_2(w_2) = \phi_1(z_1 + iz_2) + \phi_2(z_1 - iz_2), \end{aligned} \tag{4.10}$$

where $\Phi$, $\phi_1$, $\phi_2$ are entire functions in the complex plane. Therefore, $u$ is of the form (ii) of Theorem 2.1. Substituting (4.10) for $u$ in the original equation (2.1), we obtain that

$$\begin{aligned} e^{g(z)} &= (u_{z_1})^2 + (u_{z_2})^2 \\ &= (\phi_1'(z_1 + iz_2) + \phi_2'(z_1 - iz_2))^2 + (i\phi_1'(z_1 + iz_2) - i\phi_2'(z_1 - iz_2))^2 \\ &= 4\phi_1'(z_1 + iz_2)\phi_2'(z_1 - iz_2), \end{aligned}$$

which is the desired equality (2.3). This completes the proof of Theorem 2.1. $\square$

*Proof of Theorem* 3.1. The sufficiency is clear. We only prove the necessity. Let $u$ be an entire solution of (3.1). Then,

$$(u_{z_1})^m + (u_{z_2})^n = e^g,$$

or

$$\left(\frac{u_{z_1}}{e^{\frac{g}{m}}}\right)^m + \left(\frac{u_{z_2}}{e^{\frac{g}{n}}}\right)^n = 1.$$

By Cartan's theorem mentioned in the introduction, we have that $\frac{u_{z_1}}{e^{\frac{g}{m}}} = c_1$, $\frac{u_{z_2}}{e^{\frac{g}{n}}} = c_2$, i.e.,

$$u_{z_1} = c_1 e^{\frac{g}{m}}, \quad u_{z_2} = c_2 e^{\frac{g}{n}}, \tag{4.11}$$

where $c_1$ and $c_2$ are two constants, which clearly satisfy that $c_1^m + c_2^n = 1$. We discuss two different cases in the following.

*Case* (1): $m = n$. In this case, we have by (4.11) that $c_2 u_{z_1} - c_1 u_{z_2} = 0$, which is the same partial differential equation (4.8) in the proof of Theorem 2.1, from which we know that $u = f(c_1 z_1 + c_2 z_2)$, where $f$ is an entire function in $\mathbf{C}$. The solution $u$ is of the form (i) in the theorem. Substituting it for $u$ in the equation (3.1), we see that $f$ satisfies (3.2). Hence, Theorem 3.1 holds in Case (1).

*Case* (2): $m \neq n$. By (4.11), we have that

$$\frac{c_1}{m} e^{\frac{g}{m}} g_{z_2} = \frac{c_2}{n} e^{\frac{g}{n}} g_{z_1} (= u_{z_1 z_2}). \tag{4.12}$$

If $c_1 = 0$, then by (4.11), $u_{z_1} = 0$. We then have that $u = G(z_2)$ for an entire function in $\mathbf{C}$. If $c_2 = 0$, then $u_{z_2} = 0$ and thus $u = F(z_1)$ for an entire function in $\mathbf{C}$. From the original equation (3.1), we have that

$(F'(z_1))^n = e^g$ and $(G'(z_2))^m = e^g$. These are the (second and third) forms in (ii) of the theorem. In the following, we assume that $c_1c_2 \neq 0$. By (4.12), if one of $g_{z_1}$ and $g_{z_2}$ is identically zero, then both of them are identically zero, which implies that $u$ is linear, i.e., $u = c_1z_1 + c_2z_2 + c_3$ for some constants $c_1$, $c_2$, $c_3$, which clearly satisfy that $c_1^m + c_2^n = e^g$, in view of the original equation (3.1). This is the first form in (ii) of the theorem. Next, we can assume that none of $g_{z_1}$ and $g_{z_2}$ is identically zero. Then we can write (4.12) into

$$e^{\frac{g}{m} - \frac{g}{n}} = \frac{mc_2}{c_1 n} \frac{g_{z_1}}{g_{z_2}}. \tag{4.13}$$

We claim that $g$ must be a constant. Otherwise we can use our theorem in [CLY] mentioned in the proof of Theorem 3.1 again to deduce that, by virtue of (4.13), outside a set of finite Lebesgue measure,

$$T(r,g) = o\{T(r, e^{\frac{g}{m} - \frac{g}{n}})\} = o\Big\{ \frac{mc_2}{c_1 n} \frac{g_{z_1}}{g_{z_2}} \Big\} = o\{T(r,g)\},$$

which is impossible. Now that $g$ is a constant, by (4.11) we obtain that both $u_{z_1}$ and $u_{z_2}$ are constant, which implies that $u$ is linear and thus has the first form in (ii) of the theorem. This completes the proof of Theorem 3.1. ∎

## References


[BG] C. A. Berenstein and R. Gay, Complex Variables, Springer-Verlag, New York, 1991.

[Ca] H. Cartan, *Sur les zeros des combinaisons linéaires de p functions holomorphes données*, Mathematica (Cluf), **7** (1933), 5–31.

[CH] R. Courant and D. Hilbert, Methods of Mathematical Physics, Vol. II, partial differential equations, Interscience, New York, 1962.

[CLY] D. C. Chang, B. Q. Li and C. C. Yang, *On composition of meromorphic functions in several complex variables*, Forum Math., **7** (1995), 77–94.

[G] P. R. Garabedian, Partial Differential Equations, New York, Wiley, 1964.

[GI] V. Ganapathy Iyer, *On certain functional equations*, J. Indian Math. Soc., **3** (1939), 312–315.

[J] F. John, Partial Differential Equations, Springer-Verlag, New York, 1982.

[K] D. Khavinson, *A note on entire solutions of the eiconal equation*, Amer. Math. Mon., **102** (1995), 159–161.

[Kr] S. Krantz, Function Theory of Several Complex Variables, John Wiley & Sons, New York, 1982.

[Ma] J. Markushevich, Entire Functions, Amer. Elsevier Pub. Co., New York, 1966.

[Mo] P. Montel, Leçons sur les families normales de functions analytique et leurs applications, Gauthier-Villars, Paris, 1927.

[Sa] E. G. Saleeby, *Entire and meromorphic solutions of Fermat type partial differential equations*, Analysis, **19** (1999), 369–376.

[Sh] B. V. Shabat, Introduction to Complex Analysis, part II, Functions of several variables, Translation Mathematical Monographs, Vol. 110, American Mathematical Society, Providence, RI, 1992.

[St] W. Stoll, Introduction to the Value Distribution Theory of Meromorphic Functions, Springer-Verlag, New York, 1982.

[V] A. Vitter, *The lemma of the logarithmic derivative in several complex variables*, Duke Math. J., **44** (1977), 89–104.

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