PLANAR COVERINGS OF CLOSED RIEMANN SURFACES

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To Professor KIYOSHI NOSHIRO on the occasion of his 60th birthday

Several years ago, K. Oikawa, my colleague, investigated the properties of Schottky coverings of closed Riemann surfaces, leaving an interesting problem as open [3]:

Does exist a Schottky covering between the basic surface and a given planar covering?

A principal aim of this paper is to give an affirmative answer to the above problem.

In spite of the purely topological character of the problem, we must use some analytic means, namely the properties of Fuchsian group as a group of cover transformations of the universal covering of the closed surface. These are discussed in § 3. On the other hand, in § 2 will be treated a combinatorial topological problem. Results in both paragraphs will be used in § 4 to prove the main theorem 2.

The author must pay his regard to Oikawa, the proposer of the problem, and express his warmest thanks for his friends in Universidad Central de Venezuela.

§ 1. Introduction

Let W be a closed Riemann surface of genus $g>1$ and F be the fundamental group of W.

We shall consider a covering $\tilde{W}$ of W, which is normal in the sense of Ahlfors-Sario [1], namely, *unverzweigt* and *unbegrenzt fortsetzbar* in Weyl's sense [4], and possessing a normal subgroup $G \subset F$ as its fundamental group. In this way, a normal subgroup G of F and a normal covering $\tilde{W}$ of W correspond one-to-one; we shall represent the relation as follows:

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$$\tilde{W}=\tilde{W}(G), \quad G=G(\tilde{W}).$$

If another covering $\tilde{W}_1$ of W is at the same time a covering of $\tilde{W}$, $\tilde{W}_1$ is called stronger than $\tilde{W}$, and is denoted by

$$\tilde{W}_1 \geq \tilde{W}.$$

This relation is equivalent to

$$G(\tilde{W}_1) \subset G(\tilde{W}).$$

The strongest covering is the universal covering, which may be considered as the upper half plane:

$$U=\{z=x+iy\ ;\ y>0\}.$$

We shall use the letters to represent the projections as follows:
$w=p(z)$ is the projection from U onto W;
$\tilde{w}=\tilde{p}(z)$ is that from U onto $\tilde{W}$;
$w=\pi(\tilde{w})$ is that from $\tilde{W}$ onto W.

The totality of cover transformations of U w.r.t. W forms a Fuchsian group Φ, isomorphic to F. We shall denote by Γ the normal subgroup of Φ corresponding to $G \subset F$; Γ is the group of cover transformations of U w.r.t. $\tilde{W}=\tilde{W}(G)$.

When $\tilde{W}$ is a domain in the complex plane, $\tilde{W}$ is called a planar covering; in this case, let us call the corresponding groups G and Γ also "*planar*".

Let A be a subset of F or Φ. We shall denote the smallest normal subgroup containing A by the symbol $[A]$.

A base of F

$$\{a_1, b_1, \ldots, a_g, b_g\}$$

is canonical when they have only one relation

$$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1 \text{ (1 is the identity)}.$$

Following Ahlfors-Sario, the *Schottky covering* of W is defined as a covering $\tilde{W}=\tilde{W}(G)$, where

$$G=[a_1, \ldots, a_g]$$

for some canonical base of F; in this case, we shall call G and Γ also *of Schottky*.

A Schottky covering is planar [1], [3]. Moreover, Oikawa proved the following exactly [3]:

THEOREM. *A Schottky covering has no planar coverings which are strictly weaker than itself.*

We shall prove that every planar covering is a covering of some Schottky covering.

Then we shall be able to characterize a Schottky covering as a *minimal normal planar covering*, free from bases.

§ 2. A theorem on Schottky Covering

Let

$$c_1, c_2, \dots, c_h$$

be a sequence of Jordan curves on the closed surface W such that:

(A) As elements of the fundamental group F,

$$c_i \notin [c_0 = 1, c_1, \dots, c_{i-1}] \qquad (i = 1, 2, \dots, h);$$

(B) As point sets in W,

$$c_i \cap c_j = \phi \quad \text{if } i \neq j.$$

We shall prove the following:

LEMMA 1. *If a set of Jordan curves* $\{c_i\}$ *satisfies the conditions* (A), (B), *then*

$$[c_1, \dots, c_h]$$

is contained in some Schottky group.

Proof. The curve c_1 may be a non-dividing cycle or a dividing cycle; then, cutting along c_1 we obtain a bordered surface W^1 in the former case, or two bordered surfaces W^2 and W^3 in the latter case.

The curve c_2 is contained in one of W^1, W^2, W^3, say W^i; cutting along c_2 we obtain a surface W^{i1} or two surfaces W^{i2} and W^{i3}.

Continuing this process successively, we obtain at last a finite number of bordered surfaces

$$W_1, W_2, \dots, W_N.$$

Every curve c_i is divided into two "banks" c_i' and c_i''. When c_i' and c_i'' belong to a same surface W_n, we rewrite

$$c_i' = e_i, \ c_i'' = e_i^{-1};$$

when c_i' and c_i'' belong to different surfaces W_m and W_n respectively, we rewrite

$$c_i' = d_i,\ \ c_i'' = d_i^{-1}.$$

Now suppose W_1 contains the borders

$$e_1,\ e_1^{-1},\ \ldots,\ e_p,\ e_p^{-1};\ d_1,\ \ldots,\ d_q,$$

in which the indices and the directions are suitably changed. Then we can get a canonical polygon P_1 of W_1 whose boundary is of the form [1]:

$$\begin{aligned}\partial P_1 = {} & a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_r b_r a_r^{-1} b_r^{-1} \\ & \times f_1 e_1 f_1^{-1} g_1 e_1^{-1} g_1^{-1} \cdots f_p e_p f_p^{-1} g_p e_p^{-1} g_p^{-1} \\ & \times h_1 d_1 h_1^{-1} \cdots h_q d_q h_q^{-1}.\end{aligned}$$

For simplicity we shall write the above product as follows:

$$\Pi\, aba^{-1}b^{-1} \cdot \Pi\, fef^{-1}ge^{-1}g^{-1} \cdot \Pi\, hdh^{-1}.$$

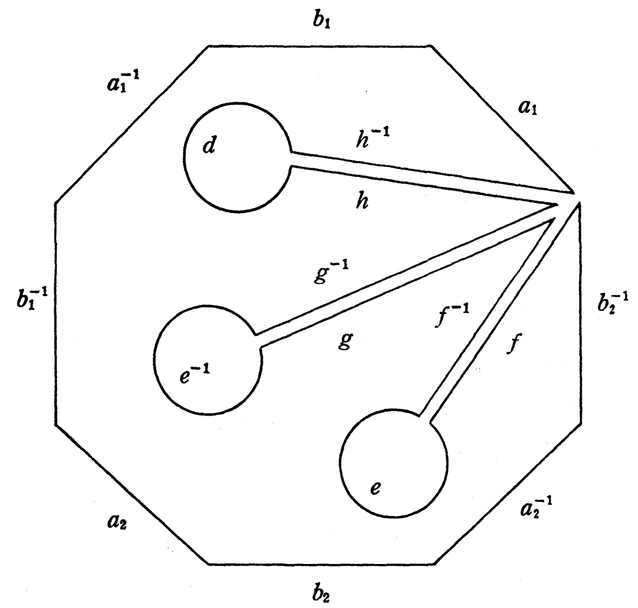

FIG. 1

We can change freely the order of the terms fef^{-1}, $ge^{-1}g^{-1}$, hdh^{-1}, which correspond to the borders of W_1, using the elementary deformations. Also can we take the same point of W_1 as the final point of g_i and f_i for each i; namely $g_i f_i^{-1}$ is a cycle on W_1.

Now put

$$a_{r+i} = f_i e_i f_i^{-1},\ \ b_{r+i} = g_i f_i^{-1}\ \ (i=1, \ldots, p),$$

then

$$a_{r+i} b_{r+i} a_{r+i}^{-1} b_{r+i}^{-1} = f_i e_i f_i^{-1} g_i e_i^{-1} g_i^{-1}.$$

Hence the boundary of P_1 can be written as follows:

$$\partial P_1 = \Pi\, aba^{-1}b^{-1} \cdot \Pi\, hdh^{-1},$$

and it is evident

$$[e_1, \ldots, e_p, d_1, \ldots, d_q] \subset [a_1, \ldots, a_{r+p}, d_1, \ldots, d_q].$$

In the same way, every W_n can be represented by the canonical polygon P_n whose boundary is similar as ∂P_1.

Now we suppose

$$d_1 \subset \partial P_1,\ \ d_1^{-1} \subset \partial P_2.$$

Let us denote

$$\partial P_2 = \prod_{i=1}^{s} a_i' b_i' a_i'^{-1} b_i'^{-1} \cdot \prod_{j=1}^{t} h_j' d_j' h_j'^{-1}$$

and suppose that d_1^{-1} is the last border d_t' without loss of generality. Identifying d_1 on ∂P_1 and d_t' on ∂P_2, we obtain a new polygon $P_1 + P_2$, whose boundary is of the form

$$\begin{aligned}\partial(P_1 + P_2) = \Pi\, aba^{-1}b^{-1} \cdot h_1 h_t'^{-1} \cdot \Pi\, a'b'a'^{-1}b'^{-1} \\ \times \Pi\, h'd'h'^{-1} \cdot h_t' h_1^{-1} \cdot \Pi\, hdh^{-1}.\end{aligned}$$

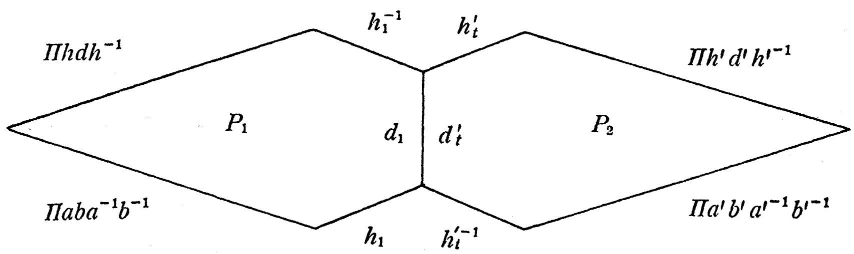

FIG. 2

We shall transform a, b, h, and d by h_1^{-1} and write

$$\bar{a} = h_1^{-1} a h_1,\ \ \bar{b} = h_1^{-1} b h_1,\ \ \bar{h} = h_1^{-1} h h_1,\ \ \bar{d} = h_1^{-1} d h_1.$$

In the similar way, let us transform a', b', h', d' by h_t'. Then we obtain the following form (with suitable change of notations):

$$\partial(P_1+P_2) = \Pi\, aba^{-1}b^{-1}\cdot \Pi\, hdh^{-1},$$

in which the common border d_1 is not contained. However, as easily seen, d_1 can be generated by a and d contained in $\partial(P_1+P_2)$. Hence

$$[e,\, d \subset \partial P_1;\ e,\, d \subset \partial P_2] \subset [a,\, d \subset \partial(P_1+P_2)]$$

Continuing this process successively, we can get at last a polygon

$$P = P_1 + P_2 + \cdots + P_N,$$

whose boundary is of the form

$$P = \Pi\, aba^{-1}b^{-1}.$$

Hence, P is nothing but a canonical polygon of W itself. Moreover we can verify inductively

$$[c_1, \ldots, c_h] \subset [a\ ;\ a \subset \partial P],$$

the right side of which is a Schottky covering group. q.e.d.

Let us note that the condition (A) is not used in the above proof. However, under this condition we can see that every bordered surface W_n (pasted along e_i) has a positive genus g_n; In fact, if the boundary ∂P_n contains no terms of the form $aba^{-1}b^{-1}$, then we can obtain a relation between $\{c_i\}$, which contradicts (A).

Since each of $\{e\}$ and $\{d\}$ does not exceed $g_1 + \cdots + g_N = g$, we conclude

$$h \leq 2g.$$

Consequently, there is no infinite sequence $\{c_n\}$ of Jordan curves satisfying (A) and (B).

Hence, when the infinite sequence $\{c_n\}$ satisfies (B) only, we can select a suitable finite subsequence $\{d_i\}_{i=1}^{h}$ which satisfies (A), (B) and

$$[d_1, \ldots, d_h] = [\{c_n\}].$$

Then we have the following:

THEOREM 1. *Let $\{c_n\}$ be a sequence of Jordan curves in the closed surface W such that*

$$c_i \cap c_j = \phi \quad (i \neq j).$$

Then there exists a Schottky group containing $[\{c_n\}]$.

§ 3. Fuchsian groups of closed Riemann surfaces

Let U be the upper half plane

$$\{z = x + iy \; ; \; y > 0\}$$

and $\mathfrak{G}$ be the group of all linear transformations which leave U invariant.

We shall introduce in U the Poincaré metric

$$ds = \frac{|dz|}{y}$$

which defines a non-Euclidean geometry in U; we use the words of elementary geometry in the sense of non-Euclid: for example, a "straight line" l is a inner arc of a circle orthogonal to ∂U. Let us denote a "directed segment" from z_1 to z_2 by $s(z_1, z_2)$, and the "distanse" between z_1 and z_2 by $\rho(z_1, z_2)$.

$\mathfrak{G}$ is the group of "motions" of plane U, leaving the distance ρ invariant.

Definition 1. Let φ be an arbitrary transformation of $\mathfrak{G}$. We shall define the norm $\|\varphi\|$ of φ as follows:

$$\|\varphi\| = \inf_{z \in U} \rho(z, \varphi(z)).$$

We can prove easily some properties of "norm" as follows:

Proposition 1.

$$\|\sigma\varphi\sigma^{-1}\| = \|\varphi\|$$

for all φ, $\sigma \in \mathfrak{G}$, *namely the norm* $\|\varphi\|$ *is invariant by the inner transformation.*

Proof. $\|\sigma\varphi\sigma^{-1}\| = \inf \rho(z, \sigma\varphi\rho^{-1}(z)) = \inf \rho(\sigma^{-1}(z), \varphi\sigma^{-1}(z)) = \inf \rho(z, \varphi(z)) = \|\varphi\|$.

Proposition 2. (i) *If* $\varphi \in \mathfrak{G}$ *is elliptic or parabolic,* $\|\varphi\| = 0$. (ii) *If* $\varphi \in \mathfrak{G}$ *is hyperbolic,*

$$\|\varphi\| = |\log K|,$$

where K *is the multiplier of* φ[1].

[1] See, for example, Ford [2].

Proof. (i) If φ is elliptic, one of the fix points a is in U. Hence

$$\|\varphi\| = \rho(a, \varphi(a)) = 0.$$

If φ is parabolic we can consider without loss of generality, the only fix point of φ is ∞ (Prop. 1). Then

$$\varphi(z) = z + c \ (c \in \mathbf{R}).$$

Uniting $z_0 = y_0$ and $\varphi(z_0)$ by a Euclidean segment C, we get

$$\|\varphi\| \leq \rho(y_0, y_0 + c) \leq \int_C \frac{|dx|}{y_0} = \frac{|c|}{y_0}.$$

Let $y_0 \to +\infty$, then we obtain $\|\varphi\| = 0$.

(ii) Let φ be hyperbolic. In this case two fix points of φ are on x-axis, which may be supposed 0 and ∞ without loss of generality. Hence

$$\varphi(z) = Kz,$$

where K is the multiplier of φ ($K > 0$, $K \neq 1$). Putting

$$z = x + iy = re^{i\theta},$$

we shall estimate Poincaré metric:

$$ds = \frac{|dz|}{y} = \frac{\sqrt{dr^2 + r^2 d\theta^2}}{r \sin\theta} \geq \frac{|dr|}{r}.$$

Suppose C is a segment from z to Kz,

$$\rho(z, \varphi(z)) = \int_C \frac{|dz|}{y} \geq \left| \int_r^{Kr} \frac{dr}{r} \right| = |\log K|,$$

where the equality holds if and only if $d\theta = 0$, $\sin\theta = 0$; namely, z is on the imaginary axis.

Definition 2. Let $\varphi \in \mathfrak{G}$ be hyperbolic. The "straight line" determined by two fix points of φ is called the *axis* of φ.

By the proof of prop. 2, the followings are evident:

Proposition 3. *For any hyperbolic transformation $\varphi \in \mathfrak{G}$,*

$$\|\varphi\| = \rho(z, \varphi(z))$$

if and only if z belongs to the axis of φ.

PROPOSITION 4. *Let φ be a hyperbolic transformation with axis l. Then $\sigma\varphi\sigma^{-1}$ is also hyperbolic whose axis is $\sigma(l)$.*

Now let us return to the closed Riemann surface W and its universal covering U.

The group Φ of cover transformations of U w.r.t. W is a special sort of Fuchsian groups, which is characterized by the following conditions:

1°) Φ is free from elliptic transformations;

2°) Φ has a compact fundamental region Δ[2].

We shall call Φ the Fuchsian group of W, for brevity.

PROPOSITION 5. *For the Fuchsian group Φ of a given closed surface W, exists a positive number r such that*

$$z_1 \neq z_2, \ \ \rho(z_1, z_2) < r \text{ implies } z_1 \not\equiv z_2(\Phi).$$

The proof is simple by the compactness of the fundamental region Δ.

PROPOSITION 6. *Every transformation of Φ is hyperbolic except $\varphi = 1$ (identity), and the set of norms*

$$\{\|\varphi\| \, ; \, \varphi \in \Phi\}$$

is discrete in **R**.

Proof. For any $\varphi \neq 1$ of Φ

$$\|\varphi\| \geq r > 0$$

by Prop. 5. Hence, Φ has no parabolic transformations.

If we take a point z on the axis of φ,

$$\|\varphi\| = \rho(z, \varphi(z)).$$

Then exists a suitable $\sigma \in \Phi$ such that $\sigma(z) = z_0$ belongs to the fixed fundamental region Δ. Put

$$\varphi_0 = \sigma\varphi\sigma^{-1},$$

and z_0 is a point on the axis of φ_0. Let us denote by Φ_0 the collection of all φ_0 whose axis pass through Δ. Then

[2] Some authors define the fundamental region as a open one. However we consider it as a closed region for convenience.

$$\{\|\varphi\| \; ; \; \varphi \in \Phi\} = \{\|\varphi_0\| \; ; \; \varphi_0 \in \Phi_0\}.$$

On the other hand, for any positive number R, the compact set

$$\{z \; ; \; \rho(z, \Delta) \leq R\}$$

is covered by a finite number of congruent figures $\varphi(\Delta)$. Hence the number of real values $\|\varphi_0\| \leq R$ is finite. q.e.d.

Let l_1, l_2 be axis of $\varphi_1, \varphi_2 \in \Phi$ respectively; there may be three cases as follows:

1°) $l_1 \cap l_2 = \phi$;

2°) $l_1 \cap l_2 = \{z_0\}$;

3°) $l_1 = l_2$.

The last case occurs if and only if φ_1 and φ_2 have common fix points. Hence we can verify easily:

PROPOSITION 7. *Let Γ be a subgroup of Φ. The totality of $\varphi \in \Gamma$, possessing a given fixed axis l, forms a free cyclic group generated by a suitable $\varphi_0 \in \Gamma$.*

§ 4. Planar coverings of closed Riemann surfaces

Let us use the same notations as in former paragraphs, and suppose $\tilde{W}$ is a normal planar covering of a closed surface W. U is also a universal covering of $\tilde{W}$ and the group Γ of cover transformations of U w.r.t. $\tilde{W}$ is a normal subgroup of Φ ; Γ is isomorphic to the fundamental group G of $\tilde{W}$.

Let C be any curve (closed or not) in U, whose terminal points are z_0, z_1. Consider the projection of C on W and $\tilde{W}$:

$$c = p(C) \text{ on } W;$$
$$\tilde{c} = \tilde{p}(C) \text{ on } \tilde{W}.$$

Evidently, c is closed if and only if

$$z_0 \equiv z_1 \qquad (\Phi);$$

$\tilde{c}$ is closed if and only if

$$z_0 \equiv z_1 \qquad (\Gamma).$$

Now let us put $\gamma_0 = 1 \in \Gamma$ and take a transformation γ_1 which has the minimum norm in $\Gamma - [\gamma_0]$. If $[\gamma_0, \gamma_1] \neq \Gamma$, we shall take γ_2 which has the minimum norm in $\Gamma - [\gamma_0, \gamma_1]$.

Continuing such processes, we obtain a finite or infinite sequence of transformations

$$\gamma_1, \gamma_2, \ldots, \gamma_n, \ldots.$$

Let us denote the totality of indices of γ_n by $N = \{1, 2, \ldots, n, \ldots\}$. Using Prop. 6 it is easily seen that γ_n has the following properties:

(I) $[\gamma_n\ ;\ n \in N] = \Gamma$.

(II) If $\varphi \in \Gamma$, $\|\varphi\| < \|\gamma_n\|$, then

$$\varphi \in [\gamma_1, \gamma_2, \ldots, \gamma_{n-1}].$$

We shall take a point z on the axis l_n of γ_n, and set

$$C_n = s(z, \gamma_n(z)),\ z \in l_n;$$
$$c_n = p(C_n);$$
$$\tilde{c}_n = \tilde{p}(C_n).$$

Let us remark that every c_n is invariant not only as a homotopy class of F but also as a point set in W, when the inicial point $z \in l_n$ is changed; the same is true for $\tilde{c}_n$.

We shall prove first

(III) *$\tilde{c}_n$ is a Jordan curve in $\tilde{W}$.*

Proof. If $\tilde{c}_n$ is not of Jordan in $\tilde{W}$, $\tilde{c}_n$ can be divided into two closed curves $\tilde{c}$ and $\tilde{c}'$ on $\tilde{W}$. Hence there exist three points z_1, z_2, z_3 on the axis l_n of γ_n in this order, such that

$$\tilde{p}(s(z_1, z_2)) = \tilde{c},$$
$$\tilde{p}(s(z_2, z_3)) = \tilde{c}'.$$

Namely there exist two transformations φ and ψ in Γ, such that

$$\varphi(z_1) = z_2,\ \psi(z_2) = z_3,$$

and

$$\|\varphi\| \leq \rho(z_1, z_2) < \rho(z_1, z_3) = \|\gamma_n\|,$$
$$\|\psi\| \leq \rho(z_2, z_3) < \rho(z_1, z_3) = \|\gamma_n\|.$$

According to (II)

$$\varphi, \psi \in [\gamma_0, \ldots, \gamma_{n-1}],$$

hence

$$\gamma_n = \psi\varphi \in [\gamma_0, \ldots, \gamma_{n-1}],$$

which is a contradiction. q.e.d.

(IV) *Every curve $\tilde{c}$ in $\tilde{W}$ whose projection is $\pi(\tilde{c}) = c_n$, is of Jordan in $\tilde{W}$.*

The proof is immediate from (III) and the normality of the covering $\tilde{W}$.

(V) *Let l_m, l_n be the axis of γ_m, γ_n respectively, and $l = \sigma(l_m)$, $l' = \tau(l_n)$, where σ, $\tau \in \Phi$. Then $l \cap l' = \phi$ or $l = l'$.*[3]

Proof. l and l' are the axis of $\gamma = \sigma\gamma_m\sigma^{-1}$ and $\gamma' = \tau\gamma_n\tau^{-1}$ respectively (Prop. 4), and $\|\gamma\| = \|\gamma_m\|$, $\|\gamma'\| = \|\gamma_n\|$.

Suppose that l and l' intersect at z_0.

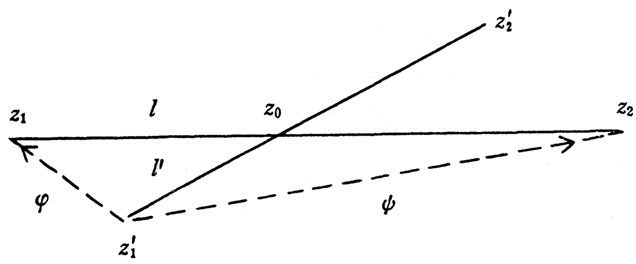

FIG. 3

Let us take (temporarily) arbitrary inicial points $z_1 \in l$, $z_1' \in l'$ and put $z_2 = \gamma(z_1)$, $z_2' = \gamma'(z_1')$. Since

$$\tilde{c} = \tilde{p}(s(z_1, z_2)), \ \tilde{c}' = \tilde{p}(s(z_1', z_2'))$$

are both Jordan curves on $\tilde{W}$ (IV), $\tilde{c}$ and $\tilde{c}'$ must have another common point $\tilde{w}_1$ than $\tilde{w}_0 = \tilde{p}(z_0)$, because of the planar character of $\tilde{W}$. Then we can select z_1, z_1' as inicial points such that

$$\tilde{p}(z_1) = \tilde{p}(z_2) = \tilde{p}(z_1') = \tilde{p}(z_2') = \tilde{w}_1,$$
$$z_0 \in s(z_1, z_2) \cap s(z_1', z_2').$$

It is clear that z_0 does not coincide with these terminal points z_1, z_2, z_1', z_2', since $\tilde{w}_0 \neq \tilde{w}_1$. Without loss of generality, we can suppose

$$\rho(z_1, z_0) \leq \rho(z_0, z_2),$$
$$\rho(z_1', z_0) \leq \rho(z_0, z_2'),$$

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[3] If $l = l'$, $m = n$ by Prop. 7.

and

$$\rho(z_1', z_0) \leq \rho(z_1, z_0) \leq \rho(z_0, z_2).$$

Then using the elementary geometry of non-Euclid, we get

$$\begin{aligned} \rho(z_1', z_1) &< \rho(z_1', z_0) + \rho(z_0, z_1) \\ &\leq \rho(z_0, z_2) + \rho(z_0, z_1) \\ &= \rho(z_1, z_2). \end{aligned}$$

Hence there exists a transformation $\varphi \in \Gamma$ such that

$$\begin{aligned} &\varphi(z_1') = z_1, \\ &\|\varphi\| \leq \rho(z_1', z_1) < \rho(z_1, z_2) = \|\gamma\| = \|\gamma_m\|. \end{aligned}$$

By the similar way we get a $\psi \in \Gamma$ such that

$$\psi(z_1') = z_2, \ \|\psi\| < \|\gamma_m\|.$$

Using (II), we conclude

$$\varphi, \psi \in [\gamma_0, \ldots, \gamma_{m-1}],$$

which implies

$$\gamma_m = \sigma^{-1}\gamma\sigma = \sigma^{-1}(\psi\varphi^{-1})\sigma \in [\gamma_0, \ldots, \gamma_{m-1}],$$

which is a contradiction. q.e.d.

The projection c_n of C_n on W is not necessarily of Jordan. However, in the subgroup of Φ consisting of all the transformations with axis l_n, there exists a generator δ_n (Prop. 7). Let

$$z \in l_n, \ D_n = s(z, \delta_n(z)), \ d_n = p(D_n).$$

It is evident that

$$c_n = d_n^{k_n} \qquad \text{for suitable } k_n.$$

(VI) *d_n is a Jordan curve in W.*

Proof. Suppose that d_n is not of Jordan in W, then by the same discussion as in the proof of (III), we can find three points z_1, z_2, z_3 on l_n in this order, such that

$$\varphi(z_1) = z_2, \ \psi(z_2) = z_3, \ \delta_n = \psi\varphi$$

for suitable $\varphi, \psi \in \Phi$. Put

$$l = l_n, \quad l' = \varphi(l_n),$$

then l' is the axis of $\delta' = \varphi\delta_n\varphi^{-1}$ (Prop. 4). If $l' = l$, namely $\varphi(l_n) = l_n$, φ has the common fix points with δ_n, which means that the axis of φ is l_n; however, this is impossible since

$$\|\varphi\| \leq \rho(z_1, z_2) < \|\delta_n\|,$$

Hence l' intersects l at z_2, which contradicts (V). q.e.d.

Thus we obtain the sequence of Jordan curves

$$\{d_n \; ; \; n \in N\},$$

each d_n of which is the projection of l_n as a point set in W, and generates c_n as a homotopy class of F.

Moreover,

$$d_m \cap d_n = \phi \text{ if } m \neq n,\text{[4)]}$$

namely, $\{d_n \; ; \; n \in N\}$ satisfies the condition (B) in §2.

Therefore, by Theorem 1, we get

THEOREM 2. *If $\tilde{W}$ is a normal planar covering of a closed Riemann surface W, there exists a Schottky covering S such that*

$$\tilde{W} \geq S > W.$$

§5. Additional remarks

Theorem 2 is applicable not only to the closed Riemann surface, but to every orientable finite surface, because the latter is always homeomorphic to some of the formers.

Moreover, we shall remark that the normality of the covering is essencial; we can make an example $\tilde{W}$ such that

(i) $\tilde{W}$ is a regular planar covering in the sense of Ahlfors-Sario;

(ii) there are no Schottky coverings $S \leq \tilde{W}$.

However, we shall not treat the problem here.

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4) See the footnote 3).

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