NOTE ON THE UNIQUENESS PROPRETY OF WEAK SOLUTIONS OF PARABOLIC EQUATIONS

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To Professor KIYOSHI NOSHIRO on the occasion of his 60th birthday

1. Aronson proved, in his paper [1], the existence and the uniqueness property of weak solutions of the initial boundary value problem for parabolic equations of second order with measurable coefficients. On the uniqueness of solutions of the Cauchy problem for such equations he also gave some interesting results in [2].

In this note we prove the uniqueness property of weak solutions of the initial boundary value problem for some equations of higher order by applying the argument used in [2].

2. We denote by x a point $(x_1, \ldots, x_n)$ in the n-dimensional Euclidean space R^n and by t a point on the real line $(-\infty, \infty)$. Let $\mathscr{D}$ be a bounded domain given in R^n and let $\overline{\mathscr{D}}$ be its closure. We denote by Ω the cylinder domain $\mathscr{D} \times (T', T'')$ in the $(n+1)$-dimensional Euclidean space $R^n \times (-\infty, \infty)$.

We introduce some function spaces.

The space $H_0^{s,2}(\mathscr{D})$ is the closure of $C_0^\infty(\mathscr{D})$ by the norm

$$\|\varphi\|_s = \Big(\int_{\mathscr{D}} \sum_{|\alpha| \leqq s} |D_x^\alpha \varphi|^2 dx\Big)^{1/2}.$$

The space $L^2[T', T''; H_0^{s,2}(\mathscr{D})]$ consists of all functions u with the following properties: i) u is measurable in Ω, ii) for almost all $t \in [T', T'']$, the function $u(x, t)$ in x belongs to $H_0^{s,2}(\mathscr{D})$ and iii) the norm $\|u\|_s$ as a function of t belongs to $L^2([T', T''])$.

We have the definition of the space $H^{1,2}[T', T''; H_0^{s,2}(\mathscr{D})]$ if, in the above definition of $L^2[T', T''; H_0^{s,2}(\mathscr{D})]$, the condition iii) is replaced by iii)′: the norm $\|u\|_s$ as a function of t belongs to $H^{1,2}([T', T''])$, which is the closure of $C^\infty((T', T''))$ by the norm

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$$|\varphi|_1=\left(\int_{T'}^{T''}\left[\varphi^2+\left(\frac{d\varphi}{dt}\right)^2\right]dt\right)^{1/2}<\infty.$$

We denote by $L^\infty[T', T''; L^2(\mathscr{D})]$ the space consisting of all functions u with the following property: u is measurable in Ω and, for almost all $t\in[T', T'']$, the function $u(x, t)$ of x belongs to $L^2(\mathscr{D})$ and finally the norm $\|u\|=\|u\|_0$ as a function in t is essentially bounded in $[T', T'']$.

3. Consider a uniformly parabolic partial differential operator L defined in Ω as follows:

$$Lu=\frac{\partial u}{\partial t}+(-1)^s\sum_{\substack{|\alpha|\leqq s\\|\beta|\leqq s}}D_x^\alpha(a_{\alpha\beta}D_x^\alpha u).$$

Here $\alpha=(\alpha_1, \ldots, \alpha_n)$ is a multi-index of non-negative integers with length $|\alpha|=\alpha_1+\cdots+\alpha_n$ and

$$D_x^\alpha=\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}$$

and further all the coefficients $a_{\alpha\beta}=a_{\alpha\beta}(x, t)$ are bounded measurable real functions in Ω such that, at almost every point in Ω,

$$\sum_{|\alpha|=|\beta|=s}a_{\alpha\beta}\xi^\alpha\xi^\beta\geqq c\left(\sum_{i=1}^n\xi_i^2\right)^s$$

for some positive constant c, where $\xi=(\xi_1, \ldots, \xi_n)$ is an arbitrary real vector and $\xi^\alpha=\xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}$.

Let f be a function with variables $x_1, \ldots, x_n, t$ and $D_x^\alpha u(0\leqq|\alpha|\leqq s)$ such that, for $u\in L^2[T', T''; H_0^{s,2}(\mathscr{D})]\cap L^\infty[T', T''; L^2(\mathscr{D})]$, f belongs to $L^2(\Omega)$ as a function in Ω.

If the function $u\in L^2[T', T''; H_0^{s,2}(\mathscr{D})]\cap L^\infty[T', T''; L^2(\mathscr{D})]$ satisfies

$$\int_{T_1}^{T_2}\int_{\mathscr{D}}\Big(-u\frac{\partial\psi}{\partial t}+(-1)^s\sum_{\substack{|\alpha|\leqq s\\|\beta|\leqq s}}(-1)^{|\beta|}a_{\alpha\beta}D_x^\alpha uD_x^\beta\psi\Big)dxdt$$
$$=\int_{\mathscr{D}}u(x, T_1)\psi(x, T_1)dx-\int_{\mathscr{D}}u(x, T_2)\psi(x, T_2)dx+\int_{T_1}^{T_2}\int_{\mathscr{D}}f\psi dxdt \tag{1}$$

for any T_1 and T_2 in $[T', T'']$ and for any $\psi\in H^{1,2}[T', T''; H_0^{s,2}(\mathscr{D})]$, then u is said to be a weak solution of the equation $Lu=f$ in Ω with the boundary value zero.

4. Now we can prove the following theorem.

THEOREM. *Let the operator* L *and the function* f *be those stated above. Suppose that all the coefficients* $a_{\alpha\beta}(|\alpha|=|\beta|=s)$ *of* L *are continuous in* Ω *and that there is a constaut* c_0 *such that in* Ω

$$f^2 \leqq c_0 \sum_{|\alpha| \leqq s} |D_x^\alpha u|^2.$$

If the weak solution u *of* $Lu=f$ *in* Ω *with bounbary value zero satisfies* $u(x, T')=0$ *in* $\mathscr{D}$, *then* u *vanishes almost everywhere in* Ω.

Proof. Let $j_\varepsilon(t)$ be an infinitely differentiable even function of a real variable t with support $|t| \leqq \varepsilon (\varepsilon > 0)$ such that

$$\int_{-\infty}^{\infty} j_\varepsilon(t) dt = 1.$$

For any T fixed in $[T', T'']$, we put

$$u^T(x, t) = \begin{cases} u(x, t), & \text{if } t \in [T', T] \\ 0 & \text{, if otherwise,} \end{cases}$$

and

$$u_\varepsilon(x, t) = \int_{-\infty}^{\infty} j_\varepsilon(t-\tau) u^T(x, \tau) d\tau.$$

It is obvious that

$$u_\varepsilon(x, t) = \int_{T'}^{T} j_\varepsilon(t-\tau) u(x, \tau) d\tau.$$

We can easily verified that $u_\varepsilon(x, t)$ belongs to $H^{1,2}[T', T''; H_0^{s,2}(\mathscr{D})]$ and that, for t fixed, $D_x^\alpha u_\varepsilon(x, t)$ tends to $D_x^\alpha u^T(x, t)$ in $L^2(\mathscr{D})$ as ε tends to zero. Hence, if u is a weak solution of $Lu=f$ in Ω with boundary value zero and if $u(x, T')=0$ in $\mathscr{D}$, then from (1) we have

$$\text{(2)} \qquad \int_{T'}^{T}\int_{\mathscr{D}} \Big(-u\frac{\partial u_\varepsilon}{\partial t} + (-1)^s \sum_{\substack{|\alpha| \leqq s \\ |\beta| \leqq s}} (-1)^{|\beta|} a_{\alpha\beta} D_x^\alpha u D_x^\beta u_\varepsilon \Big) dxdt$$

$$= -\int_{\mathscr{D}} u(x, T) u_\varepsilon(x, T)\, dx + \int_{T'}^{T}\int_{\mathscr{D}} f u_\varepsilon dxdt.$$

For the first term in the left hand side of this, we have

$$-\int_{T'}^{T}\int_{\mathscr{D}} u\frac{\partial u_\varepsilon}{\partial t} dxdt = -\int_{T'}^{T}\int_{\mathscr{D}}\int_{T'}^{T} u(x, t) j_\varepsilon'(t-\tau) u(x, \tau) d\tau dxdt.$$

Since $j_\varepsilon(t)$ is an even function, the derivative $j_\varepsilon'(t)$ is odd. Therefore the right hand side of the above equality equals zero.

On the second term of the left hand side of (2), we get the following from Gårding's inequality:

$$\lim_{\varepsilon\to 0}\int_{T'}^{T}\int_{\mathscr{D}}(-1)^s\sum_{\substack{|\alpha|\leq s\\|\beta|\leq s}}(-1)^{|\beta|}a_{\alpha\beta}D_x^\alpha u D_x^\beta u_\varepsilon dxdt$$

$$=(-1)^s\int_{T'}^{T}\int_{\mathscr{D}}\sum_{\substack{|\alpha|\leq s\\|\beta|\leq s}}(-1)^{|\beta|}a_{\alpha\beta}D_x^\alpha u D_x^\beta u dxdt$$

$$\geqq k_1\int_{T'}^{T}\int_{\mathscr{D}}\sum_{|\alpha|\leqq s}|D_x^\alpha u|^2 dxdt-k_2\int_{T'}^{T}\int_{\mathscr{D}}u^2 dxdt,$$

where k_1 and k_2 are positive constants depending only on L.

As Aronson showed in [2], the first term of the right hand side in (2) tends to

$$-\frac{1}{2}\int_{\mathscr{D}}u^2(x, T)dx,$$

as ε tends to zero.

Finally we see that

$$\lim_{\varepsilon\to 0}\int_{T'}^{T}\int_{\mathscr{D}}fu_\varepsilon dxdt=\int_{T'}^{T}\int_{\mathscr{D}}fudxdt$$

and that, for any $\delta(>0)$,

$$\int_{T'}^{T}\int_{\mathscr{D}}fudxdt\leqq\left(\int_{T'}^{T}\int_{\mathscr{D}}f^2dxdt\right)^{1/2}\left(\int_{T'}^{T}\int_{\mathscr{D}}u^2dxdt\right)^{1/2}$$

$$\leqq\sqrt{c_0}\left(\int_{T'}^{T}\int_{\mathscr{D}}\sum_{|\alpha|\leqq s}|D_x^\alpha u|^2dxdt\right)^{1/2}\left(\int_{T'}^{T}\int_{\mathscr{D}}u^2dxdt\right)^{1/2}$$

$$\leqq\frac{\delta\sqrt{c_0}}{2}\int_{T'}^{T}\int_{\mathscr{D}}\sum_{|\alpha|\leqq s}|D_x^\alpha u|^2dxdt+\frac{\sqrt{c_0}}{2\delta}\int_{T'}^{T}\int_{\mathscr{D}}u^2dxdt.$$

This follows from the Schwarz inequality and the Cauchy inequality.

Therefore, letting ε tend to zero in (2), we have the following:

$$k_1\int_{T'}^{T}\int_{\mathscr{D}}\sum_{|\alpha|\leqq s}|D_x^\alpha u|^2dxdt-k_2\int_{T'}^{T}\int_{\mathscr{D}}u^2dxdt$$

$$\leqq-\frac{1}{2}\int_{\mathscr{D}}u^2(x, T)dx+\frac{\delta\sqrt{c_0}}{2}\int_{T'}^{T}\int_{\mathscr{D}}\sum_{|\alpha|\leqq s}|D_x^\alpha u|^2dxdt+\frac{\sqrt{c_0}}{2\delta}\int_{T'}^{T}\int_{\mathscr{D}}u^2dxdt$$

for any $T\in[T', T'']$ and for any positive δ. If we choose a positive δ so small that $k_1-\frac{\delta\sqrt{c_0}}{2}>0$, then it follows that

(3) $$\frac{1}{2}\int_{\mathscr{D}} u^2(x, T)\,dx \leqq \left(k_2 + \frac{\sqrt{c_0}}{2\delta}\right)\int_{T'}^{T}\int_{\mathscr{D}} u^2 dxdt$$

for any $T \in (T', T'']$. Take $T_0 \in (T', T'']$ such that

(4) $$(T_0 - T')\left(k_2 + \frac{\sqrt{c_0}}{2\delta}\right) < \frac{1}{2}.$$

From (3), we get

$$\frac{1}{2}\int_{\mathscr{D}} u^2(x, t)\,dx \leqq \left(k_2 + \frac{\sqrt{c_0}}{2\delta}\right)\int_{T'}^{T_0}\int_{\mathscr{D}} u^2 dxdt$$

for any $t \in (T', T_0]$. Integrating both sides from T' to T_0 with respect to t, we obtain

$$\frac{1}{2}\int_{T'}^{T_0}\int_{\mathscr{D}} u^2 dxdt \leqq (T_0 - T')\left(k_2 + \frac{\sqrt{c_0}}{2\delta}\right)\int_{T'}^{T_0}\int_{\mathscr{D}} u^2 dxdt.$$

The condition (4) and the above inequality imply

$$\int_{T'}^{T_0}\int_{\mathscr{D}} u^2 dxdt = 0,$$

whence u vanishes almost everywhere in $\mathscr{D} \times (T', T_0]$. From this we see easily that u vanishes almost everywhere in Ω. Thus we have the theorem.

Remark. In the case $s = 1$, we need not to assume the continuity of $a_{\alpha\beta}(|\alpha| = |\beta| = 1)$ in $\bar{\Omega}$, since we can get (3) without use of Gårding's inequality. So, in this case, our theorem reduces to Aronson's result, a part of Theorem 3 in [1].

References

[1] D. G. Aronson, On the Green's function for second order parabolic differential equations with discontinuous coefficients, Bull. Amer. Math. Soc., **69** (1963), 841-847.

[2] D. G. Aronson, Uniqueness of positive weak solutions of second order parabolic equations, Ann. Polonici Math., **16** (1965), 285-303.

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