SIMPLE ALGEBRAS OVER A COMMUTATIVE RING

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In memory of Professor TADASI NAKAYAMA

In a previous paper [4], we studied a class of algebras over a commutative ring R which we called semisimple algebras. Here we shall study simple algebras.

In [4], we defined simple algebras over a Prüfer domain R as those semisimple algebras whose rational hulls are simple. A simple algebra A in this sense is directly indecomposable (as an algebra) and any (one-sided) ideal is A-faithful. In agreement with this, we defined simple algebras over a general commutative ring as semisimple algebras A admitting A-faithful and (A, R)-irreducible modules M (definition below), in the Symposium 1964 in Sapporo. But, in studying such simple algebras, we need very often that the natural monomorphism $A \to \mathrm{Hom}_R(M, M)$ admits an R-splitting. In this paper, we include this property in the definition of simplicity.

1. Let A be an algebra with an identity over a commutative ring R (with identity). A left A-module M is called *(A, R)-irreducible* if M has no non-trivial A-submodule which is an R-direct summand. If R is semisimple, (A, R)-irreducibility is identical with A-irreducibility. In [4], we introduced the notion of a semisimple algebra over R. If A is left semisimple over R, then (A, R)-irreducibility coincides with A-indecomposability. It follows at once the following proposition.

PROPOSITION 1. *Let A be an R-finite semisimple algebra over a Noetherian ring R. Then, a finitely generated left A-module is a direct sum of a finite number of (A, R)-irreducible modules. In particular, A itself is a direct sum of a finite number of (A, R)-irreducible left ideals.*

2. A left A-module M is called *completely faithful* if its trace ideal [1] coincides with A. This means that there exist a finite number of elements

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$x_1, \dots, x_r$ of M and A-homomorphisms $\xi_1, \dots, \xi_r : M \to A$ such that $\sum \xi_i(x_i) = 1$. In this case, M is faithful, and moreover the canonical monomorphism $A \to \mathrm{Hom}_R(M, M)$ admits a left A-splitting $f \to \sum \xi_i(f(x_i))$. Conversely, if M is R-finite and projective, left A-splitting of $A \to \mathrm{Hom}_R(M, M)$ implies the complete faithfulness of M. Indeed, let $\varphi : \mathrm{Hom}_R(M, M) \to A$ provide a left A-splitting, and let φ correspond to $\sum \xi_i \otimes x_i$ under the isomorphism [3]

$$\mathrm{Hom}_A(\mathrm{Hom}_R(M, M), A) \cong \mathrm{Hom}_A(M, A) \otimes_R M.$$

Then we observe $\sum \xi_i(x_i) = 1$.

Complete faithfulness of a right A-module is defined similarly. Let M^* be the dual right A-module of M : $M^* = \mathrm{Hom}_A(M, A)$. For any $x \in M$, there corresponds $x^{**} \in M^{**}$ by $x^{**}(\xi) = \xi(x)\ (\xi \in M^*)$, and $x \to x^{**}$ is an A-homomorphism $M \to M^{**}$. If M is completely faithful and $\sum \xi_i(x_i) = 1$ as above, then we have $\sum x_i^{**}(\xi_i) = 1$. Hence M^* is also completely faithful. If M is A-finite and projective, then M^* is A-finite and projective, and $x \to x^{**}$ gives an isomorphism $M \cong M^{**}$. We identify M with M^{**}. Then, summarizing above remarks, we have

LEMMA 1. *Let M be an A-module which is R-finite projective as well as A-finite projective. Then the dual M^* is an A-finite and projective right module, and the following statements are equivalent:*

i) *M is completely faithful.*

ii) *M^* is completely faithful.*

iii) *The canonical mapping $A \to \mathrm{Hom}_R(M, M)$ admits a left A-splitting.*

iv) *The canonical mapping $A \to \mathrm{Hom}_R(M^*, M^*)$ admits a right A-splitting.*

LEMMA 2. *For an idempotent e of A, the following three statements are equivalent:*

i) *Ae is completely faithful.*

ii) *eA is completely faithful.*

iii) *$AeA = A$.*

This lemma follows from the simple observation that $(Ae)^* \cong eA$ by the correspondence $\xi \to \xi(e)$, and $\sum \xi_i(x_i) = \sum x_i \xi_i(e)$.

3. Let R be a directly indecomposable commutative ring. An algebra A over R is called a *simple algebra* over R if A is (two-sided) semisimple over

R and if there exists a left A-module M such that

i) M is R-finite and projective,

ii) M is A-completely faithful, and

iii) M is (A, R)-irreducible.

By the semisimplicity of A, irreducibility of M coincides with indecomposability. By Lemma 1, the right A-module M^* satisfies also i), ii), iii), and we may equally define the simplicity of A by the existence of a right module satisfying i), ii), iii). Again by the semisimplicity of A, condition ii) may be replaced by the weaker one:

ii′) The canonical mapping $A \to \mathrm{Hom}_R(M, M)$ admits an R-splitting.

Further, i) and ii′) together show that *a simple algebra A is R-finite and projective.* If there exists an R-faithful M satisfying i), ii), iii), then A itself is also R-faithful.

A is called a *division algebra* if A is semisimple, R-finite and projective, and the left A-module A is (A, R)-irreducible. Since the canonical mapping $A \to \mathrm{Hom}_R(A, A)$ admits an R-splitting : $f \to f(1)$, $M = A$ satisfies the above conditions i), ii′), iii), so that *a division algebra is a simple algebra.*

THEOREM 1. *An R-faithful semisimple algebra A over R is a simple algebra if and only if $A \cong \mathrm{Hom}_D(M, M)$, where D is an R-faithful division algebra over R and M is an R-finite projective D-completely faithful D-module.*

Proof. Let A be a semisimple algebra, M an R-finite projective A-completely faithful module, and $B = \mathrm{Hom}_A(M, M)$. By [4, Th. 3, 5], B is semisimple, M is B-completely faithful, and $A = \mathrm{Hom}_B(M, M)$. Assume further that A is simple and M is (A, R)-irreducible. If B is not a division algebra, then B has a non-trivial idempotent e. But, then, M is a direct sum of non-zero A-submodules eM and $(1-e)M$, contradicting its irreducibility. This proves the one half of the Theorem. To prove the other half, assume A is a division algebra. If M is a direct sum of B-submodules M_1 and M_2, the projections $e_i : M \to M_i$, $i = 1, 2$, are idempotent elements of $\mathrm{Hom}_B(M, M) = A$ and $1 = e_1 + e_2$, $e_1 e_2 = e_2 e_1 = 0$, again contradicting the assumption on A.

Remark. In this Theorem, A is separable [resp. central separable] if and only if D is separable [resp. central separable] by [5, Th. 1] [resp. by [2, Th. 3, 3]].

4. In this section we consider the decomposition of semisimple algebras.

PROPOSITION 2. *A simple algebra is directly indecomposable (as an algebra).*

Proof. Let M be a faithful (A, R)-irreducible module. If A is the direct sum of the subalgebras A_1 and A_2, then M is the direct sum of A_1M and A_2M. Therefore one of them vanishes, say $A_1M=0$. As M is faithful, we have $A_1=0$.

PROPOSITION 3. *For a commutative semisimple algebra A over R, which is R-finite and projective, the following three properties are equivalent:*

i) *A is directly indecomposable.*

ii) *A is a simple algebra.*

iii) *A is a division algebra.*

Proof is easy and is omitted.

THEOREM 2. *Let A be a semisimple algebra over R. If a two-sided ideal I is an R-direct summand of A, then there exists a two-sided ideal J such that $A=I\oplus J$.*

Proof. By the left semisimplicity of A, I is a left A-direct summand of A. Hence there exists a left ideal J such that $A=I\oplus J$. We shall show that J is a two-sided ideal. Let e be an idempotent such that $J=Ae$. Then the right ideal eI, being an R-direct summand of I, whence of A, must be a right A-direct summand of A by the right semisimplicity of A. Hence eI is generated by an idempotent: $eI=e'A$. But $e'=(e')^2\in(eI)^2\subset e(IJ)I=0$, and we have $eI=0$. It follows that $JA=JI+J=J$, as desired.

COROLLARY. *A simple algebra A has no non-trivial two-sided ideal which is an R-direct summand of A.*

PROPOSITION 4. *A semisimple algebra A is directly indecomposable as an algebra if and only if there exists no idempotent e such that $(1-e)Ae=0$ other than 1 and 0.*

Proof. 'If' part is clear. So we assume that $e(\neq 1, 0)$ is an idempotent such that $(1-e)Ae=0$. Then $A(1-e)$ is a two-sided ideal since $A(1-e)A=A(1-e)Ae+A(1-e)=A(1-e)$. Since $A=A(1-e)+Ae$, we see that Ae is also a two-sided ideal from the proof of Theorem 2. Hence A is directly decomposable.

5. We have seen that a simple algebra A over R is R-finite and projective

and directly indecomposable as an algebra. We have no complete answer to the problem whether the converse is true or not, but we have affirmative answers in three particular cases:

1) A is commutative (Proposition 3).

2) A is separable (Theorem 4 below).

3) R is a Dedekind domain (Theorem 3 of this section).

In this section, we shall treat the case 3).

THEOREM 3. *An R-finite and torsionfree semisimple algebra A over a Dedekind domain R is simple if and only if A is directly indecomposable. In this case, every left [resp. right] ideal ($\neq 0$) is A-completely faithful.*

Proof. We have only prove the 'if' part. Thus we assume that A is directly indecomposable. Let C be the center of A. Then C is also a Dedekind domain [4, Th. 4, 5]. Let K and Z be the quotient fields of R and C, respectively. Then $A_Z(=A\otimes_C Z)$ is a central simple algebra over Z [4, Th. 4, 3], and A is a maximal order of A_Z[4, Th. 4, 6]. Now, let M be a non-zero left ideal of A. As M_K is A_K-faithful, M is A-faithful, and the natural mapping $\theta: A\to \mathrm{Hom}_R(M, M)$ is a monomorphism. Now, A coincides with the set $\{a\in A_Z \mid aM\subset M\}$, since the latter is an order of A_Z containing A. Assume $\lambda f=\theta(a)$ for some $\lambda\in R$, $\lambda\neq 0$, $f\in \mathrm{Hom}_R(M, M)$ and $a\in A$. Then $(a/\lambda)M\subset M$, so that there exists $b\in A$ satisfying $a=\lambda b$. It follows that $f=\theta(b)$. This means that the R-finite module $\mathrm{Hom}_R(M, M)/\theta(A)$ is torsionfree, and θ admits an R-splitting. Hence M is completely faithful. Taking $M=Ae$, a minimal left direct summand of A, we conclude that A is simple.

6. In this section, we consider separable algebras.

LEMMA. *In an indecomposable commutative ring R, only finitely generated idempotent ideals are* 0 *and R.*

Proof. Let $\mathfrak{a}\neq 0$ be a finitely generated ideal such that $\mathfrak{a}\mathfrak{a}=\mathfrak{a}$. There exists $a\in\mathfrak{a}$ such that $(1-a)\mathfrak{a}=0$ [6, p. 215]. It follows that a is an idempotent and $\mathfrak{a}$ is generated by a. Hence $\mathfrak{a}=R$ by the indecomposability of R.

Remark. There may exist non-finitely generated idempotent ideals with non-zero annihilators. The following example is due to S. Endo. Let $\mathfrak{o}$ be a non-discrete valuation ring, and $\mathfrak{m}$ its maximal ideal. Let a be a non-zero element in $\mathfrak{m}$. Then $\mathfrak{m}^2 = \mathfrak{m}$ and $a\mathfrak{m} \subsetneqq (a)$. Put $R = \mathfrak{o}/a\mathfrak{m}$. Then R is indecomposable, $\mathfrak{m}/a\mathfrak{m}$ is an idempotent ideal with the non-vanishing annihilator ideal $(\supset (a)/a\mathfrak{m})$.

THEOREM 4. *A separable algebra A which is R-finite and projective over an indecomposable Noetherian ring is a simple algebra if and only if its center C is indecomposable. Every non-zero left (resp. right) direct summand of a simple separable algebra A is completely faithful.*

Proof. Let e be an idempotent of A. By [2, Cor. 3, 2], two-sided ideals I of A are in one-to-one correspondence with ideals $\mathfrak{a}$ of C by the relations $I = \mathfrak{a}A$, $\mathfrak{a} = I \cap C$. Now AeA is an idempotent two-sided ideal of A. Hence, if $AeA = \mathfrak{a}A$, $\mathfrak{a}$ must be an idempotent ideal of C. Assume C is directly indecomposable. Then, by the above Lemma, we have $\mathfrak{a} = C$ so that $AeA = A$. This shows that Ae as well as eA is completely faithful (Lemma 2). In particular, a minimal left direct summand of A is completely faithful, and A is a simple algebra. Conversely, if C is directly decomposable, then A is also directly decomposable, and A is not simple.

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