ON SOME DUALITIES CONCERNING ABELIAN VARIETIES

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Dedicated to late Professor TADASI NAKAYAMA

Introduction. The group of extensions $\mathrm{Ext}(A, G_a)$ and $\mathrm{Ext}(A, G_m)$ of an abelian variety A by the additive or multiplicative group G_a, G_m have been investigated in detail ([9], [10], [GACC]). On the other hand, F. Oort [8] and M. Miyanishi [6] recently studied the groups $\mathrm{Ext}(G_a, A)$ and $\mathrm{Ext}(G_m, A)$. The purpose of the present paper is to clarify the relationship between these groups. Our results show that the latter groups can be derived from the former.

Sections 1 and 2 are preliminaries. Section 3 contains our main result, namely the existence of a canonical duality between the vector spaces $\mathrm{Ext}(A, G_a)$ and $\mathrm{Ext}(G_a, A)$. The functorial property of the pairing of duality is also proven. In Section 4 the functoriality is used to prove that the pairing behaves nicely with respect to the Frobenius operator. A proof of a theorem concerning $H^1(A, \mathcal{O})$ is added. In the last Section 5 a similar duality is defined for $\mathrm{Ext}(A, G_m)$ and $\mathrm{Ext}(G_m, A)$; this is much easier. As the consequence we see that $\mathrm{Ext}(G_m, A)$ is the character group of the Tate group $T(A^*)$ (cf. [5] Ch. VII) of the dual abelian variety A^*.

Though our results are in the negative direction (no new functors!), they seem to have some interest in view of the various duality phenomena in the theory of commutative algebraic groups, for which unified treatments are being made by several mathematicians (cf. [1], [2]).

We shall use the definitions and the results about group extensions contained in Serre's book [GACC].

1. The following results are known:

(i) $\mathrm{Ext}(A, G_m)$ is canonically isomorphic to the underlying group of the dual abelian variety (i.e. the Picard variety) A^* of A, while $\mathrm{Ext}(G_m, A)$ is

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non-canonically isomorphic to the group of points of finite order of A.

(ii) Let k be the algebraically closed ground field. The algebraic group G_a has a canonical structure of a k-module, which makes $\mathrm{Ext}(A, G_a)$ and $\mathrm{Ext}(G_a, A)$ k-modules. Let $\mathfrak{t}(A^*)$ denote the Lie algebra of the invariant derivations of A^*. Then there exist canonical isomorphisms of k-modules[1)]

$$\mathrm{Ext}(A, G_a) \cong H^1(A, \mathcal{O}) \cong \mathfrak{t}(A^*). \tag{1.1}$$

In particular we have $\dim_k \mathrm{Ext}(A, G_a) = \dim A$. On the other hand we have $\dim_k \mathrm{Ext}(G_a, A) = \dim A$ in the case of characteristic p and $\mathrm{Ext}(G_a, A) = 0$ in the case of characteristic zero. When k is of characteristic p, let $k[F]$ and $k[F^*]$ be the non-commutative k-algebras with the relations $\alpha^p F = F\alpha$, $\alpha F^* = F^*\alpha^p$ $(\alpha \in k)$. Then $k[F]$ operates on G_a, and this operation makes $\mathrm{Ext}(A, G_a)$ and $\mathrm{Ext}(G_a, A)$ a $k[F]$-module and a $k[F^*]$-module respectively. F also operates on $H^1(A, \mathcal{O})$ and on $\mathfrak{t}(A^*)$ as the "p-th power" operation, and the isomorphisms (1.1) are, in this case, isomorphisms of $k[F]$-modules[2)]. Later we shall prove that the $k[F]$-module $\mathrm{Ext}(A, G_a)$ and the $k[F^*]$-module Ext (G_a, A) are canonically dual.

2. We shall call a connected algebraic group G *elementary* if G contains two connected algebraic subgroups H and A, where H is a linear group and A is an abelian variety, satisfying $G = HA$.

LEMMA 1. *Let L be a commutative connected linear algebraic group and let A be an abelian variety.*

(*a*) *If G is an extension of L by A, i.e. if there is a strict exact sequence*

$$0 \longrightarrow A \longrightarrow G \longrightarrow L \longrightarrow 0,$$

then G is elementary.

(*b*) *If G is an extension of A by L, then G is elementary if and only if it is of finite order in the group* $\mathrm{Ext}(A, L)$.

Proof. Let H be the maximal connected linear subgroup of G. Then G/H is an abelian variety. Therefore G/AH is linear and abelian at the same time,

1) The first isomorphism is in [9]. The second one was first proved by Cartier, "Dualité des variétés abéliennes, Séminaire Bourbaki, Mai 1958. Today it is a direct consequence of the representability of Picard functor, cf. [7].

2) See section 4 below.

hence $G=AH$.

(b) Assume that G is elementary. Then $G=LB$ for some abelian subvariety B, and $L\cap B$ is a finite group. Let φ be the induced homomorphism $B\to A$. φ is surjective, and we have a commutative diagram with exact horizontal lines

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & G & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow{\scriptstyle \varphi} & & \\ 0 & \longrightarrow & L & \longrightarrow & L\times B & \longrightarrow & B & \longrightarrow & 0. \end{array}$$

This means $\varphi^*G=0$. If n is the degree of φ (which is, by the way, equal to the degree of the intersection cycle $B\cdot L$), then φ divides $n\delta_A$ ([12] Th. 27), hence $nG=0$ in $\mathrm{Ext}(A, L)$.

Conversely, assume $nG=0$. We have a commutative diagram with exact horizontal lines

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & G & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \uparrow{\scriptstyle \psi} & & \uparrow{\scriptstyle n\delta_A} & & \\ 0 & \longrightarrow & L & \longrightarrow & L\times A & \longrightarrow & A & \longrightarrow & 0. \end{array}$$

Then the homomorphism ψ must be surjective, and so we obtain $G=L\psi(A)$. Therefore G is elementary.

When G is a commutative group and n is a natural number, we put $G_n=\{x\mid x\in G,\ nx=0\}$, and we denote by G_{fini} the torsion subgroup of G.

3. Duality between $\mathrm{Ext}(A, G_a)_{\mathrm{fini}}$ and $\mathrm{Ext}(G_a, A)$

If k has characteristic zero then we have $\mathrm{Ext}(A, G_a)_{\mathrm{fini}}=0$ and $\mathrm{Ext}(G_a, A)=0$, so they are trivially dual. In the rest of this section we assume that k has characteristic p. Then $\mathrm{Ext}(A, G_a)=\mathrm{Ext}(A, G_a)_{\mathrm{fini}}$ and the dimensions of $\mathrm{Ext}(A, G_a)$ and $\mathrm{Ext}(G_a, A)$ over k are both equal to $\dim A$. We are going to show that these two groups are canonically dual to each other. Our proof is based on a result of Rosenlicht [9].

Let $\mathscr{V}$ be the category of finite-dimensional vector groups over k, where the morphisms are k-linear mappings. If $F:\mathscr{V}\to\mathbf{Ab}$ (= the category of commutative groups) is an additive functor, then for each $H\in\mathscr{V}$ the group $F(H)$ has a natural structure of k-module. F is representable if and only if

$F(k)$ has a finite dimension over k[3]. Let A be an abelian variety of dimension r. Define a functor $\Phi : \mathscr{V} \to \mathbf{Ab}$ by $\Phi(H) = \mathrm{Ext}(A, H)$, $H \in \mathscr{V}$, where in the right hand side H is considered an algebraic group. Following Rosenlicht we call an element G of $\Phi(H)$ *decomposable* if there exist a vector subspace H' of H, different from H, and an element $G' \in \Phi(H)$ such that G is obtained from G' by the injection $H' \to H$. An equivalent condition is this: G is decomposable if and only if there exists a linear mapping $u : H \to k$ such that $u \neq 0$, $F(u)G = 0$. Rosenlicht [9] proved the representablity of Φ by showing that if $G \in \Phi(H)$ is indecomposable then $\dim H \leq r = \dim A$. Thus there exist a vector space $M_A \in \mathscr{V}$ and an extension $\Delta_A \in \Phi(M_A)$:

$$0 \longrightarrow M_A \xrightarrow{\mu_A} \Delta_A \xrightarrow{j_A} A \longrightarrow 0 \tag{3.1}$$

which represent Φ, i.e. such that for any $H \in \mathscr{V}$ and for any $G \in \Phi(H)$ there exists one and only one $f \in \mathrm{Hom}_{\mathscr{V}}(M_A, H)$ satisfying $f_* \Delta_A = G$ ([9] Prop. 11). In particular we have $\mathrm{Hom}_{\mathscr{V}}(M_A, G_a) = \mathrm{Ext}(A, G_a)$, hence $\mathrm{Ext}(A, G_a)$ can be identified with the dual space of M_A.

Similarly, we define a contravariant additive functor $\Psi : \mathscr{V} \to \mathbf{Ab}$ by $\Psi(H)$ $\mathrm{Ext}(H, A)$. An element $G \in \Psi(H)$ will be called *decomposable* if there exists a linear mapping $v : k \to H$, $v \neq 0$, such that $v^* G = 0$. By Miyanishi [6] we know that $\dim \Psi(k) = \dim \mathrm{Ext}(G_a, A) = r$. Therefore Ψ is representable by a vector space $N_A \in \mathscr{V}$ and by an extension $\Gamma_A \in \Psi(N_A)$:

$$0 \longrightarrow A \xrightarrow{i_A} \Gamma_A \xrightarrow{\nu_A} N_A \longrightarrow 0. \tag{3.2}$$

For any $H \in \mathscr{V}$ we have $\Psi(H) \cong \mathrm{Hom}(H, N_A)$, in particular $N_A \cong \Psi(G_a) = \mathrm{Ext}(G_a, A)$.

Since $\mathrm{Ext}(A, H)$ and $\mathrm{Ext}(H, A)$ are functors of two variables, M_A and N_A depend functorially on A. Explicitly, if $\varphi : A \to B$ is a homomorphism of abelian varieties, then it follows from the universality of M_A that there exists a unique linear mapping $M_\varphi : M_A \to M_B$ with $(M_\varphi)_* \Delta_A = \varphi^* \Delta_B$. Similarly there exists a unique linear mapping $N_\varphi : N_A \to N_B$ satisfying $\varphi_* \Gamma_A = (N_\varphi)^* \Gamma_B$. Thus we have two functors M, N from the additive category of abelian varieties to

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[3] If F is representable by $U \in \mathscr{V}$, then $F(k) \cong \mathrm{Hom}(U\ k) \in \mathscr{V}$. Conversely, if $F(k) \in \mathscr{V}$ then let U denote the dual space of $F(k)$. Then we have natural equivalences of functors $F(H) \cong F(k) \otimes \mathrm{Hom}(k, H) \cong \mathrm{Hom}(U, k) \otimes \mathrm{Hom}(k, H) \cong \mathrm{Hom}(U, H)$. The case of a contravariant funcotr follows from the covariant case since the category $\mathscr{V}$ is self-dual.

the category $\mathscr{V}$.

THEOREM 1. *There exists a natural equivalence between the two functors* M *and* N.

The proof of Theorem 1 is the aim of this section. As a consequence of Theorem 1 we have the following

THEOREM 2. *For each abelian variety* A *there exists a canonical non-degenerate pairing of* k*-modules*

$$\mathrm{Ext}(G_a, A) \times \mathrm{Ext}(A, G_a) \longrightarrow k.$$

This pairing is functorial in A *in the following sense: if* $\varphi : A \to B$ *is a homomorphism of abelian varieties, then it holds*

$$\langle \varphi_* G, G' \rangle = \langle G, \varphi^* G' \rangle \qquad \text{for } G \in \mathrm{Ext}(G_a, A),\ G' \in \mathrm{Ext}(B, G_a).$$

Proof of Theorem 2. We have canonical isomorphisms

$$\begin{aligned} \alpha_A &: \mathrm{Ext}(G_a, A) \cong \mathrm{Hom}(G_a, N_A), \\ \beta_A &: \mathrm{Ext}(A, G_a) \cong \mathrm{Hom}(M_A, G_a), \\ \gamma_A &: N_A \cong M_A, \end{aligned}$$

which are natural in A. If $G_1 \in \mathrm{Ext}(G_a, A)$, $G_2 \in \mathrm{Ext}(A, G)$, then $(\beta_A G_2) \circ \gamma_A \circ (\alpha_A G_1) \in \mathrm{Hom}(G_a, G_a)$. Since $\mathrm{Hom}(G_a, G_a)$ is canonically isomorphic to k we may identify it with k. Thus we put

$$\langle G_1, G_2 \rangle = (\beta_A G_2) \circ \gamma_A \circ (\alpha_A G_1). \tag{3.3}$$

This is clearly a non-degenerate pairing of k-modules. If $\varphi : A \to B$ and $G \in \mathrm{Ext}(G_a, A)$ and $G' \in \mathrm{Ext}(B, G_a)$ are given, then

$$\begin{aligned} \langle \varphi_* G, G' \rangle &= (\beta_B G') \circ \gamma_B \circ (\alpha_B(\varphi_* G)) \\ &= (\beta_B G') \circ \gamma_B \circ N_\varphi \circ (\alpha_A G) \\ &= (B_B G') \circ M_\varphi \circ \gamma_A \circ (\alpha_A G) \\ &= (\beta_A(\varphi^* G')) \circ \gamma_A \circ (\alpha_A G) \\ &= \langle G, \varphi^* G' \rangle. \end{aligned}$$

This completes the proof of Theorem 2. The following commutative diagram illustrates the situation:

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, equal] & G \arrow[r] \arrow[d] & G_a \arrow[d, "g"] & & & \\
A \arrow[r] \arrow[d] & \Gamma_A \arrow[r] \arrow[d] & N_A \arrow[r, phantom, "\overset{\gamma_A}{\cong}"] \arrow[d, "N_\varphi"] & M_A \arrow[r] \arrow[d, "M_\varphi"] & \Delta_A \arrow[r] \arrow[d] & A \arrow[d, "\varphi"] \\
B \arrow[r] & \Gamma_B \arrow[r] & N_B \arrow[r, phantom, "\overset{\gamma_B}{\cong}"] & M_B \arrow[r] \arrow[d, "g'"] & \Delta_B \arrow[r] \arrow[d] & B \arrow[d, equal] \\
& & & G_a \arrow[r] & G' \arrow[r] & B
\end{tikzcd}
$$

where $g = \alpha_\varphi G$, $g' = \beta_B G'$.

If X is an algebraic variety [respectively, an algebraic group] defined over k (which we have assumed to be of characteristic p), and if $q = p^m$, $m \in Z$, then $X^{(q)}$ will denote the algebraic variety [resp. algebraic group] obtained by applying the automorphism $\pi^m : \alpha \to \alpha^q$ of k to the coefficients of the defining equations of X. Similarly, for any morphism $f : X \to Y$, a morphism $f^{(q)} : X^{(q)} \to Y^{(q)}$ is defined. There exists a canonical morphism $F = F_X : X \to X^{(p)}$ satisfying $f^{(p)} F_X = F_Y f$, which is called the Frobenius morphism and is sometimes considered a part of the structure of $X^{(p)}$. If $A = (a_{ij})$ is the jacobian matrix of $f : X \to Y$ at a point $x \in X$, then $\pi^m(A) = (a_{ij}^q)$ is the jacobian matrix of $f^{(q)}$ at $F_X^m(x) \in X^{(q)}$ (with respect to suitable coordinate systems). Therefore, if

$$0 \longrightarrow H \xrightarrow{u} G \xrightarrow{v} K \longrightarrow 0$$

is an exact sequence of algebraic groups, then the corresponding sequence

$$0 \longrightarrow H^{(q)} \xrightarrow{u^{(q)}} G^{(q)} \xrightarrow{v^{(q)}} K^{(q)} \longrightarrow 0$$

is also exact, since it is clearly exact as a sequence of abstract groups and since the associated sequence of Lie algebras is exact by what we have just seen. Hence a bijection

$$\pi^m : \mathrm{Ext}(K, H) \longrightarrow \mathrm{Ext}(K^{(q)}, H^{(q)}),$$

which is functorial in both K and H and is an isomorphism of groups.

Proof of Theorem 1. Consider the extension (3.1). It is known that the maximal abelian subvariety of Δ_A is $A^{(p)}$, and that if V_A denotes the homomorphism $A^{(p)} \to A$ induced by $\Delta_A \to A$, then $p\delta_A = V_A \circ F_A$, where F_A is the Frobenius homomorphism (see the remark on p. 707 of [9]). Transforming the extension (3.1) by the automorphism π^{-1} of k, we get an extension

(3.4) $$0 \longrightarrow M_A^{(q)} \longrightarrow \Delta_A^{(q)} \longrightarrow A^{(q)} \longrightarrow 0, \qquad (q = p^{-1}),$$

which represents the functor $H \rightsquigarrow \mathrm{Ext}(A^{(q)}, H)$. The maximal abelian subvariety of $\Delta_A^{(q)}$ is A. Putting $\Delta_A^{(q)}/A = N$ we obtain an extension

(3.5) $$0 \longrightarrow A \longrightarrow \Delta_A^{(q)} \longrightarrow N \longrightarrow 0.$$

At this point N is just an algebraic group, but we can give it a canonical structure of vector group as follows. According to p. 707 loc. cit., there exist linear coordinate functions $x_1, \ldots, x_r$ on $M_A^{(q)}$ and additive coordinate functions $y_1, \ldots, y_r$ on N such that the composed homomorphism $\theta : M_A^{(q)} \to \Delta_A^{(q)} \to N$ is given by

(3.6) $$y_i \circ \theta = x_i^p - \sum_{j=1}^{r} c_{ij} x_j, \qquad c_{ij} \in k \ (1 \leq i \leq r).$$

Now, once $x_1, \ldots, x_r$ are chosen, such a coordinate system (y) is unique: if $y_1', \ldots, y_r'$ are another additive coordinates on N such that $y_i' \circ \theta = x_i^p - \sum c_{ij}' x_t$ for some $c_{ij}' \in k$, $1 \leq i \leq r$, then $y_i = y_i'$ for all i. This is evident if we express y_i' as p-polynomials in y. If we choose another linear coordinates (x') on $M_A^{(q)}$, and if $x_i' = \sum a_{jh} x_h$, then the corresponding additive coordinate system (y') of N satisfies $y_j' = \sum a_{jh}^p x_h$. Therefore (y) defines on N a canonical structure of vector group. With this structure, we can identify N with M_A, because if we define a homomorphism $\rho : M_A^{(q)} \to N$ by $y_i \circ \rho = x_i^p$ $(1 \leq i \leq r)$, where (x) and (y) are related to each other by (3.6), then ρ is independent of the coordinate systems used in its definition.

Now we regard (3.5) as an extension of the vector group N by A, and we are going to prove that it is indecomposable. Assume the contrary. Then there exists an injection $g : G_a \to N$ such that the pull-back of (3.5) by g is trivial. We write G for $\Delta_A^{(q)}$ for simplicity. Thus we have the following commutative diagram:

(3.7) $$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & G & \longrightarrow & N & \longrightarrow & 0 \\ & & \| & & \uparrow h & & \uparrow g & & \\ 0 & \longrightarrow & A & \longrightarrow & G_a \times A & \longrightarrow & G_a & \longrightarrow & 0. \end{array}$$

Let L denote the maximal connected linear subgroup of $g^*(G) = G_a \times A$, i.e. $\mathrm{L} = G_a \times e$. After operating appropriate linear transformations on the coordinates (x) and (y) of (3.6), we may assume that the subgroup $g(G_a)$ of N' is given by $y_2 = y_3 = \cdots = y_r = 0$. Then the subgroup $\theta^{-1}(g(G_a))$ of $M_A^{(q)}$ is defined by

$$x_2^p - l_2(x) = 0, \ \ldots, \ x_r^p - l_r(x) = 0, \tag{3.8}$$

where $l_i(x) = \sum_j c_{ij}x_j$. Taking an additive coordinate function t on L, we can express the functions $x_1 \circ h, \ldots, x_r \circ h$ as p-polynomials of t. Let t^{p^d} be the highest power of t which appears in the p-polynomials $x_1 \circ h, \ldots, x_r \circ h$. Then it does not appear in $x_2 \circ h, \ldots, x_r \circ h$ since (3.8) holds on $h(L)$. Therefore t^{p^d} appears in $x_1 \circ h$, and we have

$$y_1 \circ \theta \circ h = y_1 \circ g \circ \nu = ct^{p^{d+1}} + (\text{a } p\text{-polynomial in } t \text{ of degree} \leq p^d), \tag{3.9}$$

$c \neq 0$, where $y_1 \circ g$ is an additive coordinate function on G_a. This contradicts the fact that ν induces an isomorphism between L and G_a. Thus we have proved that (3.5) is indecomposable. Let $f : N \to N_A$ be the unique linear map with $f^*(\Gamma_A) = G$. Then f is injective since G is indecomposable. Therefore f must be an isomorphism since $\dim N = \dim N_A = r$. So we may take (3.5) as the universal extension (3.2), identifying N_A with N. Then there exists a canonical isomorphism $F \circ \rho^{-1} : N_A = N \cong M_A$, where F denotes the Frobenius homomorphism $M_A^{(q)} \to M_A$ $(q = p^{-1})$.

Next we prove that the isomorphism just defined is natural in A. Let $\varphi : A \to B$ be a homomorphism of abelian varieties. We have to show $M_\varphi \circ F \circ \rho_A^{-1} = F \circ \rho_B^{-1} \circ N_\varphi$, and since $M_\varphi \circ F = F \circ M_\varphi^{(q)}$ is obvious it suffices to show the commutativity of the following diagram:

$$\begin{array}{ccc} N_A & \xleftarrow{\rho_A} & M'_A \\ {\scriptstyle N_\varphi}\downarrow & & \downarrow{\scriptstyle M'_\varphi} \\ N_B & \xleftarrow[\rho_B]{} & M'_B \end{array} \tag{3.10}$$

where $'$ stands for (p^{-1}).

Now N_φ is characterized as the unique linear map $N_A \to N_B$ such that $(N_\varphi)^* \Gamma_B = \varphi_* \Gamma_A$, and this last condition is equivalent to the existence of a homomorphism $\xi : \Gamma_A \to \Gamma_B$ such that the following diagram is commutative (cf. [GACC] p. 164-165):

$$\begin{array}{ccccc} N_A & \xleftarrow{\nu_A} & \Gamma_A & \xleftarrow{i_A} & A \\ {\scriptstyle N_\varphi}\downarrow & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \varphi} \\ N_B & \xleftarrow[\nu_B]{} & \Gamma_B & \xleftarrow[i_B]{} & B. \end{array}$$

Similarly, since $\Delta_A^{(\eta)} = \Gamma_A$, we have a commutative diagram

$$\begin{array}{ccccc} M'_A & \xrightarrow{\mu'_A} & \Gamma_A & \xrightarrow{j'_A} & A' \\ {\scriptstyle M'_\varphi}\downarrow & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \varphi'} \\ M'_B & \xrightarrow[\mu'_B]{} & \Gamma_B & \xrightarrow[j'_B]{} & B' \end{array}$$

where μ'_A, φ' etc. stand for $\mu_A^{(p^{-1})}$, $\varphi^{(p^{-1})}$ etc. Then we have

$$j'_A i_A = V'_A,\ \ j'_B i_B = V'_B,\ \ \nu_A \mu'_A = \theta_A,\ \ \nu_B \mu'_B = \theta_B.$$

Therefore

$$\begin{aligned} j'_B(\xi - \eta) i_A &= j'_B \xi i_A - j'_B \eta i_A \\ &= j'_B i_B \varphi - \varphi' j'_A i_A \\ &= V'_B \varphi - \varphi' V'_A \\ &= 0, \end{aligned}$$

hence $(\xi - \eta) i_A = 0$ since Ker $j'_B = M'_B$ is linear while A is abelian. Then we have $\nu_B \eta i_A = \nu_B \xi i_A = \nu_B i_B \varphi = 0$, therefore there exists a homomorphism $u : N_A \to N_B$ such that $\nu_B \eta = u \nu_A$. Let $(x_1, \ldots, x_r)$, $(y_1, \ldots, y_r)$, $(\bar{x}_1, \ldots, \bar{x}_s)$, $(\bar{y}_1, \ldots, \bar{y}_s)$ be linear coordinates functions on M'_A, N_A, M'_B, N_B respectively, such that

$$y_i \circ \theta_A = x_i^p - \sum c_{ij} x_j \ \ ; \ \ \bar{y}_\alpha \circ \theta_B = \bar{x}_\alpha^p - \sum \bar{c}_{\alpha\beta} \bar{x}_\beta.$$

Let the linear map M'_φ be given by

$$\bar{x}_\alpha \circ M'_\varphi = \sum_i d_{\alpha i} x_i.$$

Then, since $u \circ \theta_A = u \nu_A \mu'_A = \nu_B \eta \mu'_A = \nu_B \mu'_B M'_\varphi = \theta_B M'_\varphi$, we have

$$\begin{aligned} (\bar{y}_\alpha \circ u) \circ \theta_A &= (\bar{y}_\alpha \circ \theta_B) \circ M'_\varphi \\ &= \sum_i d_{\alpha i}^p x_i^p - \sum_\beta \bar{c}_{\alpha\beta} \sum_i d_{\beta i} x_i \\ &= \sum_i d_{\alpha i}^p (y_i \circ \theta_A) - (\text{a linear form in } x). \end{aligned}$$

Since a linear form in x which is a polynomial of $y \circ \theta_A$ must be zero, we conclude that u is a linear map defined by $\bar{y}_\alpha \circ u = \sum d_{\alpha i}^p y_i$. It then follows that $u = N_\varphi$, $\eta = \xi$. Since the matrix of N_φ is $(d_{\alpha i}^p)$, the diagram (3.10) commutes. This completes the proof of Theorem 1.

4. The operators F and F^*

In this section we compare the $k[F]$-module Ext(A, G_a) and the $k[F^*]$-

module $\mathrm{Ext}(G_a, A)$.

LEMMA 2. *Let K, H be commutative algebraic groups. Then the following diagram is commutative.*

$$
\begin{tikzcd}[column sep=small]
& \mathrm{Ext}(K^{(p)}, H) \arrow[dl, "(F_K)^*"'] \arrow[dr, "(F_H)_*"] & \\
\mathrm{Ext}(K, H) \arrow[dr, "(F_H)_*"'] & \overset{\pi}{\simeq} & \mathrm{Ext}(K^{(p)}, H^{(p)}) \\
& \mathrm{Ext}(K, H^{(p)}) \arrow[ur, "(F_K)^*"'] &
\end{tikzcd}
\tag{4.1}
$$

Proof. The commutativity of the lower triangle follows from the commutativity of the diagram

$$
\begin{tikzcd}
H \arrow[r] \arrow[d, "F_H"'] & G \arrow[r] \arrow[d, "F_G"'] & K \arrow[d, "F_K"'] \\
H^{(p)} \arrow[r] & G^{(p)} \arrow[r] & K^{(p)}
\end{tikzcd}
$$

where $G \in \mathrm{Ext}(K, H)$. As for the upper triangle, let $G' \in \mathrm{Ext}(K^{(p)}, H)$ and consider the commutative diagram

$$
\begin{tikzcd}
H \arrow[r] \arrow[d, equal] & (F_K)^*G' \arrow[r] \arrow[d] & K \arrow[d, "F_K"] \\
H \arrow[r] \arrow[d, "F_H"'] & G' \arrow[r] \arrow[d] & K^{(p)} \arrow[d, equal] \\
H^{(p)} \arrow[r] & (F_H)_*G' \arrow[r] & K^{(p)}.
\end{tikzcd}
$$

It follows from this diagram that the third horizontal line is the π-image of the first.

THEOREM 3. *Let A be an abelian variety. The pairing of Theorem 2 satisfies*

$$\langle F^*G, G'\rangle^p = \langle G, FG'\rangle$$

for $G \in \mathrm{Ext}(G_a, A)$, $G' \in \mathrm{Ext}(A, G_a)$.

Proof. By the definitions, by Theorem 1 and by Lemma 2 we have

$$
\begin{aligned}
\langle G, FG'\rangle = \langle G, (F_A)^*\pi G'\rangle = \langle (F_A)_*G, \pi G'\rangle &= \langle \pi(F_{G_a})^*G, \pi G'\rangle \\
&= \langle F^*G, G'\rangle^p.
\end{aligned}
$$

By this theorem we can say that the $k[F^*]$-module $\mathrm{Ext}(G_a, A)$ is the dual of the $k[F]$-module $\mathrm{Ext}(A, G_a)$. We can also identify $\mathrm{Ext}(G_a, A)$ with the space of the linear differential forms of the first kind on A^*, and then the operator F^* coincides with the operator of Cartier.

For the sake of completeness we add a proof of the fact that the ismorphisms

$$\mathrm{Ext}(A, G_a) \cong H^1(A, \mathcal{O}) \cong \mathfrak{t}(A^*)$$

commute with the operator F, as we could not find a suitable reference for it. The following lemma is more or less well known.

LEMMA 3. *Let A be an abelian variety and let A^* be its dual. Let $F_A : A \to A^{(p)}$ be the Frobenius homomorphism and let $V_A : A^{(p)} \to A$ be such that $V_A \cdot F_A = p \cdot \delta_A$. Then the transposed of F_A is V_{A^*}, and the transposed of V_A is F_{A^*}.*

Proof. Since ${}^tF_A {}^tV_A = {}^t(V_A F_A) = {}^t(p\delta_A) = p\delta_{A^*} = V_{A^*}F_{A^*}$, it suffices to prove ${}^tV_A = F_{A^*}$. Now ${}^tV_A : A^* \to A^{(p)*} = A^{*(p)}$ is given by the map $(V_A)^* : \mathrm{Ext}(A, G_m) \to \mathrm{Ext}(A^{(p)}, G_m)$, while the tangential linear map of ${}^tV_A$ is given by

$$(V_A)^* : \mathrm{Ext}(A, G_a) \to \mathrm{Ext}(A^{(p)}, G_a). \tag{4.2}$$

The dual of (4.2) is given by

$$M_V : M_{A^{(p)}} \to M_A.$$

But the extension (3.1) is split by $V : A^{(p)} \to A$ (cf. the beginning of the proof of our Theorem 1.). Hence $M_V = 0$, and so (4.2) is also the zero map. This means that F_{A^*} divides ${}^tV_A$. On the other hand, by the duality theorem of Nishi-Cartier we have

$$\nu({}^tV_A) = \nu(V_A) = p^r, \ \nu(F_{A^*}) = p^r \qquad (r = \dim A)$$

Hence ${}^tV_A = F_{A^*}$, Q.E.D.

If a Čech cocycle $\{f_{ij}\}$ represent a class $c \in H^1(A, \mathcal{O})$, then Fc is the cohomology class of $\{f_{ij}^p\}$. On the other hand, if

$$0 \longrightarrow G_a \xrightarrow{i} G \xrightarrow{j} A \longrightarrow 0$$

represents an element $c' \in \mathrm{Ext}(A, G_a)$, then

$$0 \longrightarrow G \xrightarrow{i \circ F^{-1}} G \xrightarrow{j} A \longrightarrow 0$$

represents Fc'. Therefore it is easy to see that the standard isomorphism between $\mathrm{Ext}(A, G_a)$ and $H^1(A, \mathcal{O})$ (cf. [9] p. 697-698) commutes with F. On the other hand F operates on $\mathfrak{t}(A^*)$ in the following way: if $\mathfrak{t}(A^*)$ is considered as the space of invariant derivations on A^*, then $FD = D^p$ for $D \in \mathfrak{t}(A^*)$. Now it holds, by [13] p. 157, that

$$D^p = (V_{A*})_*(\pi(D)),$$

where π is the isomorphism $\mathfrak{t}(A^*) \to \mathfrak{t}(A^{*(p)})$ induced by the p-th power isomorphism of the function fields $k(A^*) \cong k(A^{*(p)})$. By Lemma 2, $F = (F_{G_a})_* : \mathrm{Ext}(A, G_a) \to \mathrm{Ext}(A, G_a)$ is decomposed into

$$\mathrm{Ext}(A, G_a) \xrightarrow{\pi} \mathrm{Ext}(A^{(p)}, G_a^{(p)}) = \mathrm{Ext}(A^{(p)}, G_a) \xrightarrow{(F_A)^*} \mathrm{Ext}(A, G_a)$$

and by Lemma 3 $(F_A)^*$ corresponds to $(V_{A*})_* : \mathfrak{t}(A^{(p)*}) \to \mathfrak{t}(A_*)$. Thus the isomorphism $\mathrm{Ext}(A, G_a) \cong \mathfrak{t}(A^*)$ commutes with F.

5. Duality between $\mathrm{Exp}(A, G_m)_{\mathrm{fini}}$ and $\mathrm{Exp}(G_m, A)$.

In this section we make use of the theory of commutative algebraic group schemes (cf. [2], [3], [4]). Let k be an algebraically closed field of arbitrary characteristic $\chi(k)$. Let $\mathscr{A}$ denote the category of commutative algebraic groups over k and let $\mathscr{B}$ denote the abelian category of commutative algebraic group schemes over k. $\mathscr{B}$ coincides with $\mathscr{A}$ by a theorem of Cartier if $\chi(k) = 0$ ([2], cf. [7]). If $0 \to G \to H \to K \to 0$ is a sequence in $\mathscr{A}$, then it is exact in the sense of [GACC] if and only if it is exact in $\mathscr{B}$ in the category-theoretical sense. For G, $H \in \mathscr{B}$ one can construct the group of extensions $\mathrm{Ext}(G, H)$ in the same way as in [GACC], i.e. by the Baer construction. If $0 \to G \to H \to K \to 0$ is an exact sequence in $\mathscr{B}$ with G and K reduced, then H is reduced also. Hence if G, $H \in \mathscr{A}$ the group $\mathrm{Ext}(G, H)$ in $\mathscr{A}$ and the group $\mathrm{Ext}(G, H)$ in $\mathscr{B}$ coincide.

Let A be an abelian variety defined over k. Let n be a positive integer, and let $\mathfrak{A}_n$ and $\mathfrak{H}_n$ denote, respectively, the kernel in $\mathscr{B}$ of $n : A \to A$ and that of $n : G_m \to G_m$. Thus we have two exact sequences

$$0 \longrightarrow \mathfrak{A}_n \longrightarrow A \xrightarrow{n} A \longrightarrow 0, \tag{5.1}$$

$$0 \longrightarrow \mathfrak{H}_n \longrightarrow G_m \xrightarrow{n} G_m \longrightarrow 0. \tag{5.2}$$

From (5.1) follows the exact sequence

$$0 = \mathrm{Hom}(A, G_m) \longrightarrow \mathrm{Hom}(\mathfrak{A}_n, G_m) \longrightarrow \mathrm{Ext}(A, G_m) \xrightarrow{n} \mathrm{Ext}(A, G_m),$$

hence a canonical isomorphism

(5.3) $$\mathrm{Hom}(\mathfrak{A}_n, G_m) \cong \mathrm{Ext}(A, G_m)_n.$$

Similarly, from (5.2) we obtain a canonical isomorphism

(5.4) $$\mathrm{Hom}(\mathfrak{H}_n, A) \cong \mathrm{Ext}(G_m, A)_n.$$

It is clear that one can identify $\mathrm{Hom}(\mathfrak{A}_n, G_m)$ with $\mathrm{Hom}(\mathfrak{A}_n, \mathfrak{H}_n)$, and $\mathrm{Hom}(\mathfrak{H}_n, A)$ with $\mathrm{Hom}(\mathfrak{H}_n, \mathfrak{A}_n)$. Therefore we get a canonical pairing

(5.5) $$\mathrm{Ext}(G_m, A)_n \times \mathrm{Ext}(A, G_m)_n \longrightarrow \mathrm{Hom}(\mathfrak{H}_n, \mathfrak{H}_n)$$

via the pairing of the composition of morphisms

(5.6) $$\mathrm{Hom}(\mathfrak{H}_n, \mathfrak{A}_n) \times \mathrm{Hom}(\mathfrak{A}_n, \mathfrak{H}_n) \longrightarrow \mathrm{Hom}(\mathfrak{H}_n, \mathfrak{H}_n).$$

Now the pairing (5.6) (hence also (5.5)) is non-degenerate. This is clear when n is not divisible by $\chi(k)$, because in that case one has $\mathfrak{H}_n \cong Z/nZ$, $\mathfrak{A}_n \cong (Z/nZ)^{2r}$ (where $r = \dim A$). When $\chi(k) = p > 0$ and $n = qm$, $q = p^{\alpha}$, $(p, m) = 1$, we use the direct decomposition of the category of artinian commutative group schemes (cf. [3]). Thus

$$\begin{aligned} \mathfrak{A}_n &\cong (Z/mZ)^{2r} \times (Z/qZ)^{2f} \times (\mathfrak{A}_q)_{lr} \times (\mathfrak{A}_q)_{ll}, \\ \mathfrak{H}_n &\cong (Z/mZ) \qquad\quad \times \qquad\quad \mathfrak{H}_q. \end{aligned}$$

where f is such that the number of the points of order p in A is p^f ; f is also the dimension of the toroidal part of the completion $\hat{A}$ of A, hence $(\mathfrak{A}_q)_{lr} \cong (\mathfrak{H}_q)^f$. Therefore $\mathrm{Hom}(\mathfrak{A}_n, \mathfrak{H}_n) \cong \mathrm{Hom}(\mathfrak{A}_m, \mathfrak{H}_m) \times \mathrm{Hom}((\mathfrak{H}_q)^f, \mathfrak{H}_q)$, $\mathrm{Hom}(\mathfrak{H}_n, \mathfrak{A}_n) \cong \mathrm{Hom}(\mathfrak{H}_m, \mathfrak{A}_m) \times \mathrm{Hom}(\mathfrak{H}_q, (\mathfrak{H}_q)^f)$. It is known (and easy to show) that the linear dual $(\mathfrak{H}_q)^*$ is isomorphic to Z/qZ. Hence by the linear duality it holds that $\mathrm{Hom}(\mathfrak{H}_q, \mathfrak{H}_q) \cong \mathrm{Hom}(Z/qZ, Z/qZ)$. The non-degeneracy of (5.6) follows from these facts. A direct computation (or the argument above) shows that there is a canonical isomorphism $\mathrm{Hom}(\mathfrak{H}_n, \mathfrak{H}_n) \cong Z/nZ$, sending the identity map to 1 mod. n. Therefore we have proved

THEOREM 4. *Let A be an abelian variety and let n be a positive integer. Then there exists a canonical non-degenerate pairing*

$$\mathrm{Ext}(G_m, A)_n \times \mathrm{Ext}(A, G_m)_n \longrightarrow Z/nZ.$$

If n is not divisible by $\chi(k)$, there is a non-degenerate pairing between $A_n = \{x \in A \mid nx = 0\}$ and $A_n^* = \mathrm{Ext}(A, G_m)_n$, with values in $\mathfrak{H}_n =$ the group of n-th roots of unity in k (cf. [12] § XI, [5] ch. VII.). The relation between this pairing and that of Theorem 4 is as follows: First choose a primitive n-th root of unity ω_n in k. Since $\mathfrak{H}_n$ is generated by ω_n, one can identify Hom $(\mathfrak{H}_n, A)$ with A_n by associating $\varphi(\omega_n)$ to $\varphi \in \mathrm{Hom}(\mathfrak{H}_n, A)$. Similarly one can identify $\mathfrak{H}_n$ with Z/nZ. After these identifications, a straightforward calculation shows that the two pairings are the same.

Returning to the general case, we identify Z/nZ with the group $\{\exp(2\pi\sqrt{-1}\,\nu/n) \mid \nu = 1, \ldots, n\}$ of the unit circle U in the obvious way, and denote the value of the pairing of Th. 4 by $(x, y)_n$ $(x \in \mathrm{Ext}(G_m, A)_n$, $y \in \mathrm{Ext}(A, G_m)_n$. Thus $(x, y)_n \in U$.

Lemma 4. *Let l, n be positive integers, and let $x \in \mathrm{Ext}(G_m, A)_{ln}$, $y \in \mathrm{Ext}(A, G_m)_n$, $x' \in \mathrm{Ext}(G_m, A)_n$, $y' \in \mathrm{Ext}(A, G_m)_{ln}$. Then we have*

$$(lx, y)_n = (x, y)_{ln}, \quad (x', ly')_n = (x', y')_{ln}.$$

Proof. Let G be any object of $\mathscr{B}$ and denote by $\mathfrak{G}_n$ the kernel of the homomorphism $n : G \to G$. Then there exist a monomorphism $i : \mathfrak{G}_n \to \mathfrak{G}_{ln}$ and a homomorphism $l' : \mathfrak{G}_{ln} \to \mathfrak{G}_n$ such that the following diagrams are commutative:

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{G}_n \arrow[r] \arrow[d, "i"'] & G \arrow[r, "n"] \arrow[d, equal] & G \arrow[d, "l"] \\
0 \arrow[r] & \mathfrak{G}_{ln} \arrow[r] & G \arrow[r, "ln"'] & G,
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] & \mathfrak{G}_n \arrow[r] & G \arrow[r, "n"] & G \\
0 \arrow[r] & \mathfrak{G}_{ln} \arrow[r] \arrow[u, "l'"'] & G \arrow[r, "ln"'] \arrow[u, "l"'] & G. \arrow[u, equal]
\end{tikzcd}$$

It follows from these diagrams that $i \circ l' = l$, $l' \circ i = l$, where l denotes the multiplication by l in the respective group. Now consider the left diagram below. It gives rise to the right diagram, which is commutative.

$$\begin{tikzcd}
\mathfrak{H}_n \arrow[r] \arrow[d, "il"'] & G_m \arrow[r, "n"] \arrow[d, equal] & G_m \arrow[d, "l"] \\
\mathfrak{H}_{ln} \arrow[r] & G_m \arrow[r, "ln"'] & G_m,
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Ext}(G_m, A)_n \arrow[r, phantom, "\simeq"] & \mathrm{Hom}(\mathfrak{H}_n, A) \\
\mathrm{Ext}(G_m, A)_{ln} \arrow[r, phantom, "\simeq"] \arrow[u, "l"'] & \mathrm{Hom}(\mathfrak{H}_{ln}, A). \arrow[u]
\end{tikzcd}$$

Thus if $x \in \mathrm{Ext}(G_m, A)_{ln}$ corresponds to $\xi : \mathfrak{H}_{ln} \to A$, then lx corresponds to $\xi \circ i : \mathfrak{H}_n \to A$. Similarly, if $y \in \mathrm{Ext}(A, G_m)_n$ corresponds to $\eta : \mathfrak{A}_n \to G_m$, then the same element y, considered as an element of $\mathrm{Ext}(A, G_m)_{ln}$, corresponds to $\eta \circ l' : A_{ln} \to G_m$. On the other hand, by the identification of Hom (H_n, H_n) with a subgroup of the unit circle U, $\varphi \in \mathrm{Hom}(\mathfrak{H}_n, \mathfrak{H}_n)$ and $i \circ \varphi \circ l' \in \mathrm{Hom}(\mathfrak{H}_{ln}, \mathfrak{H}_{ln})$ corresponds to the same complex number. To show that it suffices to consider the case $\varphi=$ identity, but then the assertion is trivial. Consequently, in the first formula of the Lemma, $(lx, y)_n$ corresponds to $\eta\circ\xi'$, where ξ' is such that $\xi\circ i=i\circ\xi'$, while $(x, y)_{ln}$ corresponds to $(i\circ\eta\circ l')\circ\xi$, and the formula in question is equivalent to $i\circ(\eta\circ\xi')\circ l'=i\circ\eta\circ l'\circ\xi$. But since $i\ l'\circ\xi=l\xi=\xi\circ i\circ l'$ $=i\circ\xi'\circ l'$ and since i is monomorphism it holds $l'\circ\xi=\xi'\circ l'$, and this proves the formula. The second formula is proved in the same way.

$$\begin{tikzcd}
\mathfrak{H}_n \arrow[r, "\xi'"] \arrow[d, shift left, "i"] & \mathfrak{A}_n \arrow[r, "\eta"] \arrow[d, shift left, "i"] & \mathfrak{H}_n \arrow[d, "i"] \\
\mathfrak{H}_{ln} \arrow[r, "\xi"'] \arrow[u, shift left, "l'"] & \mathfrak{A}_{ln} \arrow[r, "i\circ\eta\circ l'"'] \arrow[u, shift left, "l'"] & \mathfrak{H}_{ln}.
\end{tikzcd}$$

Lemma 4 enables us to take the limit of the pairing of Theorem 4. In fact, let N be the set of positive integers partially ordered by means of the divisibility. The inductive system $\{\mathrm{Ext}(G_m, A)_n \mid n\in N\}$, with the inclusion mappings, has $\mathrm{Ext}(G_m, A)$ as the limit, while the projective system $\{\mathrm{Ext}(A, G_n)_n;$ $l' : \mathrm{Ext}(A, G_m)_{ln}\to\mathrm{Ext}(A, G_m)_n\}$ has the Tate group $T(A^*)$ of the dual abelian variety A^* as the limit. The inductive system and the projective system are compatible with respect to the pairing of Theorem 4 by virtue of the second formula of Lemma 4. Therefore:

THEOREM 5. *Let A be an abelian variety and let A^* be its dual. Then* $\mathrm{Ext}(G_m, A)$ *is canonically isomorphic to the character group of the Tate group* $T(A^*)$ *of* A^*.

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