REMARKS ON A THEOREM OF BOURBAKI

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Bourbaki has established the following theorem which we state without proof as

THEOREM A ([3, Theorem 6, § 4]). *Let R be a noetherian integrally closed domain and M a finitely generated torsion free R-module. Then there exists a free submodule F of M such that M/F is isomorphic to an ideal in R.*

It is our purpose in this note to present a few consequences of this theorem. Before giving these results we briefly review some terminology and known results.

We shall assume throughout this paper that all rings are noetherian. Suppose R is a local ring and M is a finitely generated, non-zero R-module. Then a sequence of elements $x_1, \ldots, x_s$ in the maximal ideal $\mathfrak{m}$ of R is called an *M-sequence* if x_i is not a zero-divisor in $M/(x_1, \ldots, x_{i-1})M$ for $i = 1, \ldots, s$. It is easily seen that if $x_1, \ldots, x_s$ is an M-sequence, then $s \leq$ Krull dim R. Thus all M-sequences can be extended to maximal M-sequences. It is well known that all maximal M-sequences have the same length and that this length is the same as the smallest integer $i \geq 0$ such that $\operatorname{Ext}_R^i(R/\mathfrak{m}, M) \neq 0$ (see [2, Proposition 2.9] for instance). We shall denote by $\operatorname{codh}_R M$ the length of a maximal M-sequence.

We now list without proof some of the well-known basic properties of $\operatorname{codh}_R M$. These can easily be derived from the characterization of $\operatorname{codh}_R M$ in terms of the functor $\operatorname{Ext}_R^*(R/\mathfrak{m},)$.

LEMMA 1. *Let R be a local ring with maximal ideal $\mathfrak{m}$ of dimension d. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of non-zero finitely generated R-modules. Then we have*

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a) $\mathrm{Codh}_R M=0$ *if and only if* $\mathfrak{m}\in \mathrm{Ass}(M)$, i.e. *if and only if there exists an exact sequence* $0\to R/\mathfrak{m}\to M$.

b) *If* $\mathrm{codh}_R M>\mathrm{codh}_R M''$, *then* $\mathrm{codh}_R M'=1+\mathrm{codh}_R M''$.

c) *If* $\mathrm{codh}_R M>\mathrm{codh}_R M'$, *then* $\mathrm{codh}_R M'=1+\mathrm{codh}_R M''$.

d) *If* $\mathrm{codh}_R M=\mathrm{codh}_R M'=t$, *then* $\mathrm{codh}_R M''\geq t-1$.

e) *If* $\mathrm{codh}_R M<\mathrm{codh}_R M'$, *then* $\mathrm{codh}_R M=\mathrm{codh}_R M''$.

We now recall that if R is an arbitrary ring and M is an R-module, then M is said to be reflexive if the natural homomorphism $M\to \mathrm{Hom}_R(\mathrm{Hom}_R(M,R),R)$ is an isomorphism. In the following lemma we give various criteria for a module to be reflexive.

LEMMA 2. *Let M be a finitely generated R-module. Then M is a reflexive R-module if and only if M satisfies the following conditions:*

a) *For each prime ideal* $\mathfrak{p}$ *in R such that* $\mathrm{codh}_{R_\mathfrak{p}} R_\mathfrak{p}\geq 1$ *we have that* $M_\mathfrak{p}$ *is* $R_\mathfrak{p}$*-reflexive.*

b) *For each prime ideal* $\mathfrak{p}$ *in R such that* $\mathrm{codh}_{R_\mathfrak{p}} R_\mathfrak{p}\geq 2$ *we have that* $\mathrm{codh}_{R_\mathfrak{p}} M_\mathfrak{p}\geq 2$.

If in addition we assume that R is an integrally closed domain, then we have the following equivalent conditions.

c) *M is reflexive.*

d) *There exists an exact sequence* $0\to M\to F_0\to F_1$ *with the* F_i *free R-modules.*

e) *M is torsion free and for each prime ideal* $\mathfrak{p}$ *in R such that* $\mathrm{ht}(\mathfrak{p})\geq 2$ (*where* $\mathrm{ht}(\mathfrak{p})=$ *Krull* dim $R_\mathfrak{p}$) *we have that* $\mathrm{codh}_{R_\mathfrak{p}} M_\mathfrak{p}\geq 2$.

Proof. Although this lemma is essentially known, we briefly sketch a proof for the convenience of the reader.

Suppose M is reflexive. Then clearly M satisfies condition a). Let $\mathfrak{p}$ be a prime ideal in R such that $\mathrm{codh}_{R_\mathfrak{p}} R_\mathfrak{p}\geq 2$. Then $M_\mathfrak{p}$ is $R_\mathfrak{p}$-reflexive and thus $M_\mathfrak{p}\approx \mathrm{Hom}_{R_\mathfrak{p}}(N, R_\mathfrak{p})$ where N is a finitely generated $R_\mathfrak{p}$-module. Then it follows from [1, Prop. 4.7] that any $R_\mathfrak{p}$-sequence of length two is also an $M_\mathfrak{p}$ sequence. Thus if M is reflexive, then M satisfies conditions a) and b).

Suppose M satisfies a) and b) and let $0\to M'\to M\to M^{**}\to M''\to 0$ be an exact sequence where $M^*=\mathrm{Hom}_R(M,R)$. If $M'\neq 0$, then $\mathrm{Ass}(M')\neq\emptyset$, i.e. there exists an exact sequence $0\to R/\mathfrak{p}\to M'$ for some prime ideal $\mathfrak{p}$. By condition a) we know that $\mathrm{codh}_{R_\mathfrak{p}} R_\mathfrak{p}\geq 2$. But then by Lemma 1, we know that

$\text{codh}_{R_\mathfrak{p}} M_\mathfrak{p} = 0$, which contradicts condition b). Thus we know that $M' = 0$. Suppose $M'' \neq 0$ and $\mathfrak{p}$ is a prime ideal in $\text{Ass}(M'')$. Then by condition a) we know that $\text{codh}_{R_\mathfrak{p}} R_\mathfrak{p} \geq 2$. Therefore by our previous proof the reflexive $R_\mathfrak{p}$-module $(M^{**})_\mathfrak{p} = (M_\mathfrak{p})^{**}$ has $\text{codh}_{R_\mathfrak{p}} M_\mathfrak{p}^{**} \geq 2$. Since $\text{codh}_{R_\mathfrak{p}} (M'')_\mathfrak{p} = 0$, we deduce from the exact sequence $0 \to M_\mathfrak{p} \to (M_\mathfrak{p})^{**} \to (M'')_\mathfrak{p} \to 0$ that $\text{codh}_{R_\mathfrak{p}} M_\mathfrak{p} = 1$ (see Lemma 1). But this contradicts condition b). Therefore it follows that M'' is also zero or that M is reflexive.

Assume now that R is integrally closed.

c) $\Rightarrow$ d). Let $F_1 \to F_0 \to M^* \to 0$ be exact with the F_i finitely generated free R-modules. Then $0 \to M^{**} \to F_0^* \to F_1^*$ is exact with F_i^* being finitely generated free R-modules. Since $M = M^{**}$ we have our desired exact sequence.

d) $\Rightarrow$ e). Let $0 \to M \to F_0 \to F_1$ be an exact sequence with the F_i free R-modules. Then M is certainly torsion free. Suppose $\text{ht}(\mathfrak{p}) \geq 2$. Since R is integrally closed, it follows that $\text{codh}_{R_\mathfrak{p}} R_\mathfrak{p} \geq 2$ and therefore that $\text{codh}_{R_\mathfrak{p}} F_0 \otimes R_\mathfrak{p} \geq 2$. Thus if $\text{codh}_{R_\mathfrak{p}} M_\mathfrak{p} \leq 1$, then by Lemma 1 c) it follows that $\text{codh}_{R_\mathfrak{p}} L_\mathfrak{p} = 0$ where $L = \text{Im}(F_0 \to F_1)$. Therefore we have that $\mathfrak{p} \in \text{Ass}(L)$. But this is impossible since L is torsion free. Thus we have shown that if $\text{ht}(\mathfrak{p}) \geq 2$, then $\text{codh}_{R_\mathfrak{p}} M_\mathfrak{p} \geq 2$.

e) $\Rightarrow$ c). Since R is integrally closed we know that $\text{codh}_{R_\mathfrak{p}} R_\mathfrak{p} \leq 1$ if and only if $\text{ht}(\mathfrak{p}) \leq 1$. For the same reason we know that $R_\mathfrak{p}$ is either a field or a discrete rank one valuation ring depending on whether $\text{ht}(\mathfrak{p})$ is zero or one. Since M is a torsion free, it follows that $M_\mathfrak{p}$ is $R_\mathfrak{p}$ free and therefore reflexive for $\text{ht}(\mathfrak{p}) \leq 1$ or, what is the same thing, for $\text{codh}_{R_\mathfrak{p}} R_\mathfrak{p} \leq 1$. Thus M satisfies part a) of the first part of the lemma. The fact that M satisfies b) is part of condition e). Thus by the first part of the lemma we know that e) $\Rightarrow$ c).

With these preliminaries out of the way we need only recall the definition of a *Cohen-Macauley* ring before stating our first result which is connected with Theorem A. A ring R is called a Cohen-Macauley ring if for each prime ideal $\mathfrak{p}$ in R we have that $\text{codh}_{R_\mathfrak{p}} R_\mathfrak{q} = \text{Krull dim } R_\mathfrak{p}$.

PROPOSITION 3. *Let R be a Cohen-Macauley ring and M a finitely generated reflexive R-module. If $0 \to F \to M \to \mathfrak{a} \to 0$ is an exact sequence of non-zero R-modules with F a free R-module and $\mathfrak{a}$ a proper, non-zero ideal in R, then* $\text{ht}(\mathfrak{p}) \leq 2$ *for each prime ideal* $\mathfrak{p}$ *in* $\text{Ass}(R/\mathfrak{a})$.

If in addition we assume that R is a unique factorization domain, then $\mathfrak{a}$ is *isomorphic to an unmixed ideal* $\mathfrak{b}$ *with* $\mathrm{ht}(\mathfrak{b}) \leq 2$ (i.e. *all the prime ideals* $\mathfrak{p}$ *in* $\mathrm{Ass}(R/\mathfrak{b})$ *have the same height which is at most* 2).

Proof. Suppose there exists a prime ideal $\mathfrak{p}$ in $\mathrm{Ass}(R/\mathfrak{a})$ with $\mathrm{ht}(\mathfrak{p}) \geq 3$. Then we know that $\mathrm{codh}_{R_\mathfrak{p}}(R/\mathfrak{a})_\mathfrak{p} = 0$. Since $\mathrm{codh}_{R_\mathfrak{p}} R_\mathfrak{p} \geq 3$, it follows from the exact sequence $0 \to \mathfrak{a}_\mathfrak{p} \to R_\mathfrak{p} \to (R/\mathfrak{a})_\mathfrak{p} \to 0$ and Lemma 1 that $\mathrm{codh}_{R_\mathfrak{p}} \mathfrak{a}_\mathfrak{p} = 1$. On the other hand since $M_\mathfrak{p}$ is $R_\mathfrak{p}$-reflexive and $\mathrm{codh}_{R_\mathfrak{p}} R_\mathfrak{p} \geq 3$, it follows from Lemma 2, that $\mathrm{codh}_{R_\mathfrak{p}} M_\mathfrak{p} \geq 2$. Therefore we deduce from the exact sequence $0 \to F_\mathfrak{p} \to M_\mathfrak{p} \to \mathfrak{a}_\mathfrak{p} \to 0$ that $\mathrm{codh}_{R_\mathfrak{p}} F_\mathfrak{p} = 2$ (see Lemma 1 b)). But since F is a free R-module we know that the $\mathrm{codh}_{R_\mathfrak{p}} F \geq 3$. This contradiction shows that if $\mathfrak{p} \in \mathrm{Ass}(R/\mathfrak{a})$, then the $\mathrm{ht}((\mathfrak{p}) \leq 2$.

The rest of the proposition follows trivially from what has been established and the following general fact concerning unique factorization domains.

Proposition 4. *Let R be a unique factorization domain and* $\mathfrak{a}$ *a non-principal ideal in R. Then* $\mathfrak{a}$ *is isomorphic to an ideal* $\mathfrak{b}$ *in R such that the* $\mathrm{ht}(\mathfrak{b}) \geq 2$ *and* $\mathrm{Ass}(R/\mathfrak{b}) \subset \mathrm{Ass}(R/\mathfrak{a})$.

Proof. If $\mathrm{ht}(\mathfrak{a}) \geq 2$ we are done. Suppose $\mathrm{ht}(\mathfrak{a}) = 1$ and $\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s \cap \mathfrak{q}_{s+1} \cap \cdots \cap \mathfrak{q}_t = \mathfrak{a}$ a primary decomposition of $\mathfrak{a}$ with $\mathrm{ht}(\mathfrak{q}_i) \geq 2$ for $i = 1, \ldots, s$ and $\mathrm{ht}(\mathfrak{q}_j) = 1$ for $j = s+1, \ldots, t$. Since R is a unique factorization domain, it follows that $q_{s+1} \cap \cdots \cap q_t = (x)$ for some x in R. Therefore we have that $\mathfrak{a} = (x) \cap \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s$. From this it easily follows that $\mathfrak{a} = x\ (\mathfrak{a}\ ;\ x)$. Thus we have $\mathfrak{a} \approx \mathfrak{a} : x$. But $\mathfrak{a} : x = (\mathfrak{q}_1 : x) \cap \cdots \cap (\mathfrak{q}_s : x)$ which shows that the $\mathrm{ht}\mathfrak{a} : x \geq 2$ and $\mathrm{Ass}(R/\mathfrak{a} : x) \subset \mathrm{Ass}(R/\mathfrak{a})$. Therefore $\mathfrak{a} : x$ is our desired ideal $\mathfrak{b}$.

Combining the above remarks with Theorem A we obtain

Theorem B. *Let R be an integrally closed Cohen-Macauley ring. Then we have*:

a) If M is a finitely generated reflexive R-module, then there exists a free submodule F of M such that M/F is isomorphic to an ideal in $\mathfrak{a}$ *such that each* $\mathfrak{p}$ *in* $\mathrm{Ass}(R/\mathfrak{a})$ *has height at most* 2.

b) If N is a finitely generated R-module, then there exists an ideal $\mathfrak{b}$ *in R such that*:

i) If $\mathfrak{p} \in \mathrm{Ass}(R/\mathfrak{b})$, *then the* $\mathrm{ht}(\mathfrak{p}) \leq 2$.

ii) There exist exact sequences of functors

$$\mathrm{Ext}^1_R(\mathfrak{b},\ ) \to \mathrm{Ext}^3_R(N,\ ) \to 0 \ \textit{and}$$

$$0 \to \mathrm{Tor}_3^R(N, \ ) \to \mathrm{Tor}_1^R(\mathfrak{b}, \ ).$$

iii) For each $i \geq 2$ *we have isomorphisms of functors*

$$\mathrm{Ext}_R^i(\mathfrak{b}, \ ) \approx \mathrm{Ext}_R^{i+2}(N, \ ) \text{ and } \mathrm{Tor}_i^R(\mathfrak{b}, \ ) \approx \mathrm{Tor}_{i+2}^R(N, \ ).$$

If in addition, we assume that R is a unique factorization domain then the ideals $\mathfrak{a}$ *and* $\mathfrak{b}$ *can be chosen to be unmixed.*

Proof. a) This part follows immediately from Theorem A and Proposition 3.

b) Let N be a finitely generated R-module and $(*)$ $0 \to M \to F_1 \to F_0 \to N \to 0$ be an exact sequence with the F_i free R-module. Then it follows from Proposition 2, that M is a reflexive R-module. Then by a) we know that there exists an exact sequence $(**)$ $0 \to F \to M \to \mathfrak{b} \to 0$ with F free and $\mathfrak{b}$ satisfying i).

From the exact sequence $(*)$ it follows that $\mathrm{Ext}_R^i(M, \ ) \approx \mathrm{Ext}_R^{i+2}(N, \ )$ and $\mathrm{Tor}_i^R(M, \ ) \approx \mathrm{Tor}_{i+2}^R(N, \ )$ for all $i \geq 1$. While the exact sequence $(**)$ yields the exact sequences $0 \to \mathrm{Tor}_1(M, \ ) \to \mathrm{Tor}_1(\mathfrak{b}, \ )$ and $\mathrm{Ext}_R^1(\mathfrak{b}, \ ) \to \mathrm{Ext}_R^1(M, \ ) \to 0$ and the isomorphisms

$$\mathrm{Tor}_i^R(\mathfrak{b}, \ ) \approx \mathrm{Tor}_i^R(M, \ ) \text{ and } \mathrm{Ext}_R^i(\mathfrak{b}, \ ) \approx \mathrm{Ext}_R^i(M, \ ) \text{ for } i \geq 2.$$

Statements ii) and iii) follow immediately from the above exact sequences and isomorphisms.

The rest of the theorem now follows trivially from what has just been established and the second part of Proposition 3.

As immediate consequences of Theorem B, we have the following corollaries:

COROLLARY 5. *Let R be an integrally closed Cohen-Macauley ring. Suppose the* $\mathrm{pd}_R(\mathfrak{a})$ *(the projective dimension of* $\mathfrak{a}$*) is finite for every ideal* $\mathfrak{a}$ *such that the* $\mathrm{ht}(\mathfrak{p}) \leq 2$ *for each* $\mathfrak{p} \in \mathrm{Ass}(R/\mathfrak{a})$. *Then R is a regular ring,* i.e. $R_\mathfrak{p}$ *is a regular local ring for each prime ideal in R.*

If, in addition, one assumes that R is a unique factorization domain, then R is regular if every unmixed ideal of height 2 has finite projective dimension.

COROLLARY 6. *Let R be a regular local ring of dimension* $d \geq 3$. *If* $\mathfrak{a}$ *is an unmixed ideal of height* 2, *then* $1 \leq \mathrm{pd}_R\mathfrak{a} \leq d-2$. *Further, given any integer i such that* $1 \leq i \leq d-2$, *then there exists an unmixed ideal of height* 2 *such that* *the* $\mathrm{pd}_R \mathfrak{a} = i$.

Proof. The first corollary does not need any explanation. As for the second corollary we first observe that if $\mathfrak{a}$ has height 2, then $\mathfrak{a}$ is not projective, thus the $\mathrm{pd}\,\mathfrak{a} \geq 1$. On the other hand since Krull dim $R = d \geq 3$ and $\mathfrak{a}$ is unmixed of height $2 < d$, then the $\mathrm{pd}_R R/\mathfrak{a} < d$. Thus we see that $1 \leq \mathrm{pd}\mathfrak{a} \leq d-2$.

Suppose $1 \leq i \leq d-2$. Then let N be a finitely generated R-module such that $\mathrm{pd}\,N = i+2$. For instance, we can take $N = R/(x_1, \ldots, x_{i+2})$ where $x_1, \ldots, x_{i+2}$ is an R-sequence of length $i+2$. Since every regular local ring is a unique factorization domain, we know by Theorem B, that there exists an unmixed ideal $\mathfrak{a}$ of height 2 in R such that $\mathrm{Ext}_R^1(\mathfrak{a}, \ ) \to \mathrm{Ext}_R^3(N, \ ) \to 0$ is exact and $\mathrm{Ext}_R^j(\mathfrak{a}, \ ) \approx \mathrm{Ext}_R^{j+2}(N, \ )$ for all $j \geq 2$. From this it follows that the $\mathrm{pd}\,\mathfrak{a} = i$.

We now give an application of Theorem A in a different direction. Before stating our concluding main result, we remind the reader of the definition of a Gorenstein ring. Namely, a ring R is a *Gorenstein* ring if the inj $\dim_{R_\mathfrak{p}} R_\mathfrak{p} < \infty$ for all prime ideals $\mathfrak{p}$ in R. For instance, a regular ring is a Gorenstein ring.

THEOREM C. *Let R be a Gorenstein ring which is also an integrally closed domain with the property that the Krull* $\dim R_\mathfrak{m} \geq 2$ *for each maximal ideal* $\mathfrak{m}$ *in R. If M is an R-module of finite length, then M can be imbedded in a cyclic module of finite length which is an essential extension of M.*

Before proceeding with the proof of Theorem C we first show that it suffices to prove the following special case of Theorem C.

THEOREM C′. *Let R be a local Gorenstein ring of dimension* $n \geq 2$ *which is also an integrally closed domain. If M is an R-module of finite length, then M can be imbedded in a cyclic R-module of finite length.*

Suppose we have established Theorem C′ and M and R satisfy the hypothesis of Theorem C. Then there are only a finite number $\mathfrak{m}_1, \ldots, \mathfrak{m}_t$ of distinct maximal ideals in R containing the annihilator of M. Further, the natural map $M \to \sum_{i=1}^{t} M_{\mathfrak{m}_i}$ (direct sum) is an isomorphism and each $M_{\mathfrak{m}_i}$ has finite length over $R_{\mathfrak{m}_i}$. Therefore by Theorem C′, we have that each $M_{\mathfrak{m}_i} \subset R_{\mathfrak{m}_i}/\mathfrak{q}(i)_{\mathfrak{m}_i}$ where each $\mathfrak{q}(i)$ is an $\mathfrak{m}_i$-primary ideal in R. Since each $\mathfrak{m}_i$ is maximal in R, we know that $R_{\mathfrak{m}_i}/\mathfrak{q}_{\mathfrak{m}_i} \approx R_{\mathfrak{q}_i}$ as R-modules. Thus we see that $M \subset \sum_{i=1}^{t} R/\mathfrak{q}_i$. But since R is the ideal generated by each $\mathfrak{q}_i$ and $\mathfrak{q}_1 \cap \cdots \cap \hat{\mathfrak{q}}_i \cap \cdots \cap \mathfrak{q}_t$, it follows by the Chinese remainder theorem that the natural map $R/\mathfrak{b} \to \sum_{i=1}^{t} R/\mathfrak{q}_i$ where $\mathfrak{b} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_t$ is an isomorphism. Thus we have that $M \subset R/\mathfrak{b}$.

Now suppose $E(M)$ is an injective envelope of M and $f: R/\mathfrak{b} \to E(M)$ is an extension of the identity map on M. Then the Im f is our desired essential cyclic extension of finite length containing M. Thus we see that in order to prove Theorem C it suffices to prove Theorem C′. The rest of this paper is devoted to this task.

From now on we will assume all our rings are local rings. Since the proof of Theorem C′ depends heavily on the properties of Gorenstein rings, we will give a brief review of those portions of the theory of Gorenstein rings which we will need before going on to the proof of Theorem C′. Our general reference for these results is [2].

Suppose R is a Gorenstein ring of Krull dimesion n and $0 \to R \to E_0 \to \cdots \to E_h \to \cdots$ is a minimal resolution of R. Then by the fundamental theorem of [2, § 1] we have that a) $E_h = 0$ for $h > n$; b) $E_n \approx E(R/\mathfrak{m})$ where $\mathfrak{m}$ is the maximal ideal in R; c) $E_h \approx \sum E(R/\mathfrak{p})$ where $\mathfrak{p}$ runs through all prime ideals of height h.

Suppose M is an R-module of finite length, then we have that $\mathrm{Hom}_R(M, E_h) = 0$ for $h < n$. From this it follows that a) $\mathrm{Ext}_R^i(M, R) = 0$ for $i < n$ and b) $\mathrm{Ext}_R^n(M, R) \approx \mathrm{Hom}_R(M, E(R/\mathfrak{m}))$, a module of finite length. Since it is well known that the natural map $M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, E(R/\mathfrak{m})), E(R/\mathfrak{m}))$ is an isomorphism if M has finite length, it follows that $\mathrm{Ext}_R^i(\mathrm{Ext}_R^n(M, R), R) = 0$ for $i < n$ and $\mathrm{Ext}_R^n(\mathrm{Ext}_R^n(M, R), R) \approx M$.

Proposition 7. *Let R be a Gorenstein ring of dimension n. If M is a non-zero finitely generated R-module such that* $\mathrm{codh}_R M = t$, *then n-t is the maximum integer i such that* $\mathrm{Ext}_R^i(M, R) \neq 0$.

Proof. We proceed by induction on t. Suppose $t = 0$. Then there exists an exact sequence $0 \to R/\mathfrak{m} \to M$ (see Lemma (a)). Since the inj $\dim_R R = n$, it follows that the induced map $\mathrm{Ext}_R^n(M, R) \to \mathrm{Ext}_R^n(R/\mathfrak{m}, R)$ is an epimorphism. Since $\mathrm{Ext}_R^n(R/\mathfrak{m}, R) \approx \mathrm{Hom}_R(R/\mathfrak{m}, E(R/\mathfrak{m})) \approx R/\mathfrak{m}$, it follows that $\mathrm{Ext}_R^n(M,R) \neq 0$

which gives us our desired result when $t=0$.

Suppose the proposition is true for $0 \leq t < s$ and suppose $\operatorname{codh}_R M = s$. Let $x \in \mathfrak{m}$ be a non-zero divisor for M. Then we know that the $\operatorname{codh}_R M/sM = s-1$. Therefore we know by the inductive hypothesis that $\operatorname{Ext}_R^{n-s+1}(M/xM, R) \neq 0$ and $\operatorname{Ext}_R^j(M/xM, R) = 0$ for $j > n-s+1$. From the long exact sequence

$$\cdots \to \operatorname{Ext}_R^i(M, R) \xrightarrow{x} \operatorname{Ext}^i(M, R) \to \operatorname{Ext}^{i+1}(Mx/M, R) \to \cdots$$

derived from the exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$, it follows that $\operatorname{Ext}_R^i(M, R) \xrightarrow{x} \operatorname{Ext}_R^i(M, R)$ is an epimorphism for all $i \geq n-s+1$. Since $x \in \mathfrak{m}$ and the $\operatorname{Ext}_R^i(M, R)$ are finitely generated R-modules, it follows from Nakayama's lemma that $\operatorname{Ext}_R^i(M, R) = 0$ for all $i \geq n-s+1$.

On the other hand, it follows now that the map $\operatorname{Ext}_R^{n-s}(M, R) \to \operatorname{Ext}_R^{n-s+1}(M/xM, R)$ is an epimorphism. Since $\operatorname{Ext}_{R+1}^{n-s}(M/xM, R) \neq 0$, we know that $\operatorname{Ext}_R^{n-s}(M, R) \neq 0$, which establishes the proposition for $t=s$ and thus completes the proof of the proposition.

COROLLARY 8. *Let R be a Gorenstein ring of dimension n and let M be a finitely generated R-module. If M' is the maximal submodule of M of finite length, then the natural map* $\operatorname{Ext}^n(M, R) \to \operatorname{Ext}^n(M', R)$ *is an isomorphism.*

Proof. Consider the exact sequence $0 \to M' \to M \to M'' \to 0$. If $M''=0$, we are done. If $M'' \neq 0$, then we know that $\operatorname{codh}_R M'' > 0$ and thus by Proposition 7 that $\operatorname{Ext}_R^n(M'', R) = 0$. Our desired result now follows from the exact sequence

$$\operatorname{Ext}^n(M'', R) \to \operatorname{Ext}^n(M, R) \to \operatorname{Ext}^n(M', R) \to 0.$$

We now return to the proof of Theorem C′. Suppose R and M satisfy the hypothesis of Theorem C′. Let $0 \to K \to G \to M \to 0$ be an exact sequence of R-modules with G a finitely generated free R-module. Then by Theorem A we know that there exists an exact sequence $0 \to F \to K \to \mathfrak{a} \to 0$ with F a free R-module and $\mathfrak{a}$ an ideal in R. Since Krull dim $R = n \geq 2$, it follows that the induced map $\operatorname{Ext}_R^{n-1}(\mathfrak{a}, R) \to \operatorname{Ext}_R^{n-1}(K, R)$ is an epimorphism. Since it follows from the exact sequences $0 \to K \to G \to M \to 0$ and $0 \to \mathfrak{a} \to R \to R/\mathfrak{a} \to 0$ that $\operatorname{Ext}_R^n(M, R) \approx \operatorname{Ext}_R^{n-1}(K, R)$ and $\operatorname{Ext}_R^n(R/\mathfrak{a}, R) \approx \operatorname{Ext}_R^{n-1}(\mathfrak{a}, R)$ (remember that $n \geq 2$), we obtain an epimorphism $\operatorname{Ext}_R^n(R/\mathfrak{a}, R) \to \operatorname{Ext}_R^n(M, R)$.

Now let $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_t$ be a primary decomposition of $\mathfrak{a}$. Since $\operatorname{Ext}_R^n(R/\mathfrak{a}, R) \to \operatorname{Ext}_R^n(M, R) \to 0$ is exact and since we may assume that M and thus

$\operatorname{Ext}_R^n(M, R)$ are not zero, it follows from Corollary 8 that one of the $\mathfrak{q}_i$ is $\mathfrak{m}$-primary. Let us assume that $\mathfrak{q}_1$ is $\mathfrak{m}$-primary. From the fact that $R/\mathfrak{q}_1$ is the maximal submodule of $\sum_{i=1}^{t} R/\mathfrak{q}_i$ of finite length, it follows from Corollary 8 that $\operatorname{Ext}_R^n(\sum_{i=1}^{t} R/\mathfrak{q}_i, R) = \operatorname{Ext}_R^n(R/\mathfrak{q}_1, R)$. Therefore the exact sequence $0 \to R/\mathfrak{a} \to \sum_{i=1}^{t} R/\mathfrak{q}_i$ gives us an epimorphism $\operatorname{Ext}_R^n(R/\mathfrak{q}_1, R) \to \operatorname{Ext}_R^n(R/\mathfrak{a}, R)$. Therefore we have an epimorphism $\operatorname{Ext}_R^n(R/\mathfrak{q}_1, R) \to \operatorname{Ext}_R^n(M, R)$, or what is the same thing, we have an epimorphism $\operatorname{Hom}_R(R/\mathfrak{q}_1, E(R/\mathfrak{m})) \to \operatorname{Hom}_R(M, E(R/\mathfrak{m}))$. Applying the exact functor $\operatorname{Hom}_R(\ , E(R/\mathfrak{m}))$ to this epimorphism we obtain the exact sequence

$$0 \to \operatorname{Hom}_R(\operatorname{Hom}_R(M, E(R/\mathfrak{m})), E(R/\mathfrak{m})) \to \operatorname{Hom}_R(\operatorname{Hom}_R(R/\mathfrak{q}_1, E(R/\mathfrak{m})), E(R/\mathfrak{m})).$$

By the duality theorem for modules of finite length, we obtain an exact sequence $0 \to M \to R/\mathfrak{q}_1$. Thus we have shown that M is a submodule of a cyclic module of finite length. This completes the proof of Theorem C′.

Remark. The fact that the condition the Krull dim $R \geq 2$ in Theorem C′ is necessary can easily be seen by considering fields and discrete rank one valuation rings for which the theorem is obviously false.

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