# EXISTENCE OF PERFECT PICARD SETS

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Dedicated to the memory of Professor TADASI NAKAYAMA

1. Let $E$ be a totally disconnected compact set in the $z$-plane and let $\Omega$ be its complement with respect to the extended $z$-plane. Then $\Omega$ is a domain and we can consider a single-valued meromorphic function $f(z)$ in $\Omega$ which has a transcendental singularity at each point $\zeta \in E$. Suppose that $E$ is a null-set of the class $W$ in the sense of Kametani [4] ( = the class $N_{\mathfrak{B}}$ in the sense of Ahlfors and Beurling [1]). Then the cluster set of $f(z)$ at each transcendental singularity is the whole $w$-plane and hence $f(z)$ has an essential singularity at each point of $E$. We shall say that a value $w$ is exceptional for $f(z)$ at an essential singularity $\zeta \in E$ if there exists a neighborhood of $\zeta$ where the function $f(z)$ does not take this value $w$. If each $f(z)$ has at most $n$ exceptional values at each singularity $\zeta \in E$, we shall call $E$ an $n$-Picard set using the terminology of Lehto [5] and call a 2-Picard set a Picard set simply. For any $E$, by Besse's theorem, there exists a single-valued regular function $g(z)$ in $\Omega$ possessing $E$ as the set of singularities. Therefore, considering the function $\exp g(z)$ in $\Omega$, we see that there exists no 1-Picard set. Thus we need consider $n$-Picard sets only for $n \geqq 2$.

For any countable $E$, every $f(z)$ has at most two exceptional values at each singularity $\zeta \in E$, because any neighborhood of $\zeta$ contains isolated points of $E$, and hence $E$ is a Picard set. But for a non-countable $E$, there needs some condition in order to be an $n$-Picard set for some $n$, even if $E$ is of logarithmic capacity zero (see Matsumoto [6]). Carleson [3] and the author [7], [8] have given sufficient conditions for sets $E$ to be $n$-Picard sets for $n$ *not smaller than* 3 and examples of perfect $E$ by means of Cantor sets. There has remained a very interesting problem unsolved. *Is there a perfect Picard set?*

The purpose of this paper is to give Cantor sets which are Picard sets. The Schottky theorem will also play important roles as in papers [3], [7], [8].

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2. We shall consider the Riemann sphere $\Sigma$ with radius 1/2 touching the $w$-plane at the origin. For any two points $w$ and $w'$ in the $w$-plane we denote by $[w, w']$ the chordal distance between them, that is,

$$[w, w'] = \begin{cases} \dfrac{|w-w'|}{\sqrt{(1+|w|^2)(1+|w'|^2)}} & \text{if } w \neq \infty \text{ and } w' \neq \infty \\ \dfrac{1}{\sqrt{1+|w|^2}} & \text{if } w' = \infty. \end{cases}$$

Further we denote by $C(w;\delta)$ $(\delta>0)$ the spherical open disc with center $w$ and with chordal radius $\delta$.

First we shall prove the following lemma which is a revised form of Carleson's [3].

LEMMA 1. *Let $w=f(z)$ be a single-valued meromorphic function on an annulus $1 \leqq |z| \leqq e^{\mu}$ $(\mu>0)$. If $f(z)$ takes there no value in a spherical disc $C(w_0;\delta)$, then there exists a positive constant $A_\delta$ depending only on $\delta$ such that the diameter of the image of $|z|=e^{\mu/2}$ by $f(z)$ with respect to the chordal distance is dominated by $A_\delta e^{-\mu/2}$ for sufficiently large $\mu$.*

*In particular, if $\delta$ is sufficiently close to 1, that is, the complementary spherical disc $C(-1/\bar{w}_0; d)$ of $C(w_0;\delta)$, $d=\sqrt{1-\delta^2}$, has a radius sufficiently small, we have*

$$A_\delta < Bd,$$

*where $B$ is a positive constant.*

*Proof.* We may assume without any loss of generality that the center $w_0$ of $C(w_0;\delta)$ is the point at infinity, for otherwise we can transform $w_0$ to the point at infinity by the linear transformation $(1+\bar{w}_0 w)/(w-w_0)$, under which the chordal distance remains invariant. Let $|w|>M$ be the domain in the $w$-plane corresponding to $C(w_0;\delta)$. Then

$$|f(z)| \leqq M \quad \text{on} \quad 1 \leqq |z| \leqq e^{\mu}.$$

By Cauchy's integral theorem, we have

$$f'(z) = \frac{1}{2\pi i}\left\{\int_{|\zeta|=e^{\mu}} \frac{f(\zeta)}{(\zeta-z)^2}d\zeta - \int_{|\zeta|=1} \frac{f(\zeta)}{(\zeta-z)^2}d\zeta\right\}$$

for every $z$ on $|z|=e^{\mu/2}$ and hence, if $\mu \geqq 2$,

$$|f'(z)| \leqq \frac{M}{2\pi}\left\{\frac{2\pi e^{\mu}}{(e^{\mu}-e^{\mu/2})^2}+\frac{2\pi}{(e^{\mu/2}-1)^2}\right\} \leqq \frac{2e^2}{(e-1)^2}Me^{-\mu}.$$

Therefore we have

$$\int_{|z|=e^{\mu/2}} |f'(z)||dz| \leqq \frac{2e^2}{(e-1)^2}Me^{-\mu}\cdot 2\pi e^{\mu/2} = \frac{4\pi e^2}{(e-1)^2}M\cdot e^{-\mu/2}.$$

The left side is the length of the image curve $f(|z|=e^{\mu/2})$, and hence the diameter of the image of $|z|=e^{\mu/2}$ by $f(z)$ with respect to the metric $|dw|$, consequently with respect to the chordal distance, is dominated by $(2\pi e^2/(e-1)^2)Me^{-\mu/2}$. We can take $(2\pi e^2/(e-1)^2)M$ as $A_\delta$, for $M$ depends only on $\delta$.

If $d<1/2$, then $M<2d$. Hence

$$B=4\pi e^2/(e-1)^2$$

is one of the wanted. Our lemma is established.

Now let $w=f(z)$ be a single-valued regular function in an annulus $1<|z|<e^{\mu}$ $(\mu>0)$ omitting two values 0 and 1. We use Bohr-Landau's theorem [2]; if $g(z)$ is regular in $|z|<1$ and $g(z)\neq 0$, 1 there, then

$$\max_{|z|=r}|g(z)| \leqq \exp\left(\frac{K\log(|g(0)|+2)}{1-r}\right) \qquad \text{for any } r,\ 0\leqq r<1,$$

where $K$ is a positive constant (a precise form of Schottky's theorem). From this we can prove the following corollary of Lemma 1.

COROLLARY. *There exists a positive constant $A$ not depending on $\mu$ and $f(z)$ such that the diameter of the image of $|z|=e^{\mu/2}$ by $f(z)$ with respect to the chordal distance is dominated by $Ae^{-\mu/2}$ for sufficiently large $\mu$.*

*Proof.* From Bohr-Landau's theorem, we can see easily that if $w=g(z)$ is a regular function in $1<|z|<e^{\sigma}$ $(\sigma>0)$ such that

$$g(z)\neq 0,\ 1 \qquad \text{and} \qquad \min_{|z|=e^{\sigma/2}}|g(z)|<a \qquad \text{for a positive } a,$$

then there is a positive constant $b$ depending only on $a$ and $\sigma$ such that

$$\max_{|z|=e^{\sigma/2}}|g(z)| \leqq b.$$

For a fixed $\sigma>0$, we shall show, supposing $\mu>\sigma$, that there exists a positive number $\delta$ not depending on $\mu$ and $f(z)$ such that the image curves of $|z|=e^{\sigma/2}$ and $|z|=e^{\mu-\sigma/2}$ by $f(z)$ lie outside at least one of three discs $C(0;\delta)$, $C(1;\delta)$

and $C(\infty;\delta)$. In fact, let $z_1$ and $z_2$ be points on $|z|=e^{\sigma/2}$ and $|z|=e^{\mu-\sigma/2}$ respectively. Then $f(z_1)$ and $f(z_2)$ lie outside at least one of three discs $C(0;\delta')$, $C(1;\delta')$ and $C(\infty;\delta')$, where $\delta'$ is a positive number such that these three discs are mutually disjoint and hence can be taken independently of $\mu$ and $f(z)$. Suppose that $f(z_1)$ and $f(z_2)$ lie outside $C(\infty;\delta')$. Then by the fact mentioned above, we can find a positive $\delta_\infty$ such that the image curves of $|z|=e^{\sigma/2}$ and $|z|=e^{\mu-\sigma/2}$ by $f(z)$ lie outside $C(\infty;\delta_\infty)$. Next suppose that $f(z_1)$ and $f(z_2)$ lie outside $C(1;\delta')$. Then we see using the linear transformation which transforms points $w=0$, $w=1$ and $w=\infty$ to points $w=1$, $w=\infty$ and $w=0$ respectively that there is a positive $\delta_1$ such that the images of $|z|=e^{\sigma/2}$ and $|z|=e^{\mu-\sigma/2}$ by $f(z)$ lie outside $C(1;\delta_1)$. Similarly we can find a positive $\delta_0$ and set

$$\delta=\min\{\delta_0,\ \delta_1,\ \delta_\infty\}>0.$$

Obviously this $\delta$ satisfies our conditions. Now by the maximum principle we see that the image of $e^{\sigma/2}\leqq|z|\leqq e^{\mu-\sigma/2}$ by $f(z)$ lies outside at least one of $C(0;\delta)$, $C(1;\delta)$ and $C(\infty;\delta)$. Hence by Lemma 1 we can conclude that the diameter of the image of $|z|=e^{\mu/2}$ by $f(z)$ with respect to the chordal distance is dominated by $A_\delta e^{-(\mu-\sigma)/2}$, so that $A=A_\delta e^{\sigma/2}$ satisfies our condition.

3. Let $E$ be a Cantor set on the closed interval $I_0$: $[-1/2,\ 1/2]$ on the real axis of the $z$-plane with successive ratios $\xi_n$, $0<\xi_n=2\ell_n<2/3$. Defining the Cantor set $E$, we repeat successively to exclude an open segment from the middle of another segment and there remain $2^n$ segments of equal length $\prod_{k=1}^{n}\ell_k$ after we repeat $n$ times, beginning with the interval $I_0$. We denote these segments by $I_{n,k}$ $(n=1,2,\ldots;\ k=1,2,\ldots,2^n)$ and denote by $S_{n,k}$ $(n=1,2,\ldots;\ k=1,2,\ldots,2^n)$ the following annuli on the complementary domain $\Omega$ of $E$:

$$S_{n,k}=\{z;\ (\textstyle\prod_{k=1}^{n}\ell_k)(1-\ell_{n+1})<|z-z_{n,k}|<(\prod_{k=1}^{n-1}\ell_k)(1-\ell_n)/2\},$$

where $z_{n,k}$ is the middle point of $I_{n,k}$. The harmonic modulus $\mu_n$ of $S_{n,k}$ is greater than $\log(2/3\,\xi_n)$. We map $S_{n,k}$ conformally onto the annulus $1<|\eta|<e^{\mu_n}$ and consider the inverse image $\Gamma_{n,k}$ of the circle $|\eta|=e^{\mu_n/2}$ on $S_{n,k}$. Supposing that $S_{n,k}$ encloses $S_{n+1,2k-1}$ and $S_{n+1,2k}$, we denote by $\Delta_{n,k}$ the triply connected domain bounded by three curves $\Gamma_{n,k}$, $\Gamma_{n+1,2k-1}$ and $\Gamma_{n+1,2k}$. We now prove the following

LEMMA 2. *Let the successive ratios $\xi_n$ satisfy the condition*

$$\lim_{n\to\infty} \xi_n = 0$$

*and let $w = f(z)$ be a single-valued regular function in $\Omega$ which omits two values* 0 *and* 1 *and has $E$ as the set of essential singularities. Then for any sufficiently small $\delta < 0$, there exists an infinite number of $\Delta_{n,k}$ such that the images of the three boundary components $\Gamma_{n,k}$, $\Gamma_{n+1,2k-1}$ and $\Gamma_{n+1,2k}$ are contained completely in the three discs $C(0;\ \delta)$, $C(1;\ \delta)$ and $C(\infty;\ \delta)$ one by one, where we assume that these three discs are mutually disjoint, and $f(z)$ takes each value outside the union of these three discs once and only once in $\Delta_{n,k}$.*

*Proof.* Contrary suppose that there exists only a finite number of $\Delta_{n,k}$ such that their three boundary components are mapped into the three discs $C(0;\ \delta)$, $C(1;\ \delta)$ and $C(\infty;\ \delta)$ one by one, and denote by $n_0$ the maximum of $n$ taken over all such $\Delta_{n,k}$. Since

$$\mu_n > \log(2/3\,\xi_n) \qquad \text{and} \qquad \lim_{n\to\infty} \xi_n = 0,$$

we can take $n_1 \geqq n_0$, for a fixed $\sigma > 0$, so large that for each $n \geqq n_1$

$$\mu_n > 2 + \sigma, \ Ae^{-\mu_n/2} < K = \min\{1/24,\ \delta/3\} \qquad \text{and} \qquad Be^{-\mu_n/2} < 1/12,$$

where $A$ and $B$ are the constants in Lemma 1 and its corollary. The diameter of the image of $\Gamma_{n,k}$ with respect to the chordal distance is dominated by $Ae^{-\mu_n/2}$, consequently by $K$, if $n \geqq n_1$, and hence there exists a spherical disc $C_{n,k}$ with chordal radius $K$ which contains completely the image $f(\Gamma_{n,k})$. For $n \geqq n_1$ take $\Delta_{n,k}$ with boundary curves $\Gamma_{n,k}$, $\Gamma_{n+1,2k-1}$ and $\Gamma_{n+1,2k}$. Then, since $K < \delta/3$, at least one of $C(0;\ \delta/3)$, $C(1;\ \delta/3)$ and $C(\infty;\ \delta/3)$, say $C(\infty;\ \delta/3)$, is disjoint from the union of $C_{n,k}$, $C_{n+1,2k-1}$ and $C_{n+1,2k}$ and hence each one of $C_{n,k}$, $C_{n+1,2k-1}$ and $C_{n+1,2k}$ cannot be disjoint from the union of the other two, for, if so for some one, there is $z_0 \in \Delta_{n,k}$ such that $f(z_0)$ lies outside the union of $C_{n,k}$, $C_{n+1,2k-1}$ and $C_{n+1,2k}$ and can be joined to the point at infinity with a curve $\Lambda$ lying outside this union, and we are led to a contradiction that the element of the inverse function $f^{-1}$ corresponding to $z_0$ can be continued analytically along $\Lambda$ up to a point arbitrarily near the point at infinity so that $f(z)$ takes the value $\infty$ in $\Delta_{n,k}$. Therefore we can conclude that

(1°) for every $\Delta_{n,k}$, $n \geqq n_1$, there is a spherical disc with chordal radius

$3K$ containing completely the image $f(\Delta_{n,k})$.

Next we shall consider $\Gamma_{n,k}$ for $n \geqq n_1+1$. Then $\Delta_{n,k}$ and some $\Delta_{n-1,k'}$ have $\Gamma_{n,k}$ as the common boundary and $\Delta_{n-1,k'} \cup \Gamma_{n,k} \cup \Delta_{n,k} \supset S_{n,k}$. From (1°), the image of $\Delta_{n-1,k'} \cup \Gamma_{n,k} \cup \Delta_{n,k}$, consequently the image of $S_{n,k}$, are contained in a spherical disc with chordal radius $6K<1/2$, so that, applying Lemma 1 in $S_{n,k}$ for $d=6K$, we see that the diameter of $f(\Gamma_{n,k})<6KBe^{-\mu_n/2}<K/2$. Hence for $n \geqq n_1+1$, each boundary component of $\Delta_{n,k}$ has the image with diameter less than $K/2$. From the same reason as above we now conclude that

(2°) for $n \geqq n_1+1$, the image of any $\Delta_{n,k}$ is contained in a spherical disc with chordal radius $3K/2$.

By induction we also see for every $p \geqq 1$ that

($p$°) for $n \geqq n_1+p-1$, the image of any $\Delta_{n,k}$ is contained in a spherical disc with chordal radius $3K/2^{p-1}$.

Let $\Omega_{n,k}$ be the part of $\Omega$ bounded by the simple closed curve $\Gamma_{n,k}$ and let $z_0$ be a point of $\Gamma_{n,k}$. Then for any $z \in \Omega_{n,k}$, there is a $\Delta_{n+p,q}$ whose closure contains $z$ and we can find a chain $\{\Delta_{n+i,j(i)}\}$ $(i=0, 1, \ldots, p;\ j(0)=k, j(p)=q)$ joining $\Delta_{n,k}$ and $\Delta_{n+p,q}$. Supposing that $n \geqq n_1$, we have by ($p$°) obtained above

$$[f(z_0), f(z)] \leqq \sum_{i=0}^{p} \text{diam. of } f(\Delta_{n+i,j(i)}) \text{ w.r.t. the chordal distance}$$

$$\leqq 2\sum_{i=0}^{p} 3K/2^i \leqq 12K < 1/2.$$

By means of a linear transformation we can consider from the above that $f(z)$ is bounded in $\Omega_{n,k}$. On the other hand, $E$ is a linear set of linear measure zero, so that $E$ is a null-set of the class $W$ (Kametani [4], Ahlfors and Beurling [1]). Hence each point of the part of $E$ contained in the interior of $\Gamma_{n,k}$ must be a removable singularity of the bounded function $f(z)$; this contradicts our assumption that each point of $E$ is an essential singularity of $f(z)$.

Next suppose that $f(z)$ takes a value $w_0$ outside the union of $C(0;\delta)$, $C(1;\delta)$ and $C(\infty;\delta)$ at two points $z'$ and $z''$ in $\Delta_{n,k}$, whose boundary components $\Gamma_{n,k}$, $\Gamma_{n+1,2k-1}$ and $\Gamma_{n+1,2k}$ are mapped into the discs $C(0;\delta)$, $C(1;\delta)$ and $C(\infty;\delta)$ one by one, and join $w_0$ to $C(0;\delta)$ and $C(1;\delta)$ with curves $\Lambda'$ and $\Lambda''$ respectively, which lie outside the union, do not intersect each other except at $w_0$ and do not pass through any projection of branch points of the

Riemannian image of $\Delta_{n,k}$ by $f(z)$. The elements of the inverse function $f^{-1}$ corresponding to $z'$ and $z''$ can be continued analytically along these curves to their end points and further from them along radii of $C(0;\delta)$ and $C(1;\delta)$ so that the curves in $\Delta_{n,k}$ corresponding to these continuations join each of $z'$ and $z''$ to $\Gamma_{n+1,2k-1}$ and $\Gamma_{n+1,2k}$, where we assume that the images $\Gamma_{n+1,2k-1}$ and $\Gamma_{n+1,2k}$ are contained in $C(0;\delta)$ and $C(1;\delta)$ respectively, and bound with parts of $\Gamma_{n+1,2k-1}$ and $\Gamma_{n+1,2k}$ a domain not containing $\Gamma_{n,k}$. Since $\Delta_{n,k}$ has no boundary other than $\Gamma_{n,k}$, $\Gamma_{n+1,2k-1}$ and $\Gamma_{n+1,2k}$, this domain must be a subdomain of $\Delta_{n,k}$ and $f(z)$ must take the value $\infty$ there; this is a contradiction. Our proof is now complete.

4. We note that for each $n$ and $k$ $(n=1,2,\dots; k=1,2,\dots,2^n)$ $\Delta_{n,k}$ is bounded by $\Gamma_{n,k}$, $\Gamma_{n+1,2k-1}$ and $\Gamma_{n+1,2k}$ and lies on the right-side of $\Delta_{n,k'}$, $1\leqq k'<k$. We now estimate the harmonic modulus of any doubly connected domain contained in $\Delta_{n,k}\cup\Gamma_{n+1,2k-1}\cup\Delta_{n+1,2k-1}$ such that one connected component of its complement contains the circles $\Gamma_{n,k}$ and $\Gamma_{n+1,2k}$ and the other contains the circles $\Gamma_{n+2,4k-3}$ and $\Gamma_{n+2,4k-2}$. Moving this domain in parallel so that the right end point of $I_{n+1,2k-1}$ comes to the origin, we see that the harmonic modulus of our domain is dominated by

$$\log\Psi((\Pi_{k=1}^{n}\ell_k)(1-\xi_{n+1})/\Pi_{k=1}^{n+1}\ell_k)=\log\Psi(2(1-\xi_{n+1})/\xi_{n+1}),$$

where $\log\Psi(P/\rho)$ $(P,\rho>0)$ denotes the harmonic modulus of the normal domain of Teichmüller, the complement of the union of the two segments, $-\rho\leqq x\leqq 0$, $y=0$ and $P\leqq x\leqq+\infty$, $y=0$ in the $z$-plane $(z=x+iy)$. It is well-known that

$$\Psi(P/\rho)<16\frac{P}{\rho}+8,$$

and we obtain thus the following

LEMMA 3. *The harmonic modulus of any doubly connected domain considered above is dominated by* $\log(32/\xi_{n+1})$.

5. Now we shall show the existence of perfect Picard sets. We shall prove the following

THEOREM. *Let $E$ be a Cantor set on the closed interval $I_0$;* $[-1/2, 1/2]$ *on the real axis of the $z$-plane with successive ratios $\xi_n$, $0<\xi_n<2/3$. If the ratios $\xi_n$ satisfy the condition*

$$\xi_{n+1} = o(\xi_n^2),$$

*then E is a Picard set.*

*Proof.* We take $\delta > 0$ so small that the discs $C(0; 2\delta)$, $C(1; 2\delta)$ and $C(\infty; 2\delta)$ are mutually disjoint. Contrary to our assertion, let us suppose that there exists a single-valued meromorphic function $f(z)$ in the complementary domain $\Omega$ of $E$ which has $E$ as the set of essential singularities and has three exceptional values at an essential singularity $\zeta \in E$, where we may assume that these values are 0,1 and $\infty$. Since our argument given in the below is applicable locally, it will not bring any loss of generality if we shall give a contradiction under the stronger assumption that $f(z)$ omits the values 0,1 and $\infty$ in $\Omega$.

Let $n_0$ be so large that $Ae^{-\mu_n/2} < \delta/2$ for any $n \geqq n_0$. By Lemma 2 there is a $\Delta_{n,k}$ $(n \geqq n_0)$ whose three boundary components are mapped into $C(0; \delta)$, $C(1; \delta)$ and $C(\infty; \delta)$ one by one, where we may assume that the boundary curve $\Gamma_{n+1,2k-1}$ of $\Delta_{n,k}$ is mapped into $C(\infty; \delta)$. Now we consider the quadruply connected domain $D = \Delta_{n,k} \cup \Gamma_{n+1,2k-1} \cup \Delta_{n+1,2k-1}$. The images of the boundary curves $\Gamma_{n+2,4k-3}$ and $\Gamma_{n+2,4k-2}$ of $D$ are contained in some spherical discs $C$ and $C'$ with radius $Ae^{-\mu_{n+2}/2} < \delta/2$ respectively, and we see that $C$ and $C'$ are contained in $C(\infty; 2\delta)$. In fact, one of them, say $C$, must contain the point at infinity, for otherwise, $f(z)$ must take the value $\infty$ in $D$, and hence is contained in $C(\infty; \delta)$. Suppose that $C'$ is not contained in $C(\infty; 2\delta)$. Then there is a point $z \in \Delta_{n+1,2k-1} \subset D$ whose image $f(z)$ lies in $C(\infty; 2\delta) - C(\infty; \delta) \cup C'$ and can be joined the origin or the point $w = 1$ with a path not intersecting the image of the boundary of $\Delta_{n+1,2k-1}$, so that $f(z)$ takes the value 0 or 1 in $\Delta_{n+1,2k-1}$. Contradiction. Thus we can find a positive $d$ such that $d < 2\delta$ and the disc $C(\infty; d)$ contains $C$ and $C'$.

Next we shall prove that $f(z)$ takes each value outside the union of the three discs $C(0; \delta)$, $C(1; \delta)$ and $C(\infty; d)$ once and only once in $D$. By Lemma 2, $f(z)$ takes each value outside the union of $C(0; \delta)$, $C(1; \delta)$ and $C(\infty; \delta)$ once and only once in $\Delta_{n,k}$, so that the inverse image $\Gamma$ of the circle $[w, \infty] = 2\delta$ on $\Delta_{n,k}$ is a simple closed curve and separates $\Gamma_{n,k}$ and $\Gamma_{n+1,2k}$ from $\Gamma_{n+2,4k-3}$ and $\Gamma_{n+2,4k-2}$. Now suppose that $f(z)$ takes a value $w_0$ outside the union of $C(0; \delta)$, $C(1; \delta)$ and $C(\infty; d)$ at two points $z'$ and $z''$ in $D$, and join $w_0$ with $C(0; \delta)$ with a curve $\Lambda$ which lies outside the union and does not pass through any projection of branch points of the Riemannian image of $D$ by $f(z)$. The elements of the inverse function $f^{-1}$ corresponding to $z'$ and $z''$ can be continued analytically along $\Lambda$ to its end point and hence we see that every value on $\Lambda$ is taken by $f(z)$ at least two times in $D$. Therefore we can assume that $w_0$ lies outside $C(\infty; 2\delta)$. Then one of $z'$ and $z''$ must lie in the domain $D'$ bounded by $\Gamma$, $\Gamma_{n+2,4k-3}$ and $\Gamma_{n+2,4k-2}$ and the corresponding element of $f^{-1}$ can be continued analytically to the origin along a curve outside $C(\infty; 2\delta)$, so that $f(z)$ takes the value 0 in $D'$; this contradicts our assumption.

Now we estimate $d$ from below. To this purpose we consider the annulus $R$: $2<|w|<\sqrt{1-d^2}/d$ corresponding to the annulus $1/\sqrt{5}>[w, \infty]>d$ on the Riemann sphere $\Sigma$, which separates $C(0; \delta)$ and $C(1; \delta)$ from $C(\infty; d)$. As we have seen above, the Riemannian image of $D$ covers $R$ univalently, the ring domain on $D$ corresponding to $R$, which has the same harmonic modulus as $R$, separates the boundary curves $\Gamma_{n,k}$ and $\Gamma_{n+1,2k}$ of $D$ from the boundary curves $\Gamma_{n+2,4k-3}$ and $\Gamma_{n+2,4k-2}$ of $D$. By Lemma 3 we have thus

$$\text{har. mod. of } R=\log(\sqrt{1-d^2}/2d)\leqq\log(32/\xi_{n+1}).$$

Since $d<2\delta<\pi/6$, we have the estimate

$$d\geqq(\sqrt{1-(\pi/6)^2}/64)\xi_{n+1}=\frac{1}{M}\xi_{n+1}.$$

This implies that $C'$ must intersect the disc $[w, \infty]\geqq\xi_{n+1}/M=m$. Consider the domain $\Delta_{n+2,4k-2}$ which, with $\Delta_{n+1,2k-1}$, has $\Gamma_{n+2,4k-2}$ as the common boundary. The images of three boundary curves $\Gamma_{n+2,4k-2}$, $\Gamma_{n+3,8k-5}$ and $\Gamma_{n+3,8k-4}$ of $\Delta_{n+2,4k-2}$ are contained in some three spherical discs $C'=C_{n+2,4k-2}$, $C_{n+3,8k-5}$ and $C_{n+3,8k-4}$ with radii less than $Ae^{-\mu_{n+2}/2}<A'\sqrt{\xi_{n+2}}$, respectively, where $A'=\sqrt{3/2}\,A$. We may suppose that $n$ is so large that for each $p\geqq1$,

$$(12\,A'M)^2\xi_{n+p+1}\leqq\xi^2_{n+p}.$$

Then $C_{n+2,4k-2}$ does not contain the point at infinity, since $A'\sqrt{\xi_{n+2}}<m/12$, so that any one of $C_{n+2,4k-2}$, $C_{n+3,8k-5}$ and $C_{n+3,8k-4}$ cannot be disjoint from the union of the other two. By the same reasoning in the proof of Lemma 2 we see that the image of $\Delta_{n+2,4k-2}$ is contained in a spherical disc with radius less than $3A'\sqrt{\xi_{n+2}}<m/4$. Since this disc intersects the disc $[w, \infty]\geqq m$, it must lie outside the disc $C(\infty; m/2)$. Next we consider the domain $\Delta_{n+3,8k-5}$. Each image of its three boundary curves is contained in a spherical disc with radius less than

$$A'\sqrt{\xi_{n+3}} \leqq \xi_{n+2}/12M < A'\sqrt{\xi_{n+2}}/12A'M < m/24,$$

where we suppose that $A' > 1/6M$; this does not bring any loss of generality. The same argument shows that the image of $\Delta_{n+3, 8k-5}$ is contained in a spherical disc with radius less than $m/8$ and lying outside the disc $C(\infty; m/4)$. The same holds for $\Delta_{n+3, 8k-4}$. Since, for each $p \geqq 2$,

$$A'\sqrt{\xi_{n+p+1}} \leqq \xi_{n+p}/12M < A'\sqrt{\xi_{n+p}}/2,$$

we can conclude by induction that the image of the domain $\Delta_{n+p+1, q}$ lying in the interior of the simple closed curve $\Gamma_{n+2, 4k-2}$ is contained in a spherical disc with radius less than $m/2^{p+1}$ and lying outside the disc $C(\infty; m/2^p)$. It follows that, in the interior of $\Gamma_{n+2, 4k-2}$, $f(z)$ takes values only in a spherical disc with radius less than

$$\sum_{p=1}^{\infty} m/2^{p+1} < m/2.$$

By means of a linear transformation we can consider that $f(z)$ is bounded in the interior of $\Gamma_{n+2, 4k-2}$; this contradicts our assumption that $f(z)$ has an essential singularity at every point of $E$. Our theorem is now established.

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