AN EXISTENCE THEOREM IN POTENTIAL THEORY

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Dedicated to the memory of Professor TADASI NAKAYAMA

1. Concerning a positive lower semicontinuous kernel G on a locally compact Hausdorff space X the following existence theorem was obtained in [3].

THEOREM A. *Assume that the adjoint kernel $\check{G}$ satisfies the continuity principle. Then for any separable compact subset K of X and any positive upper semicontinuous function $u(x)$ on K, there exists a positive measure μ, supported by K, such that*

$$G\mu(x) \geq u(x) \qquad G\text{-}p.p.p. \text{ on } K,$$
$$G\mu(x) \leq u(x) \qquad \text{on } S\mu, \text{ the support of } \mu.$$

Nakai [4] proved the theorem without assuming the separablility of K. Using Kakutani's fixed-point theorem he simplified a part of the proof. But he needed prudent considerations on topology in order to avoid the separability. In this paper we shall give a simpler proof of the theorem without assuming the separability. We shall deal with a slightly more general kernel and use Glicksberg-Fan's fixed-point theorem.

2. A lower semicontinuous function $G(x, y)$ on $X \times X$ with $0 \leq G(x, y) \leq +\infty$ is called a non-negative l.s.c. kernel on X. The kernel $\check{G}$, defined by $\check{G}(x, y) = G(y, x)$, is called the adjoint kernel of G. The potential $G\mu(x)$ of a positive measure μ is defined by $G\mu(x) = \int G(x, y)\, d\mu(y)$. The adjoint potential $\check{G}\mu(x)$ is similarly defined. The adjoint kernel $\check{G}$ is said to satisfy the continiuity principle when finite continuous is every adjoint potential $\check{G}\mu$ of a positive measure μ with compact support which is finite continuous as a function on $S\mu$.

3. We shall prove

Received April 12, 1965.

THEOREM B. *Let G be a non-negative l.s.c. kernel on a locally compact Hausdorff space X. Assume that $G(x, x)>0$ for any $x \in X$ and the adjoint kernel $\check{G}$ satisfies the continuity principle. Then for any compact subset K of X and any positive finite upper semicontinuous function $u(x)$ on K, there exists a positive measure μ, supported by K, such that*

$$G\mu \geq u \qquad G\text{-}p.p.p. \text{ on } K^{1)},$$
$$G\mu \leq u \qquad \text{on } S\mu.$$

4. First we prove

THEOREM C. *If G is a non-negative finite continuous kernel on a compact Hausdorff space K such that $G(x, x)>0$ on K, there exists a positive measure μ on K such that*

$$G\mu(x) \geq 1 \text{ on } K,$$
$$G\mu(x) = 1 \text{ on } S\mu.$$

Proof. Denote by $\mathscr{M}_1(K)$ the totality of positive unit measures on K. This, with the vague topology, is compact and convex. We define a point-to-set mapping φ on $\mathscr{M}_1(K)$ as follows: we put, for any $\mu \in \mathscr{M}_1(K)$,

$$\varphi(\mu) = \left\{\nu \in \mathscr{M}_1(K)\,;\ \int G\mu d\nu = \inf_{\lambda \in \mathscr{M}_1(K)} \int G\mu d\lambda \right\}.$$

Since $G(x, y)$ is finite continuous, $\varphi(\mu)$ is non-empty and convex, and the mapping $\varphi : \mu \to \varphi(\mu)$ is closed in the following sense: if nets $\{\mu_\alpha\,;\ \alpha \in D$, a directed set$\}$ and $\{\nu_\alpha\,;\ \alpha \in D\}$ converge vaguely to μ and ν respectively and if $\nu_\alpha \in \varphi(\mu_\alpha)$ for any $\alpha \in D$, then $\nu \in \varphi(\mu)$. Consequently by Glicksberg-Fan's fixed-point theorem[2] there exists a measure $\mu_0 \in \mathscr{M}_1(K)$ such that $\mu_0 \in \varphi(\mu_0)$. Then $m_0 = \int G\mu_0 d\mu_0 = \inf_{\lambda \in \mathscr{M}_1(K)} \int G\mu_0 d\lambda$ does not vanish, since $G(x, x)>0$ on K. The measure $\mu = m_0^{-1}\mu_0$ fulfills all the requirements.

5. Using Theorem C we prove

THEOREM D. *If G is a non-negative l.s.c. kernel on a compact Hausdorff space K such that $G(x, x)>0$ for any $x \in K$ and if the adjoint kernel $\check{G}$ satisfies*

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1) This means that every compact subset of the exceptional set $\{x \in K;\ G\mu(x) < u(x)\}$ does not support any positive measure $\lambda \neq 0$ such that $\int G\lambda d\lambda < \infty$.

2) Cf. [1] and [2].

the continuity principle, then there exists a positive measure μ *on* K *such that*

(i) $G\mu \geq 1$ *G-p.p.p. on* K,

(ii) $G\mu \leq 1$ *on* $S\mu$.

Proof. Put $m = \inf_{x\in K} G(x, x) > 0$, and take a finite number of open neighborhoods U_i $(1 \leq i \leq N)$ such that $\bigcup_1^N U_i \supset K$ and $G(x, y) > \frac{1}{2} m$ in $U_i \times U_i$. There exists an increasing net $\{G_\alpha;\ \alpha \in D$, a directed set$\}$ of non-negative finite continuous functions $G_\alpha(x, y)$ on $K \times K$ such that $G_\alpha(x, y) > \frac{1}{2} m$ in $\bigcup_1^N U_i \times U_i$ and $\lim_D G_\alpha(x, y) = G(x, y)$ at any point $(x, y) \in K \times K$. Then by Theorem B there exists a positive measure μ_α on K such that $G_\alpha \mu_\alpha \geq 1$ on K and $G_\alpha \mu_\alpha = 1$ on $S\mu_\alpha$. The net $\{\mu_\alpha;\ \alpha \in D\}$ is bounded. In fact, for a point $x \in S\mu_\alpha \cap U_i$,

$$1 = G_\alpha \mu_\alpha(x) = \int G_\alpha(x, y) d\mu_\alpha(y)$$

$$\geq \int_{U_i} G_\alpha(x, y) d\mu_\alpha(y) > \frac{1}{2} m \mu_\alpha(U_i),$$

and hence $\mu_\alpha(U_i) \leq \frac{2}{m}$ and $\mu_\alpha(K) \leq \frac{2N}{m}$. Thus there exists a cluster point μ. Put

$$D' = \{\alpha' = \langle \alpha, \omega \rangle;\ \omega, \text{ a vague neighborhood of } \mu \text{ containing } \mu_\alpha\}.$$

Then D' is a directed set with the natural order. Putting, for $\alpha' = \langle \alpha, \omega \rangle \in D'$, $\mu_{\alpha'} = \mu_\alpha$ and $G_{\alpha'} = G_\alpha$, we see that $\mu_{\alpha'} \to \mu$ vaguely and $G_{\alpha'}(x, y) \nearrow G(x, y)$ at any point $(x, y) \in K \times K$. We shall show the validity of (i) and (ii) for μ.

Proof of (i). Suppose that there exists a positive measure $\lambda \neq 0$ such that $S_\lambda \subset \{x \in K;\ G\mu(x) < 1\}$ and $\int \check{G}\lambda d\lambda < \infty$. Since $\check{G}$ satisfies the continuity principle, we may assume that $\check{G}\lambda$ is finite continuous on K. Hence

$$\int d\lambda > \int G\mu d\lambda = \int \check{G}\lambda d\mu = \lim_{D'} \int \check{G}\lambda d\mu_{\alpha'}$$

$$= \lim_{D'} \int G\mu_{\alpha'} d\lambda \geq \lim\sup_{D'} \int G_{\alpha'}\mu_{\alpha'} d\lambda \geq \int d\lambda.$$

Proof of (ii). Let x_0 be an arbitrary fixed point on $S\mu$, and put

$$D'' = \{\alpha'' = \langle \alpha', U \rangle;\ U, \text{ a neighborhood of } x_0 \text{ containing a point } x_{\alpha'} \text{ of } S\mu_{\alpha'}\}.$$

This is a directed set with the natural order. Putting, for $\alpha'' = \langle \alpha', U \rangle \in D''$, $x_{\alpha''} = x_{\alpha'}$, $\mu_{\alpha''} = \mu_{\alpha'}$ and $G_{\alpha''} = G_{\alpha'}$, we see that $x_{\alpha''} \to x_0$, $\mu_{\alpha''} \to \mu_0$ and $G_{\alpha''}(x, y) \nearrow$

$G(x, y)$ along D''. Hence for any $\alpha_0'' \in D''$

$$1 = \lim_{D''} G_{\alpha'} \mu_{\alpha''}(x_{\alpha''}) \geq \lim_{D''} G_{\alpha_0''} \mu_{\alpha''}(x_{\alpha''}) = G_{\alpha_0''} \mu(x_0).$$

Consequently $G\mu(x_0) = \lim_{D''} G_{\alpha''}\mu(x_0) \leq 1$.

6. From Theorem D follows immediately

THEOREM E. *Let G be a non-negative l.s.c. kernel on X such that $G(x, x) > 0$ for any $x \in X$ and the adjoint kernel $\check{G}$ satisfies the continuity principle. Then for any positive finite continuous function $u(x)$ on a compact set K, there exists a positive measure μ, supported by K, such that*

$$G\mu(x) \geq u(x) \qquad G\text{-}p.p.p. \text{ on } K,$$
$$G\mu(x) \leq u(x) \qquad \text{on } S\mu.$$

In fact, $G'(x, y) = G(x, y)/u(x)$ is a non-negative l.s.c. kernel on K, the adjoint kernel of which satisfies the continuity principle. Hence by Theorem D there exists a positive measure μ on K such that

$$G'\mu \geq 1 \qquad G'\text{-}p.p.p. \text{ on } K,$$
$$G'\mu \leq 1 \qquad \text{on } S\mu.$$

This μ fulfills the requirements of Theorem E.

7. Now we can prove Theorem B. Let $\{u_\alpha(x) ;\ \alpha \in D\}$ be a decreasing net of positive finite continuous functions on K such that $u_\alpha(x) \searrow u(x)$. Then there exists a positive measure μ_α on K such that

$$G\mu_\alpha(x) \geq u_\alpha(x) \qquad G\text{-}p.p.p. \text{ on } K$$
$$G\mu_\alpha(x) \leq u_\alpha(x) \qquad \text{on } S\mu_\alpha.$$

The net $\{\mu_\alpha\}$ is bounded, and similarly as in the proof of Theorem D, a subnet converges vaguely to a cluster point μ of the net $\{\mu_\alpha\}$. This μ fulfills the requirements of Theorem B.

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