A CHARACTERIZATION OF QF-3 RINGS

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To the memory of TADASI NAKAYAMA

A left QF-3 ring R is one in which ${}_RR$, *the ring considered as a left module* over itself, can be embedded in a projective injective left R module $Q({}_RR)$. QF-3 rings were introduced by Thrall [14] and have been studied and characterized by a number of authors [5, 8, 9, 12, 13, 15] usually restricted to the case of algebras over a field. In such a case, the concept of left QF-3 and right QF-3 coincide.

The study of QF-3 rings and algebras and many other such classes of rings had its origin in the now classic papers of Nakayama [10, 11]. He was an outstanding pioneer in algebra for many years, and we acknowledge our great debt to him and to his many excellent papers.

In this note we shall restrict consideration to the class of rings with minimum condition in left ideals and within the class we shall give a new characterization of left QF-3 rings. Actually, our characterization holds for a more general class of rings. See the remarks at the end of the note.

Incidently, for rings with minimum codition both on left and on right ideals, we do not know if the concepts of left QF-3 and right QF-3 coincide as they do for the case of finite dimensional algebras over fields.

Throughout the paper R will represent a ring with minimum condition on left ideals and all modules are left R modules. When R appears as a module, it is as a left module over itself. $\mathrm{Hom}(X, Y)$ will always mean $\mathrm{Hom}_R(X, Y)$ for R-modules X, Y.

Our characterization of QF-3 rings is given in terms of certain classes of modules. A module M is called ***torsionless*** if for each $m \in M$, $m \neq 0$, there exists $f \in \mathrm{Hom}(M, R)$ such that $f(m) \neq 0$. M is torsionless if there are enough homomorphisms of M to R to distinguish points of M from 0. The concept

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of a torsionless module was introduced by Bass [1] and subsequently used by a number of authors [6, 7, 16].

It is not hard to see that the class $\mathscr{L}$ of left torsionless modules is closed under taking submodules and arbitrary direct products [1]. It is not, in general, closed under taking factors and extensions.

We introduce another class of modules $\mathscr{T}$ the class of left modules M for which $\mathrm{Hom}(M, R) = 0$. This is a torsion class in the sense of S. E. Dickson [3], and it is not hard to see that it is closed under taking factors, extensions, and arbitrary direct sums. It is not, in general, closed under taking submodules.

The following theorem characterizes left QF-3 rings in terms of the classes $\mathscr{L}$ and $\mathscr{T}$.

THEOREM. *If R is a ring with minimum condition on left ideals, then R is left QF-3 if and only if*

(1) *$\mathscr{L}$ is closed under extension*

(2) *$\mathscr{T}$ is closed under taking submodules*

Proof. Let us first assume that R is QF-3 and show that it must satisfy the condition (1) of the theorem. Suppose $0 \to A \to B \to C \to 0$ is an exact sequence with A and C torsionless. Since torsionless modules can be embedded in a product of copies of the ring [1], we have an exact sequence $0 \to A \to \pi R$. Each R can be embedded in $Q(R)$ a projective injective module and this induces the embedding $0 \to \pi_R R \to \pi Q({}_R R)$. The product of injectives is always injective and the product of projectives is again projective for rings with minimum condition [2]. Combining the two above embeddings we can embed A in a projective injective module and it follows that the minimal injective $Q(A)$ for A[4] being a direct summand of a projective injective is also projective injective. The same argument shows $Q(C)$ is injective.

The exact diagram

$$\begin{array}{ccccccccc} & & 0 & & & & 0 & & \\ & & \downarrow & & & & \downarrow & & \\ 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \downarrow & & & & \downarrow & & \\ & & Q(A) & & & & Q(C) & & \end{array}$$

can be embedded in the exact commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & A & \rightarrow & B & \rightarrow & C & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & Q(A) & \rightarrow & Q(A)+Q(C) & \rightarrow & Q(C) & \rightarrow & 0 \end{array}$$

by a standard argument [5]. But then B is a submodule of the projective injective module $Q(A)+Q(C)$ and since projectives are always torsionless [1] B, being a submodule of a projective is torsionless. Thus, we have shown that $\mathscr{L}$ is closed under extension.

Now we turn our attention to condition (2). Suppose $T\supseteq M$ are R modules and $\mathrm{Hom}(M, R)\neq 0$. Let f be a non zero homomorphism from M to R. Using the injectivity of $Q(R)$ we obtain the commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & M & \longrightarrow & T \\ & & f\downarrow & & \downarrow g \\ 0 & \longrightarrow & R & \longrightarrow & Q(R) \end{array}$$

Since $Q(R)$ is assumed to be projective, it is a direct summand of a free module F. $\mathrm{Im}\ g$ can be considered a submodule of F and since $\mathrm{Im}\ g\neq 0$, there is a projection π of F onto R so that $\pi(\mathrm{Im}\ g)\neq 0$. Composing g with π (thinking of $\mathrm{Im}\ g\subseteq F$) we have a non zero homomorphism in $\mathrm{Hom}\,(T, R)$. Thus we have shown that R has condition (2) of the theorem.

Conversely, let us now assume that R satisfies (1) and (2) and we must show that R is QF-3. Let $Q=Q(R)$ be the minimal injective containing R. Since $\mathrm{Hom}\,(R, R)\neq 0$, it follows from (2) that $\mathrm{Hom}\,(Q, R)\neq 0$. Let

$$K=\bigcap_{f\in \mathrm{Hom}\,(Q,\,R)} \mathrm{Ker}\ f,$$

and note that $0\subseteq K\subsetneqq Q$. In the following we consider two cases.

Case $K\neq 0$. We shall show that this leads to a contradiction. If $K\neq 0$, it follows that $K\cap R\neq 0$ since Q is an essential extension [4] of R. Again using condition (2) and the fact that $\mathrm{Hom}\,(K\cap R, R)\neq 0$, we conclude that $\mathrm{Hom}\,(K, R)\neq 0$. Now let

$$K'=\bigcap_{f\in \mathrm{Hom}\,(K,\,R)} \mathrm{Ker}\ f$$

and we see that $K'\subsetneqq K$. Since we can identify $\mathrm{Hom}\,(K, R)$ with $\mathrm{Hom}\,(K/K', R)$ and the latter has enough homomorphisms to distinguish points of K/K'

from 0 we see that K/K' is torsionless. Using the same argument, it follows that Q/K is also torsionless. From the exact sequence

$$0 \to K/K' \to Q/K' \to Q/K \to 0$$

and we use condition (1) to conclude that Q/K' is torsionless.

Now by the natural embedding $0 \to \operatorname{Hom}(Q/K', R) \to \operatorname{Hom}(Q, R)$ and the fact that

$$\bigcap_{f \in \operatorname{Hom}(Q/K', R)} \operatorname{Ker} f = 0$$

we conclude that

$$\bigcap_{f \in \operatorname{Hom}(Q, R)} \operatorname{Ker} f \subseteq K'.$$

But this latter containment contradicts the condition that $K' \subsetneqq K$.

We now proceed with the case $K=0$. In this case Q is torsionless and we have an embedding $0 \to Q \to \pi R$ of Q into a product of copies of R. Since R has minimum condition, πR is again projective. But since Q is injective, Q is a direct summand of πR and is therefore projective. This completes the proof of the theorem.

In the following we shall give examples to show that the conditions (1) and (2) of the theorem are independent.

Consider the algebra A of matrices of the form

$$\begin{pmatrix} x & 0 & 0 \\ y & u & 0 \\ z & 0 & v \end{pmatrix}$$

with entries from a field. It is not hard to show that A has global dimension 1 [6] and therefore submodules of projectives are projective. It follows that the class $\mathscr{L}$ is precisely the class of projective modules. This class is always closed under extension so that the algebra A satisfies condition (1) of the theorem.

Now define the following left ideals

$$L = \left\{\begin{pmatrix} x & 0 & 0 \\ y & 0 & 0 \\ z & 0 & 0 \end{pmatrix}\right\} \supsetneqq M = \left\{\begin{pmatrix} 0 & 0 & 0 \\ y & 0 & 0 \\ z & 0 & 0 \end{pmatrix}\right\} \supsetneqq N = \left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ z & 0 & 0 \end{pmatrix}\right\}.$$

where in each case we mean elements of the form indicated, L is an indecomposable projective left ideal and L/M is simple. It follows that $\text{Hom}(L/M, R) = 0$ for otherwise, using the fact that R has global dimension 1, L/M would be a non trivial direct summand of L. If f were a non zero homomorphism of L/N to R, its kernel would have to be either M/N or 0. It could not be the former, for then it would be a non zero member of $\text{Hom}(L/M, R)$. Nor could it be 0, for in that case L/N would be projective and $L = L/N \oplus N$ would decompose. It follows that $\text{Hom}(L/N, R) = 0$. However, $\text{Hom}(M/N, R) \neq 0$ because M/N is isomorphic to the left ideal of elements of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ y & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus, the algebra A does not satisfy condition (2) of the theorem.

Let K be a field and $K[x, y]$, the polynominal algebra in two variables x and y. Let B be the algebra $K[x, y]/(x^2, xy, y^2)$. We shall show that B satisfies (2) of the theorem, but not condition (1).

B is a primary algebra and has only one simple module B/N where N is the radical of B. Given any non zero B module M, the non zero semi simple module M/NM has maximal submodules so M has simple factors. B has simple ideals so there are non zero homomorphisms from M/NM to B, hence $\text{Hom}(M, B) \neq 0$. Thus for this algebra, the class $\mathscr{T}$ consists only of the zero module and is clearly closed under taking submodules.

The simple B module S is torsionless, but we shall show that there is an extension E of S by S which is not torsionless. In the first place, there exist non split extensions E of S by S, for instance, $B/(\bar{x})$ where $\bar{x}$ is the image of x in B. Note that there is a unique submodule of E isomorphic to S, for otherwise E splits into the direct sum of two copies of S. We shall show that every homomorphism from E to B has kernel S (or E). If f were a homomorphism which had a kernel other than S or E, then its kernel is zero, the only other submodule of E.

If $\text{Im } f \nsubseteq N$ then $\text{Im } f = B$ because B is the only ideal of B not contained in or equal to N. However, E has dimension 2 over the field and B has dimension 3, so there can be no epimorphisms from E to B. It follows that $\text{Im } f \subseteq N$. But this implies for our monomorphism f that $\text{Im } f = N$ since both have dimension 2 over the field. It follows that E is isomorphic to N which is the direct sum of two copies of S contradicting the indecomposability of E. The contradiction shows that kernels of homomorphisms from E to B must be S or E and that E is not torsionless. So our example B fails to satisfy condition (1) of the theorem.

Remark 1. A hereditary ring always has condition (1) of our theorem as we remarked about our first example above. It follows that hereditary rings with minimum coodition are QF-3 if and only if the class $\mathscr{T}$ is closed under taking submodules. See [8] for a discussion of hereditary QF-3 rings.

Remark 2. In the proof of our theorem, we used the minimum condition on left ideals only to show that the product of projectives is again projective. S. Chase [2] has given a complete characterization of this class of rings with this property. Our theorem holds in that class of rings, the proof being the same as the one we give here.

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