A REMARK ON THE INTERSECTION OF TWO LOGICS

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The intuitionistic logic **LJ** and Curry's **LD** (cf. [1], [2]) are logics stronger than Johansson's minimal logic **LM** (cf. [3]) by the axiom schemes $\curlywedge \to x$ and $y \vee (y \to \curlywedge)$, respectively. However, **LM** can not be taken literally as the intersection of these two logics **LJ** and **LD**, which is stronger than **LM** by the axiom scheme $(\curlywedge \to x) \vee y \vee (y \to \curlywedge)$. In pointing out this situation, Prof. K. Ono suggested me to investigate the general feature of the intersection of any pair of logics. In this paper, I will show that the same situation occurs in general. I wish to express my thanks to Prof. K. Ono for his kind guidance.

Let **A** be a logic having logical constants, *implication* ($\to$) and *disjunction* ($\vee$) (and *universal quantification* ( ) for predicate logics), together with all such inference rules with respect them that are admitted in the intuitionistic logic (cf. [5], p. 81). For any logic **L**, let us denote by Π_L the class of all provable propositions in **L**.

THEOREM. *Let* **B**, **C**, *and* **D** *be the logics formed from* **A** *by adjoining the axiom schemes*

(1) $(u_1) \cdots (u_p) f(x_1, \ldots, x_s)$, $(p = 0, 1, 2, \ldots)$,

(2) $(v_1) \cdots (v_q) g(y_1, \ldots, y_t)$, $(q = 0, 1, 2, \ldots)$,

(3) $(u_1) \cdots (u_p) f(x_1, \ldots, x_s) \vee (v_1) \cdots (v_q) g(y_1, \ldots, y_t)$, $(p, q = 0, 1, 2, \ldots)$,

respectively; where u_i's and v_j's are object variables ($p = q = 0$ for proposition logics), $(u_1) \cdots (u_p) f(x_1, \ldots, x_s)$ and $(v_1) \cdots (v_q) g(y_1, \ldots, y_t)$ are expressible in **A**, *x_i's and y_j's are metalogical variables for propositions, predicates, or relations, and $s \leq t$. Then,*

I. $$\Pi_D = \Pi_B \cap \Pi_C.$$

II. **B** *and* **C** *formed from* **D** *by adjoining the axiom schemes*

$(4)_\mu$ $(w_1) \cdots (w_r)(g(y_1, \ldots, y_t) \to f(y_{\mu(1)}, \ldots, y_{\mu(s)}))$, $(r = 0, 1, 2, \ldots)$,

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$(5)_\mu \quad (w_1)\cdots(w_r)(f(y_{\mu(1)},\ldots,y_{\mu(s)})\to g(y_1,\ldots,y_t)), \qquad (r=0,1,2,\ldots),$

respectively; where w_i's are object variables ($r=0$ for proposition logics), $1\leq\mu(k)\leq t$, $k=1,\ldots,s$, and $\mu(i)=\mu(j)$ implies $i=j$.

Proof of **I**. We prove the theorem for predicate logics. For proposition logics, we can prove it as a special case of this proof.

Let us denote $(x_1,\ldots,x_s)$ and $(y_1,\ldots,y_t)$ simply by x and y, respectively. Clearly, $\Pi_D\subseteq\Pi_B\cap\Pi_C$. To show $\Pi_B\cap\Pi_C\subseteq\Pi_D$, take any proposition $\mathfrak{p}$ in $\Pi_B\cap\Pi_C$. Assume that $\mathfrak{p}$ can be proved in **B** (in **C**) by making use of propositions of the form (1) (of the form (2)) m times (n times), and let l be the maximum number of m and n. Then, propositions of the forms

(6) $$F_m(\mathfrak{p})\rightleftharpoons\bar{f}_m\to(\bar{f}_{m-1}\to(\cdots\to(\bar{f}_1\to\mathfrak{p})\cdots)),$$

(7) $$G_n(\mathfrak{p})\rightleftharpoons\bar{g}_n\to(\bar{g}_{n-1}\to(\cdots\to(\bar{g}_1\to\mathfrak{p})\cdots))$$

must be provable in **A**, hence in **D**; where $\bar{f}_i$ and $\bar{g}_j$ are propositions of the forms $(u_1)\cdots(u_p)f(a_i)$ and $(v_1)\cdots(v_q)g(b_j)$, respectively. Naturally, $F_0(\mathfrak{p})$ as well as $G_0(\mathfrak{p})$ stands for $\mathfrak{p}$. It is enough to show that any proposition of *the form*

(8) $$H_{m,n}\rightleftharpoons F_m(\mathfrak{p})\to(G_n(\mathfrak{p})\to\mathfrak{p})$$

is provable in **D** under the assumption that any propositions of *the forms* $H_{r,s}$ are provable in **D** for all r, $s<l$.

According to the practical way of description introduced by Ono (cf. [4], [5]), we have

Proof of $H_{m,n}$ /**A**, **B**→**c**.

A) Assume $F_m(\mathfrak{p})$. **B**) Assume $G_n(\mathfrak{p})$.

c)) $\mathfrak{p}$ /**ca**, **cb**, **cc** for $m>0$ and $n>0$ (**c** follows immediately from **A** for $m=0$, and from **B** for $n=0$.).

ca)) $\bar{f}_m\to\mathfrak{p}$ /**caA**→**cae**. **caA**) Assume $\bar{f}_m$.

cab) $F_{m-1}(\mathfrak{p})$ /**A**, **caA**.

cac)) $\bar{g}_n\to\mathfrak{p}$ /**cacA**→**cacd**. **cacA**) Assume $\bar{g}_n$.

cacb) $G_{n-1}(\mathfrak{p})$ /**B**, **cacA**.

cacc) $F_{m-1}(\mathfrak{p})\to(G_{n-1}(\mathfrak{p})\to\mathfrak{p})$ /Assumption of induction.

cacd) $\mathfrak{p}$ /**cacc**, **cab**, **cacb**.

cad) $F_{m-1}(\mathfrak{p})\to((\bar{g}_n\to\mathfrak{p})\to\mathfrak{p})$ /Assumption of induction for $l>1$; tautological for $l=1$.

cae) $\mathfrak{p}$ /**cad, cab, cac.**

cb)) $\bar{g}_n \rightarrow \mathfrak{p}$ /similarly as **ca.**

cc) $\bar{f}_m \vee \bar{g}_n$ /(3).

Proof of **II.** Even in **A**, (1) and (2) are equivalent to "(3) and $(4)_\mu$" and "(3) and $(5)_\mu$", respectively.

Remark. For different permutations μ and μ', $(4)_\mu$ and $(4)_{\mu'}$ ($(5)_\mu$ and $(5)_{\mu'}$) are mutually equivalent in **D.** (1) is decomposed into (3) and $(4)_\mu$, and (2) into (3) and $(5)_\mu$. However, we can decompose (1) and (2) into still *weaker* components, as Fig. 1 shows. Namely, (1) is decomposed into $(9)_\mu$ and $(4)_\mu$, and (2) into $(9)_\mu$ and $(5)_\mu$, where

$$(9)_\mu \qquad (u_1)\cdots(u_p)f(y_{\mu(1)},\ldots,y_{\mu(s)}) \vee (v_1)\cdots(v_q)g(y_1,\ldots,y_t), \quad (p,\ q=0,1,2,\ldots).$$

Motivated by this circumstance, it would be of some interest to seek for the weakest axiom scheme (or inference rule) under those which form **B** (**C**) by being added to **D.** However, it would be hard to find out anything of this kind, since such axiom scheme (or inference rule) must be equivalent to *the metalogical assumption* that *the proposition scheme* $(v_1)\cdots(v_q)g(y_1,\ldots,y_t)$ ($(u_1)\cdots(u_p)f(x_1,\ldots,x_s)$) *in the whole implies any proposition of the form* $f(x_1,\ldots,x_s)$ ($g(y_1,\ldots,y_t)$).

Example **1.** **LJ** and **LN** (named by Ono, cf. [6], [7]) are formed from **LM**

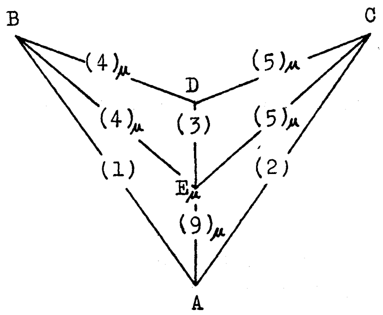

Fig. 1

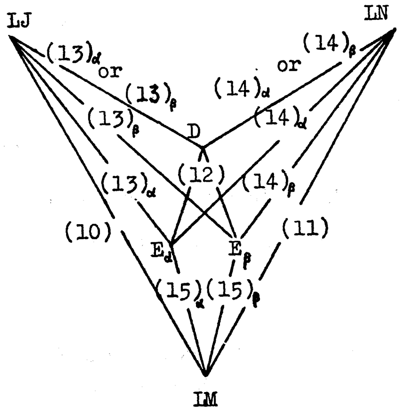

Fig. 2

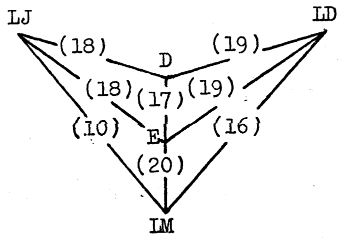

Fig. 3

by adjoining the axiom scheme

(10) $\curlywedge \to x,$

(11) $y_1 \vee (y_1 \to y_2)$ (a deformation of Peirce's rule),

respectively. The mutual relation of formulas and logics is shown in Fig. 2, where

(12) $(\curlywedge \to x) \vee y_1 \vee (y_1 \to y_2),$

$(13)_\alpha$ $(y_1 \to y_2) \to (\curlywedge \to y_1),$

$(13)_\beta$ $(y_1 \vee (y_1 \to y_2)) \to (\curlywedge \to y_2),$

$(14)_\alpha$ $(\curlywedge \to y_1) \to (y_1 \vee (y_1 \to y_2)),$

$(14)_\beta$ $(\curlywedge \to y_2) \to (y_1 \vee (y_1 \to y_2)),$

$(15)_\alpha$ $(\curlywedge \to y_1) \vee (y_1 \to y_2),$

$(15)_\beta$ $(\curlywedge \to y_2) \vee y_1 \vee (y_1 \to y_2).$

Example 2. As a special case of *Example* 1, we have Fig. 3, where

(16) $y \vee (y \to \curlywedge)$ (cf. [1], [2]),

(17) $(\curlywedge \to x) \vee y \vee (y \to \curlywedge),$

(18) $(y \to \curlywedge) \to (\curlywedge \to y),$

(19) $(\curlywedge \to y) \to (y \vee (y \to \curlywedge)),$

(20) $(\curlywedge \to y) \vee (y \to \curlywedge).$

To show characteristic feature as simply as possible, we have omitted object variables in describing above examples.

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