RELATIVE COHOMOLOGY OF ALGEBRAIC LINEAR GROUPS, II

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1. Introduction

Let G be an algebraic linear group over a field F of characteristic 0, and let H be an algebraic subgroup of G. Let A, M be rational G-modules. In [4], we defined $\mathrm{Ext}^n_{(G,H)}(A, M)$, and, in particular, relative cohomology groups $H^n(G, H, M)$ were defined as $\mathrm{Ext}^n_{(G,H)}(F, M)$.

$\mathrm{Ext}^1_{(G,H)}(A, M)$ may be identified with the space of the equivalence classes of the rational (G, H)-extensions of M by A ([4]). Moreover $\mathrm{Ext}^n_{(G,H)}(A, M)$ may be identified with the set of the equivalence classes of the rational n-fold (G, H)-extensions of M by A (Th. 2.2).

Let G be a unipotent algebraic linear group. Then there exists the natural homomorphism of $H^n(G, H, M)$ into the Lie algebra cohomology group $H^n(\mathfrak{g}, \mathfrak{h}, M)$, where $\mathfrak{g}$, $\mathfrak{h}$ are Lie algebras of G, H respectively. In Section 3, we show that, if M is finite dimensional, then the natural homomorphism $H^2(G, H, M) \to H^2(\mathfrak{g}, \mathfrak{h}, M)$ is surjective.

G. Hochschild studied the properties of rational injective modules ([3]). In Section 4, we obtain analogous results as described in [3].

2. Extensions of rational modules

Let G be an algebraic linear group over a field F, and let H be an algebraic subgroup of G. We denote by $R(G)$, or simply by R, the F-algebra of rational representative functions on G. If $f \in R$ and $x \in G$, the left and right translations, $x \cdot f$ and $f \cdot x$ of f by x are defined by $(x \cdot f)(y) = f(yx)$, $(f \cdot x)(y) = f(xy)$ for all $y \in G$. Let M be a rational G-module in the sense of [2]. We make the tensor product $R \otimes M$ over F into a G-module such that $x(f \otimes m) = f \cdot x^{-1} \otimes x \cdot m$. Then $R \otimes M$ is a rational G-module. We denote by ${}^H R$ the set consisting of the elements left fixed by left translations from H. Then ${}^H R \otimes M$ is a rationally (G, H)-injective submodule of $R \otimes M$ in the sense of [4] ([4, Prop. 2.1]).

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In [4], we defined the relative extension functor $\mathrm{Ext}^n_{(G,H)}(*, *)$.

PROPOSITION 2.1. *Let G be an algebraic linear group over a field F, and let H be an algebraic subgroup of G. Let*

$$(0)\longrightarrow A\overset{\alpha}{\longrightarrow}B\overset{\beta}{\longrightarrow}C\longrightarrow(0)$$

be a rationally (G, H)-exact sequence, where A, B, C are rational G-modules. Then, for any rational G-module M, it gives rise to exact sequences;

$$\begin{aligned}&(0)\longrightarrow \mathrm{Ext}^0_{(G,H)}(M, A)\longrightarrow \mathrm{Ext}^0_{(G,H)}(M, B)\longrightarrow \mathrm{Ext}^0_{(G,H)}(M, C)\\ &\overset{\delta_0}{\longrightarrow}\cdots \qquad\qquad \longrightarrow \mathrm{Ext}^{n-1}_{(G,H)}(M, C)\\ &\overset{\delta_{n-1}}{\longrightarrow}\mathrm{Ext}^n_{(G,H)}(M, A)\longrightarrow \mathrm{Ext}^n_{(G,H)}(M, B)\longrightarrow\cdots\end{aligned}$$

and $(0)\longrightarrow \mathrm{Ext}^0_{(G,H)}(C, M)\longrightarrow \mathrm{Ext}^0_{(G,H)}(B, M)\longrightarrow \mathrm{Ext}^0_{(G,H)}(A, M)\overset{\Delta_0}{\longrightarrow}\cdots$
$\longrightarrow \mathrm{Ext}^{n-1}_{(G,H)}(A, M)\overset{\Delta_{n-1}}{\longrightarrow}\mathrm{Ext}^n_{(G,H)}(C, M)\longrightarrow \mathrm{Ext}^n_{(G,H)}(B, M)\longrightarrow\cdots.$

Proof. We shall use the following rationally (G, H)-injective resolution $X(D)$ of a rational G-module D. For each $n\geqq 0$, $X_n(D)$ is the tensor product ${}^H R\otimes\cdots\otimes{}^H R\otimes D$, with $n+1$ factors ${}^H R$. The coboundary operator φ_n ; $X_n(D)\to X_{n+1}(D)$ is given by

$$\begin{aligned}&\varphi_n(f_0\otimes\cdots\otimes f_n\otimes d)\\ &=1\otimes f_0\otimes\cdots\otimes f_n\otimes d\\ &+\sum_{i=0}^{n-1}(-1)^{i+1}f_0\otimes\cdots\otimes f_i\otimes 1\otimes f_{i+1}\otimes\cdots\otimes f_n\otimes d\\ &+(-1)^{n+1}f_0\otimes\cdots\otimes f_n\otimes 1\otimes d.\end{aligned}$$

The augmentation $\varphi_{-1}: D\to X_0(D)$ is given by $d\to 1\otimes d$. By [4, p. 274]

$$(0)\longrightarrow D\longrightarrow X_0(D)\longrightarrow X_1(D)\longrightarrow\cdots$$

is a rationally (G, H)-injective resolution of D.
The sequence;

$$(0)\longrightarrow X_n(A)\overset{\alpha_n}{\longrightarrow}X_n(B)\overset{\beta_n}{\longrightarrow}X_n(C)\longrightarrow(0),$$

where $\alpha_n(f_0\otimes\cdots\otimes f_n\otimes a)=f_0\otimes\cdots\otimes f_n\otimes\alpha(a)$ and

$$\beta_n(f_0\otimes\cdots\otimes f_n\otimes b)=f_0\otimes\cdots\otimes f_n\otimes\beta(b),$$

is (G, H)-exact by the assumption of (α, β). Moreover, since $X_n(B)$ is rationally (G, H)-injective, $X_n(A)$ is a G-direct summand of $X_n(B)$. Hence we obtain exact sequences;

$$(0)\longrightarrow \mathrm{Hom}_G(M,\ X_n(A))\longrightarrow \mathrm{Hom}_G(M,\ X_n(B))\longrightarrow \mathrm{Hom}_G(M,\ X_n(C))\longrightarrow(0)$$

and

$$(0)\longrightarrow \mathrm{Hom}_G(X_n(C),\ M)\longrightarrow \mathrm{Hom}_G(X_n(B),\ M)\longrightarrow \mathrm{Hom}_G(X_n(A),\ M)\longrightarrow(0).$$

Therefore we get the desired results from the following commutative diagrams;

$$\begin{array}{ccccccccc}
 & & (0) & & (0) & & (0) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
(0) & \to & \tilde{X}_0(A) & \to & \tilde{X}_1(A) & \to & \tilde{X}_2(A) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
(0) & \to & \tilde{X}_0(B) & \to & \tilde{X}_1(B) & \to & \tilde{X}_2(B) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
(0) & \to & \tilde{X}_0(C) & \to & \tilde{X}_1(C) & \to & \tilde{X}_2(C) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & (0) & & (0) & & (0) & & \\
 & & (\text{exact}) & & (\text{exact}) & & (\text{exact}), & &
\end{array}$$

where $\tilde{X}_n(*)=\mathrm{Hom}_G(M,\ X_n(*))$, and

$$\begin{array}{ccccccccc}
 & & (0) & & (0) & & (0) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
(0) & \to & \bar{X}_0(C) & \to & \bar{X}_1(C) & \to & \bar{X}_2(C) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
(0) & \to & \bar{X}_0(B) & \to & \bar{X}_1(B) & \to & \bar{X}_2(B) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
(0) & \to & \bar{X}_0(A) & \to & \bar{X}_1(A) & \to & \bar{X}_2(A) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & (0) & & (0) & & (0) & & \\
 & & (\text{exact}) & & (\text{exact}) & & (\text{exact}), & &
\end{array}$$

where $\bar{X}_n(*)=\mathrm{Hom}(X_n(*),\ M)$. This completes the proof of Proposition 2.1.

A rationally $(G,\ H)$-exact sequence of rational G-modules;

$$(E_n)\quad (0)\longrightarrow C\longrightarrow X_1\longrightarrow\cdots\longrightarrow X_n\longrightarrow A\longrightarrow(0)$$

is called a rational n-fold $(G,\ H)$-extension of C by A. When a diagram of two rational $(G,\ H)$-extensions of C by A;

$$\begin{array}{ccccccccccccc}
(E_n^1) & (0) & \longrightarrow & C & \longrightarrow & X_1^1 & \longrightarrow & \cdots & \longrightarrow & X_n^1 & \longrightarrow & A & \longrightarrow(0) \\
 & & & \downarrow 1 & & \downarrow \kappa_1 & & & & \downarrow \kappa_n & & \downarrow 1 & \\
(E_n^2) & (0) & \longrightarrow & C & \longrightarrow & X_1^2 & \longrightarrow & \cdots & \longrightarrow & X_n^2 & \longrightarrow & A & \longrightarrow(0)
\end{array}$$

is commutative, the system $\kappa=\{\kappa_1,\ \ldots,\ \kappa_n\}$ of G-homomorphisms is said to be a homomorphism of (E_n^1) to (E_n^2). If $(E_n)=(E_n^0),\ \ldots,\ (E_n^r)=(E_n')$ are rational n-fold (G, H)-extensions of C by A and if there exists a homomorphism of (E_n^{i-1}) to (E_n^i), or of (E_n^i) to (E_n^{i-1}) for $1 \leqq i \leqq r$, we shall say that (E_n) is equivalent to (E_n) [5]. Let $E^n_{(G,H)}(A, C)$ be the set of the equivalence classes of rational n-fold (G, H)-extensions of C by A.

An extension (E_n) induces a homomorphism

$$\theta_{(F_n)} : \mathrm{Hom}_G(C, C) \to \mathrm{Ext}^n_{(G,H)}(A, C)$$

by Proposition 2.1. $\theta_{(E_n)}(1)$ depends only on the equivalence class of (E_n). Therefore we obtain a map

$$\theta_n : E^n_{(G,H)}(A, C) \to \mathrm{Ext}^n_{(G,H)}(A, C),$$

where θ_n (the class of $(E_n)) = \theta_{(E_n)}(1)$. In particular θ_1 is a one-one correspondence ([4]).

THEOREM 2.2. *Let G be an algebraic linear group over a field F, and let H be an algebraic subgroup of G. If A, C are rational G-modules, then* $\mathrm{Ext}^n_{(G,H)}(A, C)$ *may be identified with $E^n_{(G,H)}(A, C)$ for $n \geqq 1$.*

Proof. We may select a rationally (G, H)-exact sequence;

$$(Q) \quad (0) \longrightarrow C \longrightarrow Q_1 \longrightarrow \cdots \longrightarrow Q_{n-1} \longrightarrow B \longrightarrow (0),$$

where each Q_i is rationally (G, H)-injective. A rational (G, H)-extension of B by A;

$$(E_1) \quad (0) \longrightarrow B \longrightarrow X_n \longrightarrow A \longrightarrow (0)$$

induces an extension;

$$(Q(E_1)) \quad (0) \longrightarrow C \longrightarrow Q_1 \longrightarrow \cdots \longrightarrow Q_{n-1} \longrightarrow X \longrightarrow A \longrightarrow (0).$$

Clearly this correspondence induces a map;

$$\tilde{Q} : \quad E^1_{(G,H)}(A, B) \longrightarrow E^n_{(G,H)}(A, C).$$

On the other hand, by Proposition 1 and (Q), we obtain an isomorphism;

$$\hat{Q} : \quad \mathrm{Ext}^1_{(G,H)}(A, B) \longrightarrow \mathrm{Ext}^n_{(G,H)}(A, C).$$

Therefore we obtain a commutative diagram;

$$\begin{array}{ccc} E^1_{(G,H)}(A, B) & \xrightarrow{\tilde{Q}} & E^n_{(G,H)}(A, C) \\ \downarrow \theta_1 & & \downarrow \theta_n \\ \mathrm{Ext}^1_{(G,H)}(A, B) & \xleftarrow{\hat{Q}} & \mathrm{Ext}^n_{(G,H)}(A, C), \end{array}$$

where θ_1 and $\hat{Q}$ are isomorphisms. We shall show that $\tilde{Q}$ is surjective.

For a given rational n-fold (G, H)-extension of C by A;

$$(E_n) \quad (0)\longrightarrow C\longrightarrow X_1\longrightarrow\cdots\longrightarrow X_n\longrightarrow A\longrightarrow(0),$$

we may make commutative diagrams of rational G-modules;

$$\begin{array}{ccccccccccc} (0)\longrightarrow & C & \longrightarrow & X_1 & \longrightarrow\cdots\longrightarrow & X_{n-1} & \longrightarrow & B' & \longrightarrow(0) & ((G, H)\text{-exact}) \\ & \downarrow 1 & & \downarrow \kappa_1 & & \downarrow \kappa_{n-1} & & \downarrow \beta & & \\ (0)\longrightarrow & C & \longrightarrow & Q_1 & \longrightarrow\cdots\longrightarrow & Q_{n-1} & \longrightarrow & B & \longrightarrow(0) & ((G, H)\text{-exact}), \end{array}$$

where $B' = \mathrm{Im}(X_{n-1}\to X_n)$, and

$$\begin{array}{cccccccc} (0)\longrightarrow & B' & \longrightarrow & X_n & \overset{\varphi}{\longrightarrow} & A & \longrightarrow(0) & ((G, H)\text{-exact}) \\ & \downarrow \beta & & \downarrow \gamma & & \downarrow \alpha & & \\ (0)\longrightarrow & B & \longrightarrow & Q & \underset{\psi}{\longrightarrow} & M & \longrightarrow(0) & ((G, H)\text{-exact}), \end{array}$$

where Q is rationally (G, H)-injective. Let $A+Q$ is the direct sum as F-module. Define a mapping $\kappa : A+Q\to M$ by $\kappa(a, q) = \alpha(a) - \psi(q)$. Then $X = \mathrm{Ker}\ \kappa = \{(\varphi(x), \gamma(x)+b\ ;\ x\in X_n,\ b\in B\}$. Define a G-homomorphism $p\ ;\ X\to A$ by $p(\varphi(x), \gamma(x)+b) = \varphi(x)$. Then $\mathrm{Ker}\ p = \{(0, b)\ ;\ b\in B\}$. Therefore we get a commutative diagram

$$\begin{array}{ccccccccc} & (0)\longrightarrow & B' & \longrightarrow & X_n & \longrightarrow & A & \longrightarrow(0) & \\ & & \downarrow \beta & & \downarrow \gamma' & & \downarrow 1 & & \\ (E_1) & (0)\longrightarrow & B & \longrightarrow & X & \longrightarrow & A & \longrightarrow(0) & ((G, H)\text{-exact}), \end{array}$$

where $\gamma(x) = (\varphi(x), \gamma(x))$. It is clear from the above construction that $(Q(E_1))$ is equivalent to (E_n). Therefore $\tilde{Q}$ is surjective. This completes the proof of Theorem 2.2.

3. Relative group extensions

Let $\mathfrak{g}$ is a Lie algebra over a field F, and let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. Let $\mathfrak{Q}$ be a $\mathfrak{g}$-module. By a $(\mathfrak{g}, \mathfrak{h})$-extension of the abelian Lie algebra $\mathfrak{Q}$ we shall mean an exact sequence of Lie algebras;

$$(\mathfrak{E}) \quad (0)\longrightarrow\mathfrak{Q}\longrightarrow\mathfrak{P}\overset{\sigma}{\longrightarrow}\mathfrak{g}\longrightarrow(0),$$

satisfying the following conditions;

1) there is a linear map $\rho : \mathfrak{g}\to\mathfrak{P}$ such that $\sigma\cdot\rho =$ identity map of $\mathfrak{g}$ and ρ

$[x, y] = [\rho(x), \rho(y)]$ for all $x \in \mathfrak{h}$ and $y \in \mathfrak{g}$,

2) $[p, q] = \sigma(p)q$ for all $p \in \mathfrak{P}$ and $q \in \mathfrak{Q}$.

We shall say that two such extensions $(\mathfrak{E})$, $(\mathfrak{E}')$ are equivalent if there exists an isomorphism φ such that a diagram;

$$\begin{array}{ccccccc} & & & \mathfrak{P} & & & \\ & & \nearrow & & \searrow & & \\ (0) \longrightarrow \mathfrak{Q} & & & \downarrow \varphi & & & \mathfrak{g} \longrightarrow (0) \\ & & \searrow & & \nearrow & & \\ & & & \mathfrak{P}' & & & \end{array}$$

is commutative. We denote by $\mathfrak{E}_{(\mathfrak{g}, \mathfrak{h})}(\mathfrak{Q})$ the set of equivalence classes of $(\mathfrak{g}, \mathfrak{h})$-extensions of $\mathfrak{Q}$. As in the analogous interpretation of the ordinary Lie algebra cohomology group $H^2(\mathfrak{g}, \mathfrak{Q})$, next Proposition can be shown.

PROPOSTION 3.1. *Let $\mathfrak{g}$ be a Lie algebra over a field F, and let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. If $\mathfrak{Q}$ is a $\mathfrak{g}$-module, then the relative Lie algebra cohomology group $H^2(\mathfrak{g}, \mathfrak{h}, \mathfrak{Q})$ may be identified with the set of equivalence classes of $(\mathfrak{g}, \mathfrak{h})$-extensions of $\mathfrak{Q}$.*

Let G be an algebraic linear group over a field F of characteristic 0, and let H be an algebraic subgroup of G. Let A be a rational G-module. Let $X_n(A)$ be as in the proof of Proposition 2.1. Then, for $n \geqq 0$, the G-fixed part $X_n(A)^G$ is isomorphic, as an F-space, with $X_n(A)' = \{f \in X_{n-1}(A)\ ;\ h \cdot f(x_1, \ldots, x_n) = f(hx_1, \ldots, hx_n)$ for all $h \in H\}$; such an isomorphism is given by $g \to \phi_n(g)$, where

$$\phi_n(g)(x_1, \ldots, x_n) = g(1, x_1, \ldots, x_n),$$

its inverse being given by $f \to \phi'_{n-1}(f)$, where

$$\phi'_{n-1}(f)(x_0, \ldots, x_n) = x_0 \cdot f(x_0^{-1}x_1, \ldots, x_0^{-1}x_n).$$

The coboundary for $X(A)'$ becomes $f \to \delta f$, where

$$\begin{aligned} &(\delta f)(x_1, \ldots, x_{n+1}) \\ &= x_1 \cdot f(x_1^{-1}x_2, \ldots, x_1^{-1}x_{n+1}) \\ &+ \sum_{i=1}^{n+1}(-1)^i g(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}). \end{aligned}$$

Let Q be a finite-dimensional rational G-module. Q has the natural structure of an abelian unipotent algebraic linear group. By a rational (G, H)-extension of the abelian unipotent algebraic linear group Q we shall mean an exact sequence of algebraic linear groups;

$$(E)\quad (1)\longrightarrow Q\longrightarrow P\overset{\alpha}{\longrightarrow}G\longrightarrow(1),$$

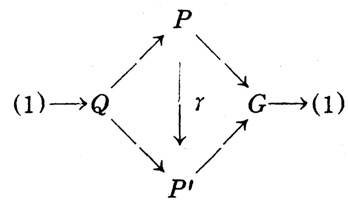
satisfying the following conditions;

1) there is a representative map $\beta: G\rightarrow P$ such that $\alpha\cdot\beta=$ identity map of G, and $\beta(xy)=\beta(x)\beta(y)$ and $\beta(yx)=\beta(y)\beta(x)$ for all $x\in H$ and $y\in G$,

2) $pqp^{-1}=\alpha(p)q$ for all $p\in P$ and $q\in Q$

3) the map $f\rightarrow f\cdot\alpha$ is an isomorphism of $R(G)$ onto the subalgebra $R(P)^{Q}$ of $R(P)$ consisting of the G-fixed elements.

We shall say that two such extensions (E), (E') are equivalent if there exists an isomorphism γ such that a diagram;



is commutative. We denote by $E_{(G,H)}(Q)$ the set of equivalence classes of these extensions.

Now, for a (G, H)-extension (E) of Q, we define $f\in X_2(Q)$ by $f(x_1, x_2)=\log\beta(x_1)\beta(x_1^{-1}x_2)\beta(x_2)^{-1}$. It is clear that $f\in X_2(Q)'$. If β' be any other map satisfying the above condition 1), then f' is cohomologous to f, where $f'=\log\beta'(x_1)\beta'(x_1^{-1}x_2)\beta'(x_2)^{-1}$. Hence a rational (G, H)-extension of Q determines a unique element of $H^2(G, H, Q)$, which depends only on the equivalence class of the given rational (G, H)-extension of Q.

PROPOSITION 3.2. *Let G be a unipotent algebraic linear group over the field F of characteristic 0, H an algebraic subgroup of G, and let $\mathfrak{g}$, $\mathfrak{h}$ be the Lie algebras of G, H respectively. If $\mathfrak{Q}$ is a finite-dimensional rational G-module, then $\mathfrak{E}_{(\mathfrak{g},\mathfrak{h})}(\mathfrak{Q})$ may be identified with $E_{(G,H)}(\mathfrak{Q})$.*

Proof. Let

$$(\mathfrak{E})\quad (0)\longrightarrow\mathfrak{Q}\longrightarrow\mathfrak{P}\overset{\sigma}{\longrightarrow}\mathfrak{g}\longrightarrow(0)$$

be a $(\mathfrak{g}, \mathfrak{h})$-extension of $\mathfrak{Q}$. Then $(\mathfrak{E})$ induces a rational $(G, 1)$-extension of Q;

$$(E)\quad (1)\longrightarrow\mathfrak{Q}\longrightarrow P\overset{\bar{\sigma}}{\longrightarrow}G\longrightarrow(1)$$

where P is the unipotent algebraic linear group consisting of the exponentials of the elements of $\mathfrak{P}$, and $\bar{\sigma}=\exp_{\mathfrak{g}}\cdot\sigma\cdot\log_P$. If $\rho:\mathfrak{g}\to\mathfrak{P}$ is a linear map satisfying the condition in the definition of the $(\mathfrak{g},\mathfrak{h})$-etention of $\mathfrak{Q}$, then, by the Campbell-Hausdorff formula, it is clear that $\bar{\rho}=\exp_{\mathfrak{P}}\cdot\rho\cdot\log_G$ satisfy the condition in the definition of the rational (G, H)-extension of $\mathfrak{Q}$. Therefore (E) is a rational (G, H)-extension of $\mathfrak{Q}$. It is clear that this correspondence induces a map of $\mathfrak{E}_{(\mathfrak{g},\mathfrak{h})}(\mathfrak{Q})$ into $E_{(G,H)}(\mathfrak{Q})$.

Conversely, let (E) be a rational (G, H)-extension, and let $\bar{\rho}:G\to P$ be a map satisfying the condition in the definition. Define $\sigma=\log_G\bar{\sigma}\exp_{\mathfrak{P}}$ and $\rho=\log_{\mathfrak{P}}\cdot\bar{\rho}\cdot\exp_{\mathfrak{g}}$. Then (E) induces a $(\mathfrak{g}, 0)$-extension of $\mathfrak{Q}$;

$$(\mathfrak{E})\quad (0)\longrightarrow\mathfrak{Q}\longrightarrow\mathfrak{P}\overset{\sigma}{\longrightarrow}\mathfrak{g}\longrightarrow(0).$$

In order to examine that $(\mathfrak{E})$ is a $(\mathfrak{g},\mathfrak{h})$-extension, we enlarge the base field F to the field F^* of the power series in one variable t with coefficients in F. Let $\mathfrak{Q}^*$, P^*, G^*, H^* be the algebraic linear groups deduced from $\mathfrak{Q}$, P, G, H by the extension of F to F^*, respectively. Let $\bar{\rho}^*$ be the extension of $\bar{\rho}$. Then

$$\bar{\rho}^*((\exp_{\mathfrak{g}^*}tX)(\exp_{\mathfrak{g}^*}tY))=\bar{\rho}^*(\exp_{\mathfrak{g}^*}tX)\,\bar{\rho}^*(\exp_{\mathfrak{g}^*}tY),$$

for all $X\in\mathfrak{h}$, $Y\in\mathfrak{g}$. Therefore

$$\begin{aligned}&(\log_{P^*}\cdot\bar{\rho}^*)((\exp_{\mathfrak{g}^*}tX)(\exp_{\mathfrak{g}^*}tY))\\&=\log_{P^*}(\bar{\rho}^*(\exp_{\mathfrak{g}^*}tX)\,\bar{\rho}^*(\exp_{\mathfrak{g}^*}tY))\\&=\log_{P^*}((\exp_{\mathfrak{P}^*}t\rho(X))(\exp_{\mathfrak{P}^*}t\rho(Y)).\end{aligned}$$

By the Campbell-Hausdorff formula, we can compare the coefficients of t^2 in the above equality. That is,

$$\rho[X, Y]=[\rho(X), \rho(Y)],\ \text{for all } X\in\mathfrak{h},\ Y\in\mathfrak{g}.$$

Hence $(\mathfrak{E})$ is a $(\mathfrak{g},\mathfrak{h})$-extension of $\mathfrak{Q}$. Clearly, this correspondence of (E) to $(\mathfrak{E})$ induces the inverse of the above map of $\mathfrak{E}_{(\mathfrak{g},\mathfrak{h})}(\mathfrak{Q})$ to $E_{(G,H)}(\mathfrak{Q})$. This completes the proof of Proposition 3.2.

By Proposition 3.1, 3.2, there exists the map of $H^2(\mathfrak{g},\mathfrak{h},\mathfrak{Q})$ to $H^2(G, H, \mathfrak{Q})$. On the other hand there exists the canonical homomorphism; $H^n(G, H, \mathfrak{Q})\to H^n(\mathfrak{g},\mathfrak{h},\mathfrak{Q})$ ([4. Th. 3.5]). By the same way as in [2, p. 518] we can verify that the composition of the above maps; $H^2(\mathfrak{g}, \mathfrak{h}, \mathfrak{Q}) \to H^2(G, H, \mathfrak{Q})) \to H^2(\mathfrak{g}, \mathfrak{h}, \mathfrak{Q})$ is the identity map of $H^2(\mathfrak{g}, \mathfrak{h}, \mathfrak{Q})$. Thus we obtained the next result

THEOREM 3.3. *Let G be a unipotent algebraic linear group over the field F of characteristic* 0, *H an algebraic subgroup of G, and let $\mathfrak{g}$, $\mathfrak{h}$ be the Lie algebras of G, H, respectively. Let $\mathfrak{Q}$ be a finite dimensional rational G-module. Then the canonical homomorphism: $H^2(G, H, \mathfrak{Q}) \to H^2(\mathfrak{g}, \mathfrak{h}, \mathfrak{Q})$ is surjective. Moreover the canonical homomorphism induces a map of $H^2(G, H, \mathfrak{Q})$ onto the set of the equivalence classes of the rational (G, H)-extensions of $\mathfrak{Q}$.*

4. Relatively injective modules

Let G be an algebraic linear group over a field F, and let H be an algebraic subgroup of G. Let M be a rationally (G, H)-injective module. It is known that, for every rational G-module A, the tensor product $A \otimes M$ is rationally (G, H)-injective ([4, Prop. 2.1]). As in the analogous interpretation of [3, Prop. 2.1], the following result can be shown by using Proposition 2.1 and [4, Prop. 2.3].

PROPOSITION 4.1. *Let G be an algebraic linear group over a field F, H an algebraic subgroup of G, and let M be a rational G-module. Suppose that, for every finite-dimensional G-module U, $H^1(G, H, U \otimes M) = (0)$. Then M is rationally (G, H)-injective*

Next Proposition is a generalization of [4, Prop. 2.1].

PROPOSITION 4.2. *Let G, H be as in Proposition* 4.1, *and let L be an algebraic subgroup of G such that there is a rational representative map $\rho : G \to L$ satisfying $\rho(yx) = y\rho(x)$ for all $y \in L$ and $x \in G$. Suppose that $\rho(x)^{-1}\rho(xh) \in L \cap H$ for all $h \in H$ and $x \in G$. Let M be a rational L-module. Then ${}^HR \otimes M$ is rationally $(L, L \cap H)$-injective. If A is any rationally (G, H)-injective module, then $A \otimes M$ is rationally $(L, L \cap H)$-injective.*

Proof. Let $(0) \longrightarrow C \xrightarrow{p} B \longrightarrow A' \longrightarrow (0)$ be a rational $(L, L \cap H)$-exact sequence, where A', B, C are rational L-modules, and let γ be an L-module homomorphism of C into ${}^HR \otimes M$. Let φ be an $L \cap H$-module homomorphism of B onto C such that $\varphi \cdot p$ is the identity map of C. We shall identify elements of ${}^HR \otimes M$ with naturally corresponding maps of G into M. For $b \in B$, define the map

$\beta(b) : G \to M$ by

$$\beta(b)(x) = \rho(x)[\gamma(\varphi(\rho(x)^{-1}\cdot b))(\rho(x)^{-1}x)].$$

By [2, Prop. 2.2], $\beta(b) \in R \otimes M$ and $\gamma = \beta\cdot p$. By assumption, for any $h \in H$ and any $x \in G$, there is $h' \in H \cap L$ such that $\rho(xh) = \rho(x)h'$. By the definition of β,

$$\begin{aligned}\beta(b)(xh) &= \rho(xh)[\gamma(\varphi(\rho(xh)^{-1}\cdot b))(\rho(xh)^{-1}xh)]\\ &= \rho(x)h'[\gamma(\varphi(h'^{-1}\rho(x)^{-1}\cdot b))(h'^{-1}\rho(x)^{-1}xh)]\\ &= \rho(x)[\gamma(\varphi(\rho(x)^{-1}\cdot b))(xh)]\\ &= \beta(b)(x).\end{aligned}$$

Hence $\beta(b) \in {}^{H}R \otimes M$.

The second part of Proposition is shown by the same way as [2, Prop. 2.2]. This completes the proof of Proposition 4.2.

Now we shall assume that the base field F is of characteristic 0. Let L be a unipotent normal algebraic subgroup of G. Then there is a rational representative map $\rho : G \to L$ such that $\rho(yx) = y\rho(x)$ for all $x \in G$ and $y \in L$ ([2. Th. 3.1]). Proposition 4.2 gives the following result.

PROPOSITION 4.3. *Let G be an algebraic linear group over the field F of chara cteristic 0, H an algebraic subgroup of G, and let L be a unipotent normal algebraic subgroup of G. Suppose that $\rho(x)^{-1}\rho(xh) \in L \cap H$ for all $x \in G$ and $h \in H$, where ρ is a rational representative map of G into L such that $\rho(yx) = y\rho(x)$ for all $x \in G$ and $y \in L$. Let M be a rationally (G, H)-injective module. Then M is rationally $(L, L \cap H)$-injective.*

Now we prove the main result in this section.

THEOREM 4.4. *Let P be an algebraic linear group over the field F of chara-cteristic 0, Q an algebraic subgroup of P, and G be a normal algebraic subgroup of P. Let N be the maximal unipotent normal algebraic subgroup of G. Suppose that there is a maximal fully reducible subgroup K of G contained in the normalizer of $N \cap Q$ in G and that $\rho(x)^{-1}\rho(xq) \in M \cap Q$ for all $x \in P$ and $q \in Q$, where ρ is a rational representative map of P into N such that $\rho(np) = n\rho(p)$ for all $p \in P$ and $n \in N$. Let M be a rationally (P, Q)-injective module and let K' be a fully reducible algebraic subgroup of K. Then M is rationally (G, H)-injective, where $H = K'\cdot(N \cap Q)$.*

Proof. By Proposition 4.3, M is rationally $(N, N\cap Q)$-injective. Let U be any rational G-module. Then $U\otimes M$ is rationally $(N, N\cap Q)$-injective. By [4, Th. 2.5], for every rational G-module A,

$$H(G, H, A)=H(N, N\cap Q, A)^{G/N}.$$

In particular, it follow that $H^1(G, H, U\otimes M)=(0)$. Hence, by Proposition 4.1, M is rationally (G, H)-injective. This completes the proof of Theorem 4.4.

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