HOMOLOGY OF NON-COMMUTATIVE POLYNOMIAL RINGS

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§ 1. Introduction

Let Γ be a ring with unit element and let Λ be the Ore extension of Γ with respect to a derivation d of Γ [4, 3]. It is shown in [3] that l.gl. dim Λ $\leqq 1 +$ l.gl. dim Γ. It is not in general possible to replace this inequality by equality.

We consider here the special case where Γ is the polynomial ring in n variables over a commutative ring K. If d is a K-derivation of Γ then Λ becomes a K-algebra and we prove that if further Λ is a supplemented K-algebra, we have l.gl.dim $\Lambda = 1 +$ l.gl.dim Γ (Theorem 1). The proof consists first in constructing a Λ-free complex of length $n+1$ for K, which we prove to be acyclic (Proposition 2) by putting a suitable filtration on this complex and passing to the associated graded. We use this resolution to prove that $\mathrm{l.dim}_\Lambda K = n+1$. We then employ a spectral sequence argument to complete the proof of Theorem 1. If Λ is not supplemented, Theorem 1 is not necessarily valid [5].

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§ 2

Let K be a commutative ring with 1 and let $\Gamma = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over K. Let d be a K-derivation of Γ into itself. Clearly d is uniquely determined by its values f_i on x_i. Conversely, given n polynomials $f_i \in \Gamma$, $1 \leq i \leq n$, there exists a K-derivation d of Γ into itself with $d(x_i) = f_i$, $1 \leq i \leq n$.

Let Λ be the non-commutative polynomial ring in one variable x_{n+1} over Γ with respect to d. Then Λ is the K-algebra with generators $x_1, \ldots, x_{n+1}$ and relations given by

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$$x_i x_j - x_j x_i = 0,\ 1 \leq i,\ j \leq n \text{ and } x_{n+1} x_i - x_i x_{n+1} = f_i,\ 1 \leq i \leq n.$$

PROPOSITION 1. *The K-algebra Λ is a supplemented algebra if and only if there exist* $\alpha_1, \ldots, \alpha_n \in K$ such that $f_i(\alpha_1, \ldots, \alpha_n) = 0,\ 1 \leq i \leq n$.

Proof. Let $\varepsilon : \Lambda \to K$ be a supplementation and let $\varepsilon(x_i) = \alpha_i,\ 1 \leq i \leq n$. We have

$$\begin{aligned} f_i(\alpha_1, \ldots, \alpha_n) = f_i(\varepsilon(x_1), \ldots, \varepsilon(x_n)) &= \varepsilon(f_i(x_1, \ldots, x_n)) \\ &= \varepsilon(x_{n+1} x_i - x_i x_{n+1}) \\ &= \varepsilon(x_{n+1})\varepsilon(x_i) - \varepsilon(x_i)\varepsilon(x_{n+1}) \\ &= 0. \end{aligned}$$

Conversely, let $\alpha_j \in K,\ 1 \leq j \leq n$ with $f_i(\alpha_1, \ldots, \alpha_n) = 0,\ 1 \leq i \leq n$. Define $\varepsilon(x_j) = \alpha_j,\ 1 \leq j \leq n,\ \varepsilon(x_{n+1}) = 0$. It is easily verified that ε can be extended to a K-algebra homomorphism of Λ onto K.

From now onwards, we assume that Λ is a supplemented algebra, that is, there exist $\alpha_j \in K,\ 1 \leq j \leq n$ with $f_i(\alpha_1, \ldots, \alpha_n) = 0,\ 1 \leq i \leq n$. It is easy to verify that there exists a K-algebra automorphism ϕ of Λ such that $\phi(x_i) = x_i + \alpha_i,\ 1 \leq i \leq n$ and $\phi(x_{n+1}) = x_{n+1}$. Thus, we may assume without loss of generality that $\alpha_j = 0,\ 1 \leq j \leq n$ and the supplementation ε is given by $\varepsilon(x_j) = 0,\ 1 \leq j \leq n+1$. We may now write

$$f_i = \sum_{1 \leq j \leq n} f_{ji} x_j,\ f_{ji} \in \Gamma.$$

The matrix (f_{ij}) defines a Γ-linear map δ_1 of the first homogeneous component $E_1^{\Gamma}(y_1, \ldots, y_n)$ of the exterior algebra over Γ in the variables $y_1, \ldots, y_n$, given by

$$\delta_1(y_i) = \sum_{1 \leq j \leq n} f_{ji} y_j.$$

Let δ denote the extension of δ_1 to a derivation of $E^{\Gamma}(y_1, \ldots, y_n)$ into itself.

We write $\overline{X}_i = \Lambda \otimes_K E_i(y_1, \ldots, y_{n+1}),\ (i \geq 0)$, where

$E_i(y_1, \ldots, y_{n+1})$ is the i^{th} component of the exterior algebra over K in the variables $y_1, \ldots, y_{n+1}$. We identify $\overline{X}_0$ with Λ. We define the left Λ-homomorphisms $\overline{d}_k : \overline{X}_k \to \overline{X}_{k-1}\ (k \geq 1)$ as follows:

$$\overline{d}_1(1 \otimes y_i) = x_i,\ 1 \leq i \leq n+1,$$

For $i \geq 2$,

$$\bar{d}_i(1\otimes y_{j_1}\cdots y_{j_i}) = \sum_{1\leq k\leq i}(-1)^{k+1}x_{j_k}\otimes y_{j_1}\cdots\hat{y}_{j_k}\cdots y_{j_i}\ ;\ j_1<\cdots<j_i<n+1$$

and

$$\bar{d}_i(1\otimes y_{j_1}\cdots y_{j_{i-1}}y_{n+1}) = \bar{d}_{i-1}(1\otimes y_{j_1}\cdots y_{j_{i-1}})y_{n+1} + (-1)^{i-1}x_{n+1}\otimes y_{j_1}\cdots y_{j_{i-1}} + (-1)^i\delta(y_{j_1}\cdots y_{j_{i-1}})$$

where $\delta(y_{j_1}\cdots y_{j_{i-1}})\in E^{\Gamma}_{i-1}(y_1,\ldots,y_n) = \Gamma\otimes_K E_{i-1}(y_1,\ldots,y_{n+1}) \subset \Lambda\otimes_K E_{i-1}(y_1,\ldots,y_{n+1})$.

PROPOSITION 2. *The sequence*

$$(*)\qquad 0\longrightarrow \bar{X}_{n+1}\xrightarrow{\bar{d}_{n+1}}\bar{X}_n\longrightarrow\cdots\longrightarrow\bar{X}_1\xrightarrow{\bar{d}_1}\bar{X}_0\xrightarrow{\varepsilon}K\longrightarrow 0$$

is a left Λ-free resolution of K considered as a left Λ-module through ε.

Proof. Since $\varepsilon\bar{d}_1(1\otimes y_i) = \varepsilon(x_i) = 0$ for $1\leq i\leq n+1$, it follows that $\varepsilon\circ\bar{d}_1 = 0$. We now verify that $\bar{d}_{i-1}\circ\bar{d}_i = 0$, $1<i\leq n+1$. We write $z = y_{j_1}\cdots y_{j_i}$. If $j_i<n+1$, we have $\bar{d}_{i-1}\circ\bar{d}_i(1\otimes z) = 0$ since, in this case, $\bar{d}_i$ is the usual boundary homomorphism in the Koszul-resolution for K considered as a Γ-module [1, p. 151].

Let $j_i = n+1$. We write $y = y_{j_1}\cdots y_{j_{i-1}}$ and $\hat{y}_k = y_{j_1}\cdots\hat{y}_{j_k}\cdots y_{j_{i-1}}$. We have

$$\bar{d}_{i-1}\circ\bar{d}_i(1\otimes yy_{n+1}) = \bar{d}_{i-1}(\bar{d}_i(1\otimes y)y_{n+1}) + (-1)^{i-1}x_{n+1}\bar{d}_{i-1}(1\otimes y) + (-1)^i\bar{d}_{i-1}\delta(y).$$

Now

$$\begin{aligned}
\bar{d}_{i-1}(\bar{d}_i(1\otimes y)y_{n+1}) &= \sum_{1\leq k\leq i-1}(-1)^{k+1}x_{j_k}\bar{d}_{i-1}(1\otimes\hat{y}_k y_{n+1})\\
&= \sum_{1\leq k\leq i-1}(-1)^{k+1}x_{j_k}\{\bar{d}_{i-1}(1\otimes\hat{y}_k)y_{n+1} + (-1)^{i-2}x_{n+1}\otimes\hat{y}_k + (-1)^{i-1}\delta\hat{y}_k\}\\
&= \bar{d}_{i-1}\Big(\sum_{1\leq k\leq i-1}(-1)^{k+1}x_{j_k}\otimes\hat{y}_k\Big)y_{n+1} + (-1)^{i-2}\sum_{1\leq k\leq i-1}(-1)^{k+1}x_{n+1}x_{j_k}\otimes\hat{y}_k +\\
&\quad + (-1)^{i-1}\sum_{1\leq k\leq i-1}(-1)^{k+1}f_{j_k}\otimes y_k + (-1)^{i-1}\delta\Big(\sum_{1\leq k\leq i-1}(-1)^{k+1}x_{j_k}\otimes\hat{y}_k\Big).\\
&= \bar{d}_{i-1}\circ\bar{d}_i(1\otimes y)y_{n+1} + (-1)^{i-2}\sum_{1\leq k\leq i-1}(-1)^{k+1}x_{n+1}x_{j_k}\otimes\hat{y}_k +\\
&\quad + (-1)^{i-1}\sum_{1\leq k\leq i-1}(-1)^{k+1}f_{j_k}\otimes\hat{y}_k + (-1)^{i-1}\delta\bar{d}_{i-1}(1\otimes y).\\
&= (-1)^{i-2}x_{n+1}\bar{d}_{i-1}(1\otimes y) + (-1)^{i-1}\sum_{1\leq k\leq i-1}(-1)^{k+1}f_{j_k}\otimes\hat{y}_k +\\
&\quad + (-1)^{i-1}\delta\bar{d}_{i-1}(1\otimes y).
\end{aligned}$$

Hence

$$\bar{d}_{i-1}\circ\bar{d}_i(1\otimes yy_{n+1}) = (-1)^i\{(\bar{d}_{i-1}\delta - \delta\bar{d}_{i-1})(1\otimes y) - \sum_{1\leq k\leq i-1}(-1)^{k+1}f_{j_k}\otimes\hat{y}_k\},$$

Since δ is a derivation of $E^{\Gamma}(y_1, \ldots, y_n)$ and $\bar{d} = (\bar{d}_i)$ restricted to $E^{\Gamma}(y_1, \ldots, y_n)$ is an antiderivation, it follows that $\bar{d}\delta - \delta\bar{d}$ is an antiderivation of $E^{\Gamma}(y_1, \ldots, y_n)$. Further,

$$(\bar{d}\delta - \delta\bar{d})(y_i) = \bar{d}(\sum_{1\leq j\leq n} f_{ji}y_j) = \sum_{1\leq j\leq n} f_{ji}x_j = f_i, \quad 1\leq i\leq n.$$

Hence it is clear that

$$(\bar{d}_{i-1}\delta - \delta\bar{d}_{i-1})(1\otimes y) = \sum_{1\leq k\leq i-1} (-1)^{k+1} f_{j_k}\otimes \hat{y}_k.$$

Thus $\bar{d}_{i-1}\circ\bar{d}_i(1\otimes yy_{n+1}) = 0$.

Thus $(*)$ is a complex of left Λ-modules and it is clear that $\operatorname{Ker}\varepsilon = \operatorname{Im}\bar{d}_1$. To prove the exactness of $(*)$ we define a suitable filtration of the complex

$$0\to \bar{X}_{n+1}\to \cdots \to \bar{X}_0$$

whose associated graded complex is exact. By a well-known lemma on filtered complexes, the exactness follows immediately.

Let $F_p\Lambda$ be the K-submodule of Λ consisting of all elements of Λ of degree less than or equal to p in x_{n+1}. Then Λ is a filtered ring whose associated graded ring $E^\circ(\Lambda)$ is isomorphic to $K[x_1, \ldots, x_{n+1}]$ (See [5]). We define a gradation on $E_i(y_1, \ldots, y_{n+1})$ by assigning the degree 0 to y_i, $1\leq i\leq n$ and the degree 1 to y_{n+1}. Moreover,

$$E_i(y_1, \ldots, y_{n+1}) = E_i(y_1, \ldots, y_n)\oplus E_{i-1}(y_1, \ldots, y_n)y_{n+1}.$$

We define

$$F_p\bar{X}_i = [F_p\Lambda\otimes E_i(y_1, \ldots, y_n)]\oplus[F_{p-1}\Lambda\otimes E_{i-1}(y_1, \ldots, y_n)y_{n+1}].$$

It is easily seen that $\{F_p\bar{X}_i\}_{p\geq 0}$ is a filtration of $\bar{X}_i$ and that $\bar{d}_i(F_p\bar{X}_i)\subset F_p\bar{X}_{i-1}$. We thus get the complex

$$0\longrightarrow E_p^\circ(\bar{X}_{n+1})\xrightarrow{E^\circ(\bar{d}^{\,p}_{n+1})} E_p^\circ(\bar{X}_n)\longrightarrow\cdots\xrightarrow{E^\circ(\bar{d}^{\,p}_1)} E_p^\circ(\bar{X}_0).$$

We have

$$E_p^\circ(\bar{X}_i)\approx[E_p^\circ(\Lambda)\otimes E_i(y_1, \ldots, y_n)]\oplus[E_{p-1}^\circ(\Lambda)\otimes E_{i-1}(y_1, \ldots, y_n)y_{n+1}].$$

Let now (X_i, d_i) be the Koszul resolution for K as a $K[x_1, \ldots, x_{n+1}]$-module. We define a gradation on $K[x_1, \ldots, x_{n+1}]$ by assigning degrees 0 to x_i, $1\leq i\leq n$ and degree 1 to x_{n+1}. We introduce a gradation on X_i by setting

$$X_i^p = [K[x_1, \ldots, x_{n+1}]_p \otimes E_i(y_1, \ldots, y_n)] \oplus [K[x_1, \ldots, x_{n+1}]_{p-1} \otimes E_{i-1}(y_1, \ldots, y_n)y_{n+1}],$$

where $K[x_1, \ldots, x_{n+1}]_p$ is the p^{th} homogeneous component in the gradation of $K[x_1, \ldots, x_{n+1}]$ defined above. It is easily seen that $d_i(X_i^p) \subset X_{i-1}^p$, and that the sequence

$$0 \longrightarrow X_{n+1}^p \xrightarrow{d_{n+1}^p} X_n^p \longrightarrow \cdots X_1^p \xrightarrow{d_1^p} X_0^p,$$

is exact for every p.

Clearly $E_p^\circ(\overline{X}_i) \approx X_i^p$. Since for any $\varphi \in F_{p-1}\Lambda$ and $y_{j_1} \cdots y_{j_{i-1}} \in E_{i-1}(y_1 \cdots y_n)$, we have $\varphi\delta(y_{j_1} \cdots y_{j_{i-1}}) \in F_{p-1}\overline{X}_{i-1}$, it follows that $E_p^\circ(\overline{d}_i) = d_i^p$. Thus the complex $(E_p^\circ(\overline{X}_i), E_p^\circ(\overline{d}_i))$ is isomorphic to (X_i^p, d_i^p). Since (X_i^p, d_i^p) is exact, it follows that $(E_p^\circ(\overline{X}_i), E_p^\circ(\overline{d}_i))$ is exact and hence $(*)$ is exact. Since $\overline{X}_i$ is clearly a free left Λ-module, the proposition is proved.

THEOREM 1. *Let K be a commutative ring with 1 and let Λ be the K-algebra generated by $x_1, \ldots, x_{n+1}$ with the relations $x_ix_j - x_jx_i = 0$, $1 \leq i, j \leq n$, and $x_{n+1}x_i - x_ix_{n+1} = f_i$, $f_i \in K[x_1, \ldots, x_n]$, $1 \leq i \leq n$. If Λ is a supplemented K-algebra, and K is considered as a left Λ-module through the supplementation, we have* $\text{l.dim}_\Lambda K = n+1$. *Further* $\text{l.gl.dim}\,\Lambda = n+1+\text{gl.dim}\,K$.

Proof. As remarked earlier, we may assume that there exists a supplementation ε with $\varepsilon(x_i) = 0$, $1 \leq i \leq n+1$. It follows from Proposition 2 that $\text{l.dim}_\Lambda K \leq n+1$. We now prove that $\text{l.dim}_\Lambda K = n+1$. For this we first compute $\overline{d}_{n+1}$. Let $w = 1 \otimes y_1 \cdots y_{n+1} \in \overline{X}_{n+1}$ and $w_i = (-1)^{i+1} 1 \otimes y_1 \cdots \hat{y}_i \cdots y_{n+1} \in \overline{X}_n$, $1 \leq i \leq n+1$. We have

$$\begin{aligned}\overline{d}_{n+1}(w) &= \sum_{1 \leq i \leq n} x_iw_i + x_{n+1}w_{n+1} - \sum_{1 \leq i \leq n} f_{ii}w_{n+1} \\ &= \sum_{1 \leq i \leq n} x_iw_i + (x_{n+1} - \sum_{1 \leq i \leq n} f_{ii})\, w_{n+1}.\end{aligned}$$

Let θ be the automorphism of Λ given by

$$\theta(x_i) = x_i, \quad 1 \leq i \leq n$$
$$\theta(x_{n+1}) = x_{n+1} - \sum_{1 \leq i \leq n} f_{ii}.$$

We have

$$\overline{d}_{n+1}(w) = \sum_{1 \leq i \leq n+1} \theta(x_i)\, w_i.$$

Thus $\mathrm{Ext}_\Lambda^{n+1}(K, M) = H_{n+1}(\mathrm{Hom}_\Lambda(\overline{X}, M)), \ ({}_\Lambda M)$
$= \mathrm{Hom}_\Lambda(\overline{X}_{n+1}, M)/B^{n+1}$

where $B^{n+1} = \{g \in \mathrm{Hom}_\Lambda(\overline{X}_{n+1}, M) \mid g(w) = \sum_{1 \leq i \leq n+1} \theta(x_i) h(w_i)$, for some $h \in \mathrm{Hom}(\overline{X}_n, M)\}$. It is clear that the K-isomorphism $\mathrm{Hom}_\Lambda(\overline{X}_{n+1}, M) \approx M$ given by $g \to g(w)$ induces an isomorphism

$$\mathrm{Ext}_\Lambda^{n+1}(K, M) \approx M/\theta(I)M$$

where $I = \mathrm{Ker}\ \varepsilon$. In particular $\mathrm{Ext}_\Lambda^{n+1}(K, {}_{\theta^{-1}}M) \approx M/IM$. Taking M to be any K-module and considering it as a left Λ-module through ε, we find that

$$(*) \qquad \mathrm{Ext}_\Lambda^{n+1}(K, {}_{\theta^{-1}}M) \approx M.$$

Choosing M to be nonzero we find that $\mathrm{l.dim}_\Lambda K = n+1$.

We now prove that $\mathrm{l.gl.dim}\ \Lambda = n + 1 + \mathrm{gl.dim}\ K$. If $\mathrm{gl.dim}\ K = \infty$, since $\mathrm{l.gl.dim}\ \Lambda \geq \mathrm{gl.dim}\ K$ [2, p. 74, Prop. 2], we have $\mathrm{l.gl.dim}\ \Lambda = \infty$ and we have the required equality. Suppose then that $\mathrm{gl.dim}\ K = m < \infty$. Let $\Gamma = K[x_1, \ldots, x_n]$. In view of [6, Th. 1] or [3], it follows that $\mathrm{l.gl.dim}\ \Lambda \leq 1 + \mathrm{l.gl.dim}\ \Gamma = n + 1 + \mathrm{gl.dim}\ K$. To prove equality, we need the "maximum term principle" [2] for spectral sequences. The map $\varepsilon : \Lambda \to K$ gives rise to a spectral sequence (see [1, p. 349]).

$$\mathrm{Ext}_K^p(A, \mathrm{Ext}_\Lambda^q(K, C)) \Rightarrow \mathrm{Ext}_\Lambda^n(A, C), \ ({}_K A, {}_\Lambda K_K, {}_\Lambda C).$$

Since $\mathrm{gl.dim}\ K = m$ and $\mathrm{l.dim}_\Lambda K = n+1$, we have that $\mathrm{Ext}_K^p(A, \mathrm{Ext}_\Lambda^q(K, C)) = 0$ if $p > m$ or $q > n+1$. Thus $\mathrm{Ext}_\Lambda^r(A, C) = 0$ for $r > m+n+1$ and we have an isomorphism

$$\mathrm{Ext}_K^m(A, \mathrm{Ext}_\Lambda^{n+1}(K, C)) \approx \mathrm{Ext}_\Lambda^{m+n+1}(A, C).$$

Let A be a K-module such that $\mathrm{l.dim}_K A = m$. Then, there exists a ${}_K C'$ such that $\mathrm{Ext}_K^m(A, C') \neq (0)$. Consider C' as a left Λ-module through ε and take $C = {}_{\theta^{-1}}C'$, where θ is the automorphism of Λ defined above. We have by $(*)$, $\mathrm{Ext}_\Lambda^{n+1}(K, C) \approx C'$ and thus

$$\mathrm{Ext}_\Lambda^{m+n+1}(A, C) \approx \mathrm{Ext}_K^m(A, C') \neq (0).$$

Hence $\mathrm{gl.dim}\ \Lambda \geq m + n + 1$. This completes the proof of the theorem.

Remark. If Λ is not a supplemented algebra, we may have $\mathrm{l.gl.dim}\ \Lambda < n + 1 + \mathrm{gl.dim}\ K$. In fact, let $\Gamma = K[x_1]$, where K is a field of characteristic 0.

The K-algebra Λ on generators x_1, x_2 with the relation $x_2x_1 - x_1x_2 = 1$ is an Ore-extension of Γ and l.gl.dim $\Lambda = 1$ [5].

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