ON THE BOUNDARY BEHAVIOR OF HOLOMORPHIC FUNCTIONS IN THE UNIT DISK

SISTER CLAUDE MARIE FAUST, C. C. V. I.

I. Introduction

1. Let $f(z)$ be a holomorphic function defined in the unit disk $|z|<1$, which we shall denote by D. Let Σ be a subset of D, whose closure has at least one point in common with C, the circumference of the unit disk. The set of all values a such that the equation $f(z)=a$ has infinitely many solutions in Σ is called the *range of* $f(z)$ *in* Σ, and is denoted by $R(f, \Sigma)$. Let τ be a point of C, and let $\{z_n\}$ be a sequence of points in D with the properties: $z_n = r_n \tau$, $0<r_n<1$, $\lim_{n\to\infty} r_n = 1$. The non-Euclidean (hyperbolic) distance $\rho(z_n, z_{n+1})$ between two points z_n and z_{n+1} of the sequence is defined to be equal to

$$\frac{1}{2}\log\frac{1+u}{1-u}, \quad u = \left|\frac{z_n - z_{n+1}}{1-\bar{z}_n z_{n+1}}\right|$$

(cf. [3], Ch. II).

We shall abbreviate the expression "non-Euclidean" to *n-E*. For a discussion of the *n-E* geometrical matters involved in this paper, the reader is referred to [3].

Given a point τ on C, the set of all points z in D for which

$$-\frac{\pi}{2}<\alpha<\arg(1-\bar{\tau}z)<\beta<\frac{\pi}{2}, \quad |z-\tau|<\varepsilon,$$

where α and β are given angles and ε is so small that the boundary of the resulting set has only the point τ in common with C shall be called a *Stolz angle at* τ. If $\alpha=-\beta$, the resulting set is called a *symmetric Stolz angle with vertex* τ *and of opening* 2β, and will be denoted by $\Delta_{\tau,\beta}$.

It is the purpose of the present paper to study the boundary behavior of

Received June 14, 1961.

This paper is part of the author's doctoral dissertation (University of Notre Dame, 1961). She would like to express her gratitude to Professor Wladimir Seidel for his many helpful suggestions in directing this dissertation.

a holomorphic function in the neighborhood of the point τ, $|\tau|=1$. We shall arrive at a generalization of a theorem of W. Seidel. The concepts and method used in proving it are essentially the same that were employed by Seidel (cf. [9], pp. 159-171).

2. The following notations will also be used in the formulation of the theorem:

(a) For every r with $0<r<1$, we shall let

$$D_r=\{z \mid |z|\angle r\} \text{ and } \overline{D}_r=\{z \mid |z|\leqq r\}.$$

We shall denote the open and closed n-E circular disks with n-E center z and n-E radius ρ by $D(z, \rho)$ and $\overline{D}(z, \rho)$, respectively. We shall also denote the circumference of the n-E circular disk with n-E center z and n-E radius ρ by $C(z, \rho)$.

(b) Given $f(z)$ a holomorphic function in D. For each z_n in the sequence $\{z_n\}$, we shall denote the function $f\left(\frac{z+z_n}{1+\bar{z}_n z}\right)$, holomorphic in D, by $f(z; z_n)$.

(c) For any angle α, $0<\alpha<\frac{\pi}{2}$, we let

$$\sigma=\frac{1}{2}\log\cot\left(\frac{\pi}{4}-\frac{\alpha}{2}\right).$$

If Ω is the diameter of the unit disk connecting τ and $-\tau$, where $|\tau|=1$, then

$$H_{\tau,\alpha}\doteq\bigcup_{z\in\Omega}D(z, \sigma)$$

is the lens-shaped region bounded by two hypercycles (cf. [3], Ch. II) symmetric in the diameter Ω and forming at τ the angles α and $-\alpha$ with Ω.

II. A Theorem

3. We now prove the following generalization of a theorem given by W. Seidel ([9], pp. 166-169, Theorem 4):

THEOREM. *Let $f(z)$ be holomorphic in D, let τ be a point of C, and let $z_n=r_n\tau$, $0<r_n<1$, $\lim_{n\to\infty} r_n=1$, be a sequence of points for which*

(1) $$\rho(z_n, z_{n+1})<M$$

where M is a positive constant, and $n=1, 2, \ldots$, and

(2) $$\lim_{n\to\infty} f(z_n) = \infty .$$

Then, there exists a real number α_τ, *with* $0 \leqq \alpha_\tau \leqq \frac{\pi}{2}$, *such that*

1. $f(z)$ *tends to infinity in every Stolz angle* $\Delta_{\tau,\beta}$, *where* $\beta < \alpha_\tau$;

2. *The complement of the range of the function in the Stolz angle* $\Delta_{\tau,\beta}$, $\mathfrak{C}R(f, \Delta_{\tau,\beta})$, *consists of at most one point for every Stolz angle* $\Delta_{\tau,\beta}$, *where* $\beta > \alpha_\tau$.

Note. The extreme case $\alpha_\tau = 0$ must be interpreted to mean that conclusion 2 holds for every Stolz angle $\Delta_{\tau,\beta}$, while the extreme case $\alpha_\tau = \frac{\pi}{2}$ must be interpreted to mean that conclusion 1 holds for every Stolz angle $\Delta_{\tau,\beta}$.

The above theorem differs from the theorem of Seidel only in the restriction imposed upon the sequence of points $\{z_n\}$. In his theorem, Seidel specifies that $\lim_{n\to\infty} \rho(z_n, z_{n+1}) = 0$.

4. In order to establish the theorem, we shall first prove the following lemmas:

LEMMA 1. *Let* $f(z)$ *be holomorphic in D, let* τ *be a point of C, and let* $\{z_n\}$ *be a sequence of points with the same properties as in the theorem. Let the family* $\{f(z; z_n), n = 1, 2, \ldots\}$ *be normal in D. Then the point* τ *is a Fatou point* (*cf.* [7], *p.* 59) *of* $f(z)$ *with the limit* ∞.

Proof. For each z_n, the function $f(z; z_n)$ is holomorphic in D. We have

$$f(0; z_n) = f(z_n)$$

so that, by (2), we have

(3) $$\lim_{n\to\infty} f(0; z_n) = \infty .$$

Let $\Delta_{\tau,\beta}$ be any given symmetric Stolz angle with vertex τ and of opening 2β, $0 < \beta < \frac{\pi}{2}$. We want to find a sequence of closed n-E disks $\bar{D}(z_n, \gamma)$ with γ large enough so that the union $\bigcup_{n=1}^{\infty} \bar{D}(z_n, \gamma)$ will contain in its interior the intersection of some neighborhood of τ with $\Delta_{\tau,\beta}$. It is clear that this construction is always possible.

Now, by hypothesis, the family $\{f(z; z_n)\}$ is normal in D, so that (3) implies that

$$\lim_{n\to\infty} f(z; z_n) = \infty$$

uniformly on every disk $\overline{D}_r$, $r<1$. In particular, setting $r=\tanh\gamma$ and noting that $f(z)$ assumes the same values in $D(z_n, \gamma)$ as $f(z;\ z_n)$ does in D_r, we see that $f(z)$ tends to infinity on the sequence of the disks $\overline{D}(z_n, \gamma)$. Hence, we infer that $f(z)$ tends to infinity as $z\to\tau$ in $\Delta_{\tau,\beta}$. Since the symmetric Stolz angle $\Delta_{\tau,\beta}$ was taken to be arbitrary, $0<\beta<\frac{\pi}{2}$, we arrive at the conclusion that τ is a *Fatou point of* $f(z)$ *with the limit* ∞.

LEMMA 2. *Let* $f(z)$ *be holomorphic in D, let* τ *be a point of C, and let* $z_n=r_n\tau$, $0<r_n<1$, $\lim_{n\to\infty} r_n=1$ *be a sequence of points in D. Let the point* $z=0$ *be an irregular point* (*cf.* [6], *p.* 37) *of the family of functions* $\{f(z;\ z_n)\}$. *Then* $\mathfrak{C}R(f, \Delta_{\tau,\alpha})$ *consists of at most one point for every Stolz angle* $\Delta_{\tau,\alpha}$.

Proof. Since the point $z=0$ is an irregular point of the family $\{f(z;\ z_n)\}$, the family fails to be normal at $z=0$. Hence, in every neighborhood D_λ, $\lambda<1$, of $z=0$, every value, except perhaps one, is assumed by infinitely many of the functions of the family ([6], p. 61). Now, $f(z;\ z_n)$ assumes in the disk $D(0, \sigma)$, *where* $\sigma=\frac{1}{2}\log\frac{1+\lambda}{1-\lambda}$, the same values as $f(z)$ assumes in the disk $D(z_n, \sigma)$. The *n-E* disks are all contained within the region $H_{\tau,\alpha}$ bounded by two hypercycles symmetric in the diameter connecting the points τ and $-\tau$ and forming at τ angles α and $-\alpha$ with the diameter, where $\alpha=2\arctan\lambda$. But in a neighborhood of τ, the region $H_{\tau,\alpha}$ is contained within the Stolz angle $\Delta_{\tau,\alpha}$. Hence, $\mathfrak{C}R(f, \Delta_{\tau,\alpha})$ consists of at most one point for every Stolz angle $\Delta_{\tau,\alpha}$.

LEMMA 3. *Let* $f(z)$ *be holomorphic in D, and let* τ *be a point of C. We associate with every sequence* $\{\zeta_n\}$, $\zeta_n=r_n\tau$, $0<r_n<1$, $\lim_{n\to\infty} r_n=1$, *a non-negative number* Γ *in the following manner:* Γ *is the l.u.b. of the n-E lengths of the radii of all disks* $\overline{D}_c$, $c<1$, *within which the family* $\{f(z;\ \zeta_n)\}$ *is normal. If there exists at least one sequence of sequences* $\{z_n^{(\nu)}\}$ *such that the associated numbers* $\Gamma_\nu\to 0$, *then* $\mathfrak{C}R(f, \Delta_{\tau,\alpha})$ *consists of at most one point for every Stolz angle* $\Delta_{\tau,\alpha}$, *and so* $\alpha_\tau=0$.

Proof. Let $\Delta_{\tau,\alpha}$ be a given symmetric Stolz angle with vertex τ and of opening 2α, where α is an arbitrarily small fixed number. Since we are given a sequence of sequences $\{z_n^{(\nu)}\}$ with the associated numbers Γ_ν, such that $\Gamma_\nu\to 0$, we know that there exists a sequence $\{z_n^{(\nu_0)}\}$ with the associated number $\Gamma_{\nu_0}<\tan\frac{\alpha}{2}$. The family $\{f(z;\ z_n^{(\nu_0)})\}$ fails to be normal in the disk D_σ, $\Gamma_{\nu_0}<\sigma<$

$\tan\frac{\alpha}{2}$. Thus, there exists a point z_0 with $|z_0|<\sigma$, such that every value, except perhaps one, is assumed by infinitely many of the functions of the family $\{f(z;\ z_n^{(\nu_0)})\}$ in every n-E disk with n-E center z_0. Choose the n-E radius of such a disk so small that the disk lies wholly within the disk D_σ. Now setting $r=\frac{1}{2}\log\frac{1+\sigma}{1-\sigma}$, $f(z;\ z_n^{(\nu_0)})$ assumes in D_σ the same values as $f(z)$ assumes in $D(z_n^{(\nu_0)},\ r)$. Then, setting $\alpha^*=2$ arc $\tan\Gamma_{\nu_0}$, it follows by the same argument as in Lemma 2, that $\mathfrak{C}R(f,\ \Delta_{\tau,\beta})$ consists of at most one point for every Stolz angle $\Delta_{\tau,\beta}$, $\beta>\alpha^*$. Since $\alpha^*<\alpha$, and since α was given to be an arbitrarily small number, it follows that $\mathfrak{C}R(f,\ \Delta_{\tau,\alpha})$ will consist of at most one point for every Stolz angle $\Delta_{\tau,\alpha}$, and so $\alpha_\tau=0$.

5. We can now proceed with the proof of the theorem. For each z_n consider the function $f(z;\ z_n)$ holomorphic in D.

We shall now examine the family $\{f(z;\ z_n)\}$ for normality. There are altogether three mutually exclusive cases to be considered:

I. The family $\{f(z;\ z_n)\}$ is normal in D;

II. The family $\{f(z;\ z_n)\}$ is not normal in D, but is normal at $z=0$;

III. The family $\{f(z;\ z_n)\}$ is not normal at $z=0$.

Consider Case I. In this case, the family $\{f(z;\ z_n)\}$ is normal in D. By Lemma 1 we arrive at the conclusion that in Case I the point τ is a *Fatou point of* $f(z)$ *with the limit* ∞, and we have $\alpha_\tau=\frac{\pi}{2}$.

Let us next consider Case III. In this case, the family $\{f(z;\ z_n)\}$ fails to be normal at the point $z=0$, and, according to Lemma 2, $\mathfrak{C}R(f,\ \Delta_{\tau,\alpha})$ consists of at most one point for every Stolz angle $\Delta_{\tau,\alpha}$, and we have $\alpha_\tau=0$.

Finally, in Case II, let $0<q<1$ be the smallest modulus of all those points in D at which the family $\{f(z;\ z_n)\}$ fails to be normal. Since the set of such points is closed relative to D ([6], p. 38), such a smallest positive modulus exists. Setting $\sigma=\frac{1}{2}\log\frac{1+q}{1-q}$ construct the open disks $D(z_n,\ \sigma)$, $n=1,2,\ldots$

Consider now the family of all sequences $\{z_n^{(\nu)}\}_{\nu\in I}$ where I is an uncountable index set, such that

$$z_n^{(\nu)}=r_n^{(\nu)}\tau,\qquad 0<r_n^{(\nu)}<1,\qquad \lim_{n\to\infty} r_n^{(\nu)}=1.$$

For each $\nu\in I$, let Γ_ν be the l. u. b. of the radii of all circles D_c, $c<1$, within which the family $\{f(z;\ z_n^{(\nu)})\}$ is normal.

It is clear from Lemma 2 that if any $\Gamma_\nu = 0$ we have $\alpha_\tau = 0$. Also, if there exists at least one sequence $\Gamma_{\nu_k} \to 0$, we have, according to Lemma 3, $\alpha_\tau = 0$.

Hence, we may confine ourselves to the case that there exists a positive number a such that all $\Gamma_\nu > a$. Now take a point $\zeta_n^{(1)}$ in $D(z_n, \sigma)$ on $\overline{0\tau}$ whose n-E distance from that point of intersection of $C(z_n, \sigma)$ with the radius $\overline{0\tau}$ which is farther from 0 is equal to $\frac{1}{4} \log \frac{1+a}{1-a} = \lambda$. Since the family $\{f(z; \zeta_n^{(1)})\}$ is normal in $D(0, 2\lambda)$, we know, by what has been shown in Lemma 1, that $f(z)$ tends to infinity on the sequence of the disks $D(\zeta_n^{(1)}, 2\lambda)$. Now, take a point $\zeta_n^{(2)}$ in $D(\zeta_n^{(1)}, 2\lambda)$ on $\overline{0\tau}$ whose n-E distance from the farther point of intersection of $C(\zeta_n^{(1)}, 2\lambda)$ with $\overline{0\tau}$ is equal to λ. As before, it follows that in the disks $D(\zeta_n^{(2)}, 2\lambda)$, $f(z) \to \infty$. Proceeding in this manner, it is clear that since $\rho(z_n, z_{n+1}) < M$, after a finite number of steps k, the point $\zeta_n^{(k)}$ will fall in the disk $D(z_{n+1}, \sigma)$. This shows that $f(z) \to \infty$ as $z \to \tau$ along $\overline{0\tau}$. Now, Seidel ([9], p. 170, Corollary 5) has shown that if $f(z)$ is holomorphic in D and τ a point on C for which $\lim_{r \to 1} f(r\tau) = \infty$, then there e:

$0 \leqq \alpha_\tau \leqq \frac{\pi}{2}$, for which the conclusion of the theorem

theorem is now complete.

III. Counterexamples

6. In this section we shall investigate three questions. First, we shall consider the possibility of drawing a conclusion for the Stolz angle $\Delta_{\tau,\beta}$ in the theorem when $\beta = \alpha_\tau$. Secondly, we shall consider the possibility of proving the theorem by allowing the given sequence of points $\{z_n\}$ to have the property that $\lim_{n \to \infty} f(z_n) = c$, where c is a value assumed by $f(z)$ in the unit disk. Finally, we shall investigate the possibility of removing the condition that the n-E distances between the pairs of consecutive points of the given sequence are bounded by some positive constant M as required in the theorem, and not imposing any other condition upon the sequence, other than that $f(z_n) \to \infty$ as $z_n \to \tau$.

Let us consider the first problem. We claim that no conclusion can be drawn for $\Delta_{\tau,\alpha_\tau}$ itself. The following example shows that this is the case:

Example 1. Let $f(z) = e^w$, $(z = x + iy)$, where

$$w = e^{-(\pi/4)i}\frac{1+z}{1-z}.$$

The function $f(z)$ is holomorphic in D and $\lim_{x\to 1^-} f(x) = \infty$. It is easily seen that for $\tau = 1$, $\alpha_\tau = \frac{\pi}{4}$. The function $w = e^{-(\pi/4)i}\frac{1+z}{1-z}$ maps D onto the half-plane $-\frac{3}{4}\pi < \arg w < \frac{\pi}{4}$. Also, the ray $\arg w = -\frac{\pi}{2}$ is a Julia line (cf. [5]) for e^w. The region bounded by the two hypercycles through -1, $+1$ and making angles $\frac{\pi}{4}$ and $-\frac{\pi}{4}$ with the diameter $(-1, 1)$ of D is carried by the mapping $w = e^{-(\pi/4)i}\frac{1+z}{1-z}$ onto a region in the w-plane given by $-\frac{\pi}{2} < \arg z < 0$, and $\Delta_{1,\pi/4}$ is mapped onto a region in the w-plane whose every point satisfies the inequality $\Re w > -\frac{1}{\sqrt{2}}$, since the two sides of $\Delta_{1,\pi/4}$ go into the straight half-lines $\Re w > \frac{1}{\sqrt{2}}$, $\Im w = \frac{1}{\sqrt{2}}$ and $\Re w = -\frac{1}{\sqrt{2}}$, $\Im w < -\frac{1}{\sqrt{2}}$. Consequently, $|f(z)| > e^{-1/\sqrt{2}}$ throughout $\Delta_{1,\pi/4}$ and $f(z)$ does not tend to ∞ as $z \to 1$ in $\Delta_{1,\pi/4}$. Thus neither one of the conclusions 1 and 2 holds for $\Delta_{1,\pi/4}$.

7. Let us now consider the second problem. We note that in the theorem we assume that $\lim_{n\to\infty} f(z_n) = \infty$. Since $f(z)$ is given to be a holomorphic function in D, we know that the value ∞ is not assumed by this function there. It is easy to see that the conclusion of the theorem also holds, with obvious modification, if condition (2) is replaced by the condition $\lim_{n\to\infty} f(z_n) = c$, where the value c is either omitted or assumed at most a finite number of times by $f(z)$ in D. If, however, $\lim_{n\to\infty} f(z_n) = c$, where $f(z)$ assumes the value c in the unit disk infinitely many times, then it may be shown by an example that the theorem fails to be true. This example is taken from a recent paper of F. Bagemihl and W. Seidel ([1], pp. 11-13), and is as follows:

Example 2. Let

$$B(z) = \prod_{n=1}^{\infty} \frac{z_n - z}{1 - z_n z}$$

where $z_n = 1 - e^{-n}$, $n = 1, 2, \ldots$.

Since $z_n \to 1$ and $\prod_{n=1}^{\infty} z_n > 0$, by a theorem of Blaschke ([2], p. 202), the product converges uniformly in every closed subregion of D and thus defines a bounded holomorphic function $B(z)$ in D. We have $\lim_{n\to\infty} \rho(z_n, z_{n+1}) = \frac{1}{2}$.

We note, then, that the function $B(z)$ possesses the following properties:

(A) $B(z)$ is holomorphic and bounded in D;

(B) $\lim_{n\to\infty} B(z_n)=0$ where $\{z_n\}$ is a sequence of points for which $z_n \to 1$ and $\rho(z_n, z_{n+1}) < M < \infty$, $n=1, 2, \ldots$;

(C) The value 0 is assumed by the function $B(z)$ infinitely often in D.

The function $B(z)$ shows that it is not possible to replace condition (2) in the theorem by the condition $\lim_{n\to\infty} f(z_n)=c$, where c is a value assumed by $f(z)$ infinitely often in D. Indeed, F. Bagemihl and W. Seidel have proved that the function $B(z)$ does not possess a radial limit at the point $\tau=1$ ([1], pp. 11-13). If the theorem, as modified, were true, this would imply that $\alpha_\tau=0$. On the other hand, conclusion (2) of the theorem can not hold since $B(z)$ is bounded in D.

8. We shall now investigate the third problem as stated in § 6. We shall show by an example that if no condition is imposed upon the sequence, other than the fact that $f(z_n) \to \infty$ as $z_n \to 1$, the theorem is no longer true.

Example 3. Let R be a simply connected region in the w-plane whose boundary contains a prime end P of the *third* or *fourth kind* (cf. [4], pp. 7-9), the set of principal points B of whose impression[1)] contains the point at infinity. Since R is a simply connected region which is not the whole w-plane, we know, by the Riemann mapping theorem and the fundamental theorem on prime ends (cf. [4], p. 18), that there exists a univalent and holomorphic function $z=\Psi(w)$ which maps the region R onto the unit disk D in the z-plane so that the prime end P corresponds to the point $z=1$.

Let us now investigate the inverse function $w=f(z)$ which is univalent and holomorphic in D. The image of the radius $\overline{01}$ in D is a Jordan arc which approaches arbitrarily near the set of points B. It follows that there exists a sequence of real points $\{x_n\}$ on the radius $\overline{01}$ of D such that $\lim_{n\to\infty} f(x_n)=\infty$. By a theorem of Lindelöf ([4], p. 23) the cluster set (cf. [7], p. 61) of $f(z)$ in any Stolz angle with vertex at $\tau=1$ must be the set of principal points of the impression of the prime end. Since the set B of principal points does not consist of one point, the function $f(z)$ can not tend to infinity in any symmetric Stolz angle with vertex 1. Also, since $f(z)$ is univalent in D, the function can

1) The term "impression" of a prime end was introduced by G. Piranian. (Cf. [8], pp. 45-55).

not take any value infinitely often in any Stolz angle. Hence, according to the theorem, we conclude that $\overline{\lim_{n\to\infty}}\, \rho(x_n,\ x_{n+1}) = \infty$.

The function constructed above shows that such an extension of the theorem as stated in §6 is not possible even for a univalent function.

Finally it may be mentioned that by means of our theorem one may likewise generalize the following results of W. Seidel: Corollaries 1, 3 and 4, and Theorem 5 (cf. [9], pp. 163, 169-170).

References

[1] F. Bagemihl and W. Seidel, *Seqnential and continuous limits of meromorphic functions*, Annales Academiae Scientiarum Fennicae, Series A I, No. 280 (1960), pp. 1–17.

[2] W. Blaschke, *Eine Erweiterung des Satzes von Vitali über Folgen analytischer Funktionen*, Leipziger Berichte vol. **67** (1915), pp. 194–200.

[3] C. Carathéodory, *Conformal Representation*, second edition, Cambridge, University Press, 1952, Chapter II.

[4] C. Gattegno and A. Ostrowski, *Représentation Conforme à la Frontière. Domaines Généraux*, Mémorial des Sciences Mathématiques, No. 109, (1949).

[5] G. Julia, *Leçons sur les fonctions uniformes à point singulier essentiel isolé*, Paris, 1924, p. 102 ff.

[6] Paul Montel, *Leçons sur les familles normales de fonctions analytiques et leurs applications*, Paris: Gauthiers-Villars, 1927.

[7] Kiyoshi Noshiro, *Cluster Sets*, Berlin-Göttingen-Heidelberg, 1960.

[8] G. Piranian, *The boundary of a simply connected domain*, Bulletin of the American Mathematical Society, **64**, (1958), pp. 45–55.

[9] W. Seidel, *Holomorphic functions with spiral asymptotic paths*, Nagoya Mathematical Journal, vol. **14** (1959), pp. 159-171.

University of Notre Dame, Notre Dame, Indiana

and

Incarnate Word College, San Antonio, Texas