ON META-ABELIAN FIELDS OF A CERTAIN TYPE

YOSHIOMI FURUTA

Let k be an algebraic number field of finite degree, and l a rational prime (including 2); k and l being fixed throughout this paper. For any power l^n of l, denote by ζ_n an arbitrarily fixed primitive l^n-th root of unity, and put $k_n = k(\zeta_n)$. Let r be the maximal rational integer such that $\zeta_r \in k$ i.e. $k_r = k$ and $k_{r+1} \neq k$.

S. Kuroda [7] shows that the decomposition law of rational primes in some absolute non-abelian normal extension is determined by the rational 2^2-th power residue symbol of Dirichlet, to which A. Fröhlich [1] gives a more general apprehension. L. Rédei defined in [8] a new symbol, which he called "bedingtes Artinsches Symbol" (restricted Artin symbol), and he established in [9] a theory concerning Pell's equations by means of this symbol.

In the present paper, we define in § **1** the "restricted l^n-th power residue symbol", which is related to the restricted Artin symbol in the same manner as the ordinary power residue symbol to the ordinary Artin symbol. The restricted l^n-th power residue symbol is a generalization of Dirichlet's symbol mentioned above. So we investigate some meta-abelian extensions over k, for which the decomposition law of prime ideals of k is given by means of the restricted l^n-th power residue symbol. More precisely, let A/k be an abelian extension over k and $\mathfrak{K}/A$ a kummerian extension of A obtained by adjoining to A the l^{n_i}-th roots ω_i of numbers a_i in k $(i = 1, \ldots, t)$. We call a normal subfied M of $\mathfrak{K}$ a *k-meta-abelian l-field over k*, or simply *k-meta-aeblian*, if M contains all the l^{n_i}-th roots of unity. Then the decomposition law of prime ideals of k in a k-meta-abelian l-field is determined. This result is a generalization of that of Kuroda [7] concerning P-meta-abelian 2-field over P, P being the rational number field.

Received, July 10, 1958.

The writer would like to express his gratituede to Prof. S. Kuroda and Dr. T. Kubota for their valuable advices.

§ 1. Preliminaries

For a prime ideal $\mathfrak{p}$ of k prime to l,[1] a number a of k prime to $\mathfrak{p}$, and a rational integer n, the *restricted l^n-th power residue symbol* $\left[\frac{a}{\mathfrak{p}}\right]_n$ is defined recursively as follows:

For $n \leqq 0$ we set $\left[\frac{a}{\mathfrak{p}}\right]_n = 1$.[2] For $n \geqq 1$ and $r > 0$ the symbol $\left[\frac{a}{\mathfrak{p}}\right]_n$ is defined only when $\left[\frac{a}{\mathfrak{p}}\right]_{n-r} = 1$, and, if this condition is fulfilled, we put $\left[\frac{a}{\mathfrak{p}}\right]_n = \zeta_r^x$ where $a^{(N\mathfrak{p}^\rho - 1)/l^n} \equiv \zeta_r^x \pmod{\mathfrak{p}}$, ρ being the smallest natural number with $l^n \mid N\mathfrak{p}^\rho - 1$ (we denote here by N, as well hereafter, the absolute norm). For $n \geqq 1$ and $r = 0$ the symbol $\left[\frac{a}{\mathfrak{p}}\right]_n$ is defined only when $a^{(N\mathfrak{p}^\rho - 1)/l^n} \equiv 1 \pmod{\mathfrak{p}}$, and in this case we put $\left[\frac{a}{\mathfrak{p}}\right]_n = 1$. Since all the l^r-th roots of unity are incongruent each other mod. $\mathfrak{p}$ owing to $\zeta_r \in k$, the symbol $\left[\frac{a}{\mathfrak{p}}\right]_n$ is uniquely defined. For an ideal $\mathfrak{m}$ of k prime both to a and l with the prime ideal decomposition $\mathfrak{m} = \mathfrak{p}_1^{l_1} \ldots \mathfrak{p}_t^{l_t}$, we set $\left[\frac{a}{\mathfrak{m}}\right]_n = \left[\frac{a}{\mathfrak{p}_1}\right]_n^{l_1} \cdots \left[\frac{a}{\mathfrak{p}_t}\right]_n^{l_t}$, when each $\left[\frac{a}{\mathfrak{p}_i}\right]_n$ $(i = 1, \ldots, t)$ is defined.

Now, from the definition follows immediately

LEMMA 1. *We have* $\left(\frac{a}{\mathfrak{p}}\right)_{l^t} = \left[\frac{a}{\mathfrak{p}}\right]_t$ *for* $1 \leqq t \leqq r$, *where the left-hand-side is the ordinary l^t-th power residue symbol mod. $\mathfrak{p}$ in k.*

LEMMA 2. *Let Ω be a normal extension over k; $\mathfrak{p}$ a prime ideal in k, not ramified in Ω; and $\mathfrak{P}$ a prime divisor of $\mathfrak{p}$ in Ω. Let further f' and f'' be the degrees*[3] *of $\mathfrak{p}$ with respect to Ω_n/k, and to k_n/k, respectively. If a number a in k satisfies $\left[\frac{a}{\mathfrak{p}}\right]_{n-r} = 1$ in Ω, then putting $\kappa = f'/f''$, we have*

$$\left[\frac{a}{\mathfrak{P}}\right]_n = \left[\frac{a}{\mathfrak{p}}\right]_n^\kappa, \tag{1}$$

---

1) Throughout this paper we always assume that $\mathfrak{p}$ is prime to l.

2) The symbol $\left[\frac{a}{\mathfrak{p}}\right]_n$ is defined for $n \leqq 0$ only for the sake of simplifying the definion.

3) By the degree of a prime ideal $\mathfrak{p}$ of k with respect to a normal extension Ω/k we mean, as usual, the number f such that $N_{\Omega/k}\mathfrak{P} = \mathfrak{p}^f$, $\mathfrak{P}$ being a prime divisor of $\mathfrak{p}$ in Ω.

where the left- and right-hand-sides are the restricted l^n-th power residue symbol in Ω, and in k, respectively.

Proof. For $n \leqq 0$, (1) is clear. For $n \geqq 1$ we have $\left[\frac{a}{\mathfrak{P}}\right]_n = \zeta_x^r$, where x is determined by $a^{(N\mathfrak{p}^{f_1}-1)/l^n} \equiv \zeta_r^x$ (mod. $\mathfrak{P}$), f_1 being the degree of $\mathfrak{P}$ with respect to Ω_n/Ω. Since both sides of the congruence are numbers of k, $a^{(N\mathfrak{p}^{f_1}-1)/l^n} \equiv \zeta_r^x$ (mod. $\mathfrak{p}$). Putting $N_{\Omega/k}\mathfrak{P} = \mathfrak{p}^{f_2}$, we have $f' = f_1 f_2$. Putting further $N\mathfrak{p}^{f''} = 1 + sl^n$, $(N\mathfrak{P}^{f_1} - 1)/l^n = (N\mathfrak{p}^{f'} - 1)/l^n = ((N\mathfrak{p}^{f''})^\kappa - 1)/l^n = ((1 + sl^n)^\kappa - 1)/l^n \equiv s\kappa = \kappa(N\mathfrak{p}^{f''} - 1)/l^n$ (mod. sl^n). Since $a^{sl^n} = a^{N\mathfrak{p}^{f''}-1} \equiv 1$ (mod. $\mathfrak{p}$), by definition, $a^{(N\mathfrak{P}^{f_1}-1)/l^n} \equiv a^{\kappa(N\mathfrak{p}^{f''}-1)/l^n} \equiv \left[\frac{a}{\mathfrak{p}}\right]_n^\kappa$ (mod. $\mathfrak{p}$), which proves (1).

Now, if χ is a character of the Galois group of an abelian extension A/k, we call simply χ a *character of* A/k, and set $\chi(\mathfrak{m}) = \chi\left(\left(\frac{A/k}{\mathfrak{m}}\right)\right)$, where $\left(\frac{A/k}{\mathfrak{m}}\right)$ is the Artin symbol.

Let $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ be subgroups of an abelian group $\mathfrak{G}$; and $\mathfrak{A}$ a subgroup of $\mathfrak{B}\mathfrak{C}$. We call $\mathfrak{A}$ an *l^r-subgroup of* $\mathfrak{B}\mathfrak{C}$, if for any $a \in \mathfrak{A}$ there exist $b \in \mathfrak{B}$ and $c \in \mathfrak{C}$ such that $a = bc$ and $b^{l^r} \in \mathfrak{A}$.

Now let A and B be two abelian extensions over k; $\mathfrak{A}$ and $\mathfrak{B}$ their Galois groups; and Φ and Ψ their character groups, respectively. Then the Galois group $\mathfrak{G}$ of AB/k is isomorphic to a subgroup of the direct product of $\mathfrak{A}$ and $\mathfrak{B}$, and the isomorphism is given by $\sigma \rightarrow (\sigma_A, \sigma_B)$, where $\sigma \in \mathfrak{G}$, $\sigma_A = \text{rest}_{AB\rightarrow A}\sigma$ and $\sigma_B = \text{rest}_{AB\rightarrow B}\sigma$. By setting

$$(2) \qquad \varphi\psi(\sigma) = \varphi(\sigma_A)\psi(\sigma_B) \qquad \text{for } \varphi \in \Phi,\ \psi \in \Psi,$$

we can imbed Φ and Ψ in the character group X of $\mathfrak{G}$. If we define the homomorphism ι of $\Phi \times \Psi$ (direct[4]) onto X by

$$(3) \qquad \iota(\varphi \times \psi) = \varphi\psi \qquad \text{for } \varphi \times \psi \in \Phi \times \Psi$$

the character group X of $\mathfrak{G}$ is induced from $\Phi \times \Psi$ by the homomorphism ι. Furthermore the character group of $\mathfrak{A}\mathfrak{B}/\mathfrak{B}$ is induced from Φ by the isomorphism $\lambda = \lambda_{k\rightarrow B}$ of $\Phi/\Phi \cap \Psi \cong \Phi\Psi/\Psi$, i.e.

$$(4) \qquad (\lambda_{k\rightarrow B}\varphi)\ (\bar{\alpha}) = \varphi(\alpha_A)$$

where $\alpha_A = \text{rest}_{AB\rightarrow A}\bar{\alpha}$ for $\bar{\alpha} \in \mathfrak{G}(AB/B)$.

---

[4] Throughout this paper the notation $\times$ means the direct product.

LEMMA 3. *Notations being as above, if X_1 is an l^r-subgroup of $\Phi\times\Psi$, then $X_1^*=\iota(X_1)$ is an l^r-subgroup of $\Phi\Psi$. If X_1^* is an l^r-subgroup of $\Phi\Psi$, then there exists an l^r-subgroup X_1 of $\Phi\times\Psi$ such that $\iota(X_1)=X_1^*$.*

Proof. (i) Let X_1 be an l^r-subgroup of $\Phi\times\Psi$. Let further $\chi^*=\varphi\psi\in X_1^*$, $\varphi\in\Phi$, $\psi\in\Psi$: $\chi^*=\iota(\chi)$, $\chi=\varphi_1\times\psi_1\in X_1$, $\varphi_1\in\Phi$, $\psi_1\in\Psi$; and $\iota(\varphi_1)=\varphi\varphi_0$. Then $\iota(\psi_1)=\psi\varphi_0^{-1}$ and by the assumption $\varphi_1^{l^r}\in X_1$. Hence $\chi^*=\varphi\psi=(\varphi\varphi_0)(\psi\varphi_0^{-1})$, $\varphi\varphi_0\in\Phi$, $\psi\varphi_0^{-1}\in\Psi$ and $(\varphi\varphi_0)^{l^r}\in X_1^*$. Therefore $X_1^*=\iota(X_1)$ is an l^r-subgroup of $\Phi\Psi$.

(ii) Conversely, let X_1^* be an l^r-subgroup of $\Phi\Psi$. If we denote by X_1 the group consisting of all $\varphi\times\psi\in\Phi\times\Psi$ such that $\varphi\psi\in X_1^*$ and $\varphi^{l^r}\in X_1^*$, then obiousely $\iota(X_1)=X_1^*$, and X_1 is an l^r-subgroup of $\Phi\times\Psi$.

§ 2. Fundamental lemma

Let $K=k_n(\omega_1,\ldots,\omega_t)$, where ω_i is an l^{n_i}-th root of $a_i\in k$ $(i=1,\ldots,t)$ and $n=\max(n_1,\ldots,n_t)$; A an abelian extension over k containing k_n; Φ and Ψ the character groups of A/k_n and of K/k_n respectively; and $X=\Phi\Psi$ the character group of AK/k_n in the sense of (3). If we define ψ_i by

$$\psi_i(\alpha)=\omega_i^\alpha/\omega_i \qquad \text{for every } \alpha\in\mathfrak{G}(K/k_n), \tag{5}$$

the character group Ψ of K/k_n is generated by all such ψ_i $(i=1,\ldots,t)$.

Let U_σ be a representative of $\sigma\in\mathfrak{G}(k_n/k)$ in $\mathfrak{G}(AK/k)$. Put $U_\sigma^{-1}\alpha U_\sigma=\alpha^\sigma$ for $\alpha\in\mathfrak{G}(AK/k_n)$, and $\chi^\sigma(\alpha)=\chi(\alpha^\sigma)$ for $\chi\in X$. If $\chi=\varphi\psi$, $\varphi\in\Phi$, $\psi\in\Psi$, then we have

$$\chi^\sigma(\alpha)=\varphi\psi^\sigma(\alpha) \tag{6}$$

since $\chi^\sigma(\alpha)=\varphi\psi(\alpha^\sigma)=\varphi(\alpha_A^\sigma)\psi(\alpha_K^\sigma)=\varphi(\alpha_A)\psi^\sigma(\alpha_K)$. On the other hand we may write $\omega_i^{U_\sigma}=\omega_i b_{i,\sigma}$ for some $b_{i,\sigma}\in k_n$, because $(\omega_i^{U_\sigma}/\omega_i)^{l^{n_i}}=a_i^{U_\sigma}/a_i=1$. By comparing $\omega_i^{U_\sigma\alpha^\sigma}$ and $\omega_i^{\alpha U_\sigma}$, we see $\psi_i(\alpha^\sigma)=\psi_i(\alpha)^{\iota_\sigma}$, hence $\psi^\sigma(\alpha)=\psi(\alpha)^{\iota_\sigma}$ for any $\psi\in\Psi$. Let l^μ be the order of ψ, and c an integer determined by $\zeta_\mu^\sigma=\zeta_\mu^c$. Then we have

$$\psi^\sigma(\alpha)=\psi^c(\alpha). \tag{7}$$

Put $m=\mu-r$, and assume $m>0$. Then $(\zeta_\mu^{l^m})^\sigma=\zeta_\mu^{l^m}$ for any $\sigma\in\mathfrak{G}(k_n/k)$. Hence we have

(8) $c-1\equiv 0 \pmod{l^r}$ for any $\sigma\in\mathfrak{G}(k_n/k)$.

It is clear that (8) holds for $m\leqq 0$. Now, again assume $m>0$. Then since $\zeta_\mu^{l^x}\notin k$ for any positive integer $x<m$, there exists $\sigma\in\mathfrak{G}(k_n/k)$ such that $(\zeta_\mu^{l^x})^\sigma \neq\zeta_\mu^{l^x}$ for any positive integer $x<m$. Hence if $\mu>r$,

(9) $c-1\not\equiv 0 \pmod{l^{r+1}}$ for some $\sigma\in\mathfrak{G}(k_n/k)$.

Now we prove the following fundamental

LEMMA 4. *Notations A, K, Φ, Ψ and X being as above, let M be a subfield of AK over k_n, and X_0 the subgroup of the character group $X=\Phi\Psi$ of AK/k_n corresponding to M. If M is a k-meta-abelian l-field over k, then X_0 is an l^r-subgroup of $\Phi\Psi$, and conversely.*

Proof. By the assumption, $AK\supset M\supset k_n$, therefore in order that M is a k-meta-abelian l-field over k, it is necessary and sufficient that M is normal over k. Put $\mathfrak{H}=\mathfrak{G}(AK/M)$.

(i) Suppose that M is normal over k, i.e. $\mathfrak{H}^\sigma=\mathfrak{H}$ for any $\sigma\in\mathfrak{G}(k_n/k)$. Then, by (6), $\varphi\psi\in X_0$ implies $\varphi\psi^\sigma\in X_0$, hence $\psi^\sigma\psi^{-1}\in X_0$. Let l^μ be the order of ψ. If $\mu>r$, then by (7), (8) and (9), $\psi^\sigma\psi^{-1}=\psi^{c-1}=(\psi^{l^r})^y$ for some $\sigma\in\mathfrak{G}(k_n/k)$, where $(y,l)=1$. Hence $\psi^{l^r}\in X_0$. If $\mu\leqq r$, trivially $\psi^{l^r}\in X_0$. Therefore X_0 is an l^r-subgroup of $\Phi\Psi$.

(ii) Conversely, suppose that X_0 is an l^r-subgroup of $\Phi\Psi$. Let $\chi\in X_0$, then there exist $\varphi\in\Phi$ and $\psi\in\Psi$ such that $\chi=\varphi\psi$ and $\psi^{l^r}\in X_0$. On the other hand by (6) $\chi^\sigma=\varphi\psi^\sigma$ for any $\sigma\in\mathfrak{G}(k_n/k)$, and, by (7) and (8), $\psi^\sigma=\psi^c$, where $c-1\equiv 0 \pmod{l^r}$. Hence $\chi^\sigma=\varphi\psi^\sigma\in X_0$. Therefore $\mathfrak{H}^\sigma=\mathfrak{H}$ for any $\sigma\in\mathfrak{G}(k_n/k)$. Thus the lemma is proved.

§ 3. Theorems

Denote by $\langle a\rangle$ the cyclic group generated by $a\in k$, and set $\langle a\rangle_n=\langle a\rangle/k_n^{l^n}\cap\langle a\rangle$. Let ψ be a generating character of $k_n(\omega)/k_n$, ω being an l^n-th root of a. If we denote by $\langle\psi\rangle$ the character group of $k_n(\omega)/k_n$, we see $\langle a\rangle_n\cong\langle\psi\rangle$. Thus, denoting by $[a]_n$ a generating class of $\langle a\rangle_n$, we can identify $[a]_n$ with ψ.

Let $K=k_n(\omega_1,\ldots,\omega_t)$, where ω_i is an l^{n_i}-th root of $a_i\in k$ $(i=1,\ldots,t)$ and $n=\max(n_1,\ldots,n_t)$; $\Psi=\langle a_1\rangle_{n_1}\times\ldots\times\langle a_t\rangle_{n_t}$; A an abelian extension over k with the character group Φ. Put $X=\Phi\times\Psi$. Then the character group

X^* of AK/k_n and the group X correspond each other by means of (2) and (3), restricting Φ to Ak_n/k_n. Therefore by lemma 3 and lemma 4 we have

THEOREM 1. *Every k-meta-abelian l-field M over k corresponds to an* l^r*-subgroup* X_0 *of* $\Phi^* \times \Psi$, *where* Φ^* *is the restriction to* Ak_n/k_n *of the character group* Φ *of an abelian extension* A/k *and* $\Psi = \{a_1\}_{n_1} \times \ldots \{a_t\}_{n_t}$ *for* $a_i \in k$ *and for natural numbers* n_i $(i = 1, \ldots, t)$; *and conversely.*

Notations being as in theorem 1, let $\mathfrak{p}$ be a prime ideal of k not ramified in M/k; $\mathfrak{P}$ a prime divisor of $\mathfrak{p}$ in k_n. If f_0 is the degree of $\mathfrak{p}$ with respect to k_n/k, then by the translation theorem of the class field theory we have for any integer x

$$\varphi^*(\mathfrak{P}^x) = (\lambda_{k\to k_n}\varphi)\ (\mathfrak{P}^x) = \varphi(N_{k_n/k}\mathfrak{P}^x) = \varphi(\mathfrak{p}^{f_0 x}), \tag{10}$$

$\lambda_{k\to k_n}$ being as (4). For $\psi = \psi_1^{x_1} \times \ldots \times \psi_t^{x_t} \in \Psi$, $\psi_i = [a_i]_{n_i}$ $(i = 1, \ldots, t)$, put $n = \max(n_1, \ldots, n_t)$ and $a = \prod_{i=1}^{t} a_i^{x_i l^{n-n_i}}$. Put further $K = k_n(\omega_1, \ldots, \omega_t)$, ω_i being an l^{n_i}-th root of a_i $(i = 1, \ldots, t)$, and $\psi^* = \iota(\psi)$, ι being the homomorphism of Ψ onto the character group of K/k_n by means of (3). Then $\psi^*(\mathfrak{P}^x) = \left(\frac{a}{\mathfrak{P}^x}\right)_{l^n}$. Moreover by lemma 1 $\psi^*(\mathfrak{P}^x) = \left[\frac{a}{\mathfrak{P}}\right]_n^x$ in k_n. If $\psi^{*l^r}(\mathfrak{P}^x) = 1$, then by lemma 1 $\psi^{*l^r}(\mathfrak{P}^x) = \left[\frac{a}{\mathfrak{P}}\right]_{n-r}^x = 1$, hence by lemma 2

$$\psi^*(\mathfrak{P}^x) = \left[\frac{a}{\mathfrak{P}}\right]_n^x = \left[\frac{a}{\mathfrak{p}}\right]_n^x, \tag{11}$$

where the last is the symbol in k. Now we define $\psi(\mathfrak{p}^x)$ by

$$\psi(\mathfrak{p}^x) = \left[\frac{a}{\mathfrak{p}}\right]_n^x, \tag{12}$$

and, for $\chi = \varphi \times \psi \in \Phi \times \Psi$, $\chi(\mathfrak{p}^x)$ by

$$\chi(\mathfrak{p}^x) = \varphi^{f_0 x}(\mathfrak{p})\psi^x(\mathfrak{p}). \tag{13}$$

THEOREM 2. *Let M be a k-meta-abelian l-field over k corresponding by theorem* 1 *to an* l^r*-subgroup* X_0 *of* $\Phi \times \Psi$. *Then the degree of a prime ideal* $\mathfrak{p}$ *of k, not ramified in M/k, is equal to* $f = f_0 f_1$ *where* f_0 *and* f_1 *are the smallest integers such that* $l^n | N\mathfrak{p}^{f_0} - 1$ *and* $\chi(\mathfrak{p}^{f_1}) = 1$ *for all* $\chi \in X_0$, *respectively.*

Proof. $\mathfrak{P}$ being a prime divisor of $\mathfrak{p}$ in k_n, the degree of $\mathfrak{p}$ with respect to M/k is equal to the product of the degrees of $\mathfrak{p}$ with respect to k_n/k and of $\mathfrak{P}$ to M/k_n. Since the former is equal to f_0, we have only to show that the degree of $\mathfrak{P}$ with respect to $\dot{M}/k_n$, i.e. the smallest number x such that $\chi^*(\mathfrak{P}^x)=1$ for all $\chi^* \in X_0^* = \iota(X_0)$ is equal to f_1. By theorem 1 and lemma 3, $\chi^* \in X_0^*$ implies $\chi^* = \varphi^* \psi^*$ and $\psi^{*l^r} \in X_0^*$ for some $\varphi \in \Phi$ and for some $\psi \in \Psi$. On the other hand, by (10), (11), (12), and (13) we see that $\chi^*(\mathfrak{P}^x)=1$ under the condition $\psi^{*l^r}(\mathfrak{P}^x)=1$ if and only if $\chi(\mathfrak{p}^x)=1$. Furthermore, by (11) and (12) $\psi^{*l^r}(\mathfrak{P}^x)=1$ if and only if $\psi^{l^r}(\mathfrak{p}^x)=1$. Whence the theorem follows immediately.

References

[1] A. Fröhlich, Non abelian laws of prime decomposition, Proc. Int. Congress Math. 1954, Amsterdam. pp. 20–21.

[2] A. Fröhlich, On fields of class two, Proc. London Math. Soc. (3), **4** (1954), pp. 235-256.

[3] Y. Furuta, A reciprocity law of the power residue symbol, J. Math. Soc. Japan, **10** (1958), pp. 46–54.

[4] W. H. Mills, The *m*-th power residue symbol, Amer. J. Math., **73** (1951), pp. 59–64.

[5] W. H. Mills, Reciprocity in algebraic number fields, *ibid.*, pp. 65–77.

[6] H. Hasse, Invariante Kennzeichnung relativ-abelscher Zahlkörper mit vorgegebener Galois Gruppe über einem Teilkörper des Grundkörpers, Abh. Deutsch. Akad. Wiss. Berlin, 8 (1947).

[7] S. Kuroda, Über die Zerlegung rationaler Primzahlen in gewissen nicht-abelschen galoischen Körpern, J. Math. Soc. Japan, **3** (1951), pp. 148-156.

[8] L. Rédei, Bedingtes Artinsches Symbol mit Anwendung in der Klassenkörpertheorie, Acta Math. Acad. Sci. Hungaricae, **4** (1954), pp. 1-29.

[9] L. Rédei, Die 2-Ringklassengruppe des quadratischen Zahlkörpers und die Theorie der Pellschen Gleichung, *ibid.*, pp. 31-85.

Mathematical Institute
Nagoya University