ON REAL IRREDUCIBLE REPRESENTATIONS OF LIE ALGEBRAS

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§ 1. Introduction

Let us consider the following two problems:

Problem A. *Let* $\mathfrak{g}$ *be a given Lie algebra over the real number field* R. *Then find all real, irreducible representations of* $\mathfrak{g}$.

Problem B. *Let* n *be a given positive integer. Then find all irreducible subalgebras of the Lie algebra* $\mathfrak{gl}(n, R)$ *of all real matrices of degree* n.

In a beautiful and fundamental paper [1], E. Cartan solved completely the Problem B, in the sense that he gave a method to determine all the subalgebras of $\mathfrak{gl}(n, R)$ by a finite process, and determined them actually for the case $n \leqq 12$ for which he gave a table. As we shall see in § 6, 7, the Problem A is reduced to the one to find all complex irreducible representations and to distinguish among them those representations which are of the first class, and then the Problem A is easily reduced to the reductive case, i.e. to the case where $\mathfrak{g}$ is reductive. As a reductive Lie algebra is a direct sum of simple Lie algebras, the Problem A can be further reduced to the case where $\mathfrak{g}$ is simple, as we shall see later. Now if the Problem A could be solved for every Lie algebra $\mathfrak{g}$, then one has only to look at the table to solve B. In analysing [1] closely, we notice that E. Cartan solved the Problem B by this principle. In several places of [1], E. Cartan has recourse to verifications for each type of simple Lie algebras A, B, C, D and the results of verifications for exceptional cases are stated without proof.

In the present paper, we shall solve the Problem A by the above mentioned principle and reestablish the results of [1]. The knowledge of [1] is not presupposed for the reader. Where E. Cartan had recourse to verifications for each type of simple algebras, we shall be able to obtain the corresponding results by general considerations.

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§ 2. Complex conjugates of complex vector spaces

For later use, we state here some facts about "complex conjugates" of complex vector spases. Let V, U be finite dimensional vector spaces over the complex number field C. A mapping $f: V\rightarrow U$ is called *anti-linear* if

$$f(\alpha x+\beta y)=\bar{\alpha}f(x)+\bar{\beta}f(y)$$

for every α, $\beta\in C$ and x, $y\in V$. In particular an anti-linear mapping from V into C is called an *anti-linear form* on V. Let us denote by $V^{(*)}$ the set of all anti-linear forms on V. Then by the operations

$$(f_1+f_2)\ (x)=f_1(x)+f_2(x),\ (\alpha f)\ (x)=\alpha\cdot f(x)$$

for f_1, f_2, $f\in V^{(*)}$, $x\in V$, $\alpha\in C$, $V^{(*)}$ becomes a complex vector space and dim $V^{(*)}$ = dim V.

Now let us denote by $\overline{V}$ the dual vector space of the complex vector space $V^{(*)}$, i.e. the vector space consisting of all linear forms on $V^{(*)}$. Then every $x\in V$ determines an element $\bar{x}\in\overline{V}$ as follows:

$$(\bar{x},f)=(x,f)=f(x)\qquad\text{for every } f\in V^{(*)}$$

and the mapping $x\rightarrow\bar{x}$ is a one-to-one, anti-linear mapping from V onto $\overline{V}$. Moreover, if A is a linear endomorphism of the vector space V, then A determines a linear endomorphism $\bar{A}$ of $\overline{V}$ as follows: $\bar{A}\bar{x}=\overline{Ax}$ for every x in V. Then the mapping $A\rightarrow\bar{A}$ is a one-to-one anti-linear mapping from the vector space $\mathfrak{gl}(V)$ of all linear endomorphisms of the complex vector space V onto $\mathfrak{gl}(\overline{V})$.

We note that if (e_i) is a base of V, then $(\bar{e}_i)$ is a base of $\overline{V}$ and the matrix of $A\in\mathfrak{gl}(V)$ with respect to (e_i) is the complex conjugate of the matrix of $\bar{A}\in\mathfrak{gl}(\overline{V})$ with respect to $(\bar{e}_i)$. We shall call $\overline{V}$, $\bar{x}$, $\bar{A}$ the *complex conjugates* of V, x, A respectively.

§ 3. Scalar restrictions and scalar extentions

Let V be a vector space over C. Then V can be regarded in a natural way also a vector space over R. We denote this real vector space by V_R and call it the *scalar restriction* of V to the real number field R. Note that V and V_R coincide as a set. Now if A is a linear endomorphism of the complex vector space V, then A induces naturally a linear endomorphism A_R of the real vector space V_R.

If (ρ, V) is a complex representation of a real Lie algebra $\mathfrak{g}$, then (ρ_R, V_R) is a real representation of $\mathfrak{g}$, where $\rho_R(X) = (\rho(X))_R$ for every $X \in \mathfrak{g}$. We shall call the real representation (ρ_R, V_R) the scalar restriction of the complex representation (ρ, V).

Now let E be a vector space over R. Then we denote by E^C the complex vector space which is obtained from E by extending the ground field R to C. If A is a linear endomorphism of E, then A is extended uniquely to a linear endomorphism A^C of E^C.

If (d, E) is a real representation of a real Lie algebra $\mathfrak{g}$, then (d^C, E^C) is a complex representation of $\mathfrak{g}$, where $d^C(X) = (d(X))^C$ for every $X \in \mathfrak{g}$. We shall call the complex representation (d^C, E^C) the scalar extension of the real representation (d, E).

§ 4. Conjugate representations

Let (ρ, V) be a complex representation of a *real* Lie algebra $\mathfrak{g}$. Then we can form another complex representation $(\bar{\rho}, \bar{V})$ of $\mathfrak{g}$, where $\bar{V}$ is the complex conjugate of the complex vector space V, and $\bar{\rho}$ is defined by $\bar{\rho}(X) = \rho(X)$ for every $X \in \mathfrak{g}$. Since $\mathfrak{g}$ is real Lie algebra, $(\bar{\rho}, \bar{V})$ becomes a complex representation of $\mathfrak{g}$. We note that the scalar restrictions ρ_R, $\bar{\rho}_R$ are equivalent real representation of $\mathfrak{g}$. In fact the mapping $x \to \bar{x}$ from V onto $\bar{V}$ gives the equivalence of V_R and $\bar{V}_R$. Now let (ρ, V), (σ, U) be two complex representations of $\mathfrak{g}$. If $(\bar{\rho}, \bar{V})$ is equivalent to (σ, U), then we shall say that (ρ, V) is *conjugate* to (σ, U) and denote it by $\bar{\rho} \sim \sigma$. In particular, if $\bar{\rho} \sim \rho$, then we say ρ *self-conjugate*. If $\bar{\rho} \sim \sigma$, then we have easily $\rho \sim \bar{\sigma}$, so the relation of "conjugate" is symmetric. Let us note that *a complex representation* (ρ, V) *is conjugate to* (σ, U) *if and only if there exists a one-to-one anti-linear mapping* f *from* V *onto* U *such that*

$$f \circ \rho(X) = \sigma(X) \circ f$$

for every $X \in \mathfrak{g}$. In fact, if $\bar{\rho} \sim \sigma$, then there is a linear isomorphism $\varphi: \bar{V} \to U$ such that $\varphi \circ \bar{\rho}(X) = \sigma(X) \circ \varphi$ for every $X \in \mathfrak{g}$. Define the mapping f by $f(x) = \varphi(\bar{x})$, then f has all the desired properties. The converse is shown analogously.

In particular a complex representation (ρ, V) is self-conjugate if and only if there is a one-to-one anti-linear mapping J from V onto itself (we shall call such a mapping J anti-linear automorphism of V) such that

$$J \circ \rho(X) = \rho(X) \circ J$$

for every $X \in \mathfrak{g}$, i.e. J is commutative with every $\rho(X)$ $(X \in \mathfrak{g})$. In this case we say also that J is *invariant* by ρ or that ρ leaves J invariant.

Now let us remark that our notion of conjugate or self-conjugate representation coincides with the notion of "correlatif" or "auto-correlatif" of E. Cartan [1] respectively, if $\mathfrak{g}$ is a semi-simple Lie algebra over R. To this purpose we shall prove the following

LEMMA 1. *Let $\mathfrak{g}$ be a semi-simple Lie algebra over R and (ρ, V), (σ, U) be two complex representations of $\mathfrak{g}$. Then (ρ, V) is equivalent to (σ, U) if and only if the characteristic polynomials of both representations coincide, i.e.*

$$\det\,(tI - \rho(X)) = \det\,(tI - \sigma(X)) \tag{1}$$

for every $X \in \mathfrak{g}$, where t is an indeterminate and I is the identity operator on V or U.

Proof. Assume that (1) hold for every $X \in \mathfrak{g}$ and let us prove that $\rho \sim \sigma$. Let $\mathfrak{g}^C$ be the complex form of $\mathfrak{g}$ and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Then $\mathfrak{h}^C$ is a Cartan subalgebra of $\mathfrak{g}^C$. Now every complex representation (ρ, V) of $\mathfrak{g}$ can be extended uniquely to the complex representation of $\mathfrak{g}^C$ which we also denote by (ρ, V). Then as is easily seen, (1) holds for every $X \in \mathfrak{g}^C$. Now let $\Lambda_1, \ldots, \Lambda_r$ and $\Lambda'_1, \ldots, \Lambda'_s$ be the system of weights of representation (ρ, V), (σ, U) respectively with respect to the Cartan subalgebra $\mathfrak{h}^C$. Then by (1) we have

$$\prod_{i=1}^{r} (t - \Lambda_i(H))^{m_i} = \prod_{j=1}^{s} (t - \Lambda'_j(H))^{n_j}$$

for every $H \in \mathfrak{h}^C$, where m_i, n_j are the multiplicities of Λ_i, Λ'_j respectively. Then we have $r = s$ and $\Lambda_1, \ldots, \Lambda_r$ coincide with $\Lambda'_1, \ldots, \Lambda'_r$ together with their multiplicities up to their order. Then, the highest weights of every irreducible component of (ρ, V) and (σ, U) must coincide together with their multiplicities. Thus we have $\rho \sim \sigma$ as representations of $\mathfrak{g}^C$. Then we have $\rho \sim \sigma$ as representations of $\mathfrak{g}$.

The converse assertion is trivial. So we have completed the proof.

COROLLARY. *Let $\mathfrak{g}$ be a semi-simple Lie algebra over R, and (ρ, V), (σ, U) be two complex representation of $\mathfrak{g}$. Then $\bar{\rho} \sim \sigma$ if and only if the coefficients of the characteristic polynomials of $\rho(X)$, $\sigma(X)$ are complex conjugate of each other for every $X \in \mathfrak{g}$. In particular, ρ is self-conjugate if and only if the coefficients of the characteristic polynomial of $\rho(X)$ are all real numbers.*

In [1], E. Cartan has defined the notion of "correlatif" or "auto-correlatif" using the characteristic polynomials of representations. The relation of this notion to our notion of conjugateness or self-conjugateness is shown in the above corollary.

§ 5. Fundamental theorem of E. Cartan

We are now in an appropriate position to explain the fundamental theorem of E. Cartan connecting real, irreducible representations with complex, irreducible representations. Now let $\mathfrak{g}$ be a Lie algebra over R. Let us denote by $R_n(\mathfrak{g})$ the set of all real, irreducible representation classes of $\mathfrak{g}$ of degree n, and by $C_n(\mathfrak{g})$ the set of all complex, irreducible representation classes of $\mathfrak{g}$ of degree n. We also denote by $R_n^I(\mathfrak{g})$, $R_n^{II}(\mathfrak{g})$, the following subsets of $R_n(\mathfrak{g})$:

$$R_n^I(\mathfrak{g}) = \{[d] \in R_n(\mathfrak{g})\ ;\ d^C \text{ is irreducible}\}$$
$$R_n^{II}(\mathfrak{g}) = \{[d] \in R_n(\mathfrak{g})\ ;\ d^C \text{ is reducible}\}$$

where $[d]$ means the representation class containing d. If $[d] \in R_n^I(\mathfrak{g})$ or $\in R_n^{II}(\mathfrak{g})$, then $[d]$ and d are called of *first class* or *second class* respectively. We also denote by $C_n^I(\mathfrak{g})$, $C_n^{II}(\mathfrak{g})$ the following subsets of $C_n(\mathfrak{g})$:

$$C_n^I(\mathfrak{g}) = \{[\rho] \in C_n(\mathfrak{g})\ ;\ \rho_R \text{ is reducible}\}$$
$$C_n^{II}(\mathfrak{g}) = \{[\rho] \in C_n(\mathfrak{g})\ ;\ \rho_R \text{ is irreducible}\}$$

If $[\rho] \in C_n^I(\mathfrak{g})$ or $\in C_n^{II}(\mathfrak{g})$, then $[\rho]$ and ρ are called of *first class* or *second class* respectively. Then we have obviously

$$R_n(\mathfrak{g}) = R_n^I(\mathfrak{g}) \cup R_n^{II}(\mathfrak{g}),\ R_n^I(\mathfrak{g}) \cap R_n^{II}(\mathfrak{g}) = \text{empty set},$$
$$C_n(\mathfrak{g}) = C_n^I(\mathfrak{g}) \cup C_n^{II}(\mathfrak{g}),\ C_n^{II}(\mathfrak{g}) \cap C_n^{II}(\mathfrak{g}) = \text{empty set}.$$

Now let us associate to an irreducible real representation (d, E) of first class, an irreducible complex representation (d^C, E^C). Since from $d_1 \sim d_2$, we have

$d_1^C \sim d_2^C$, we have a mapping

$$\Psi_1: [d] \to [d^C]$$

from $R_n^I(\mathfrak{g})$ into $C_n(\mathfrak{g})$.

If (d, E) is an irreducible real representation of second class, then (d^C, E^C) is reducible. Let V be any invariant subspace of E^C such that $V \neq E^C$, $V \neq (0)$. Then, denoting by $x \to \bar{x}$ the anti-linear automorphism of E^C determined by E (i.e. if $x = y + \sqrt{-1}\ z$, $y \in E$, $z \in E$, then $\bar{x} = y - \sqrt{-1}\ z$) and by $\bar{V}$ the image of V under this mapping $x \to \bar{x}$, we have

$$E^C = V + \bar{V},\ V \cap \bar{V} = (0). \tag{2}$$

In fact, since $\overline{V+\bar{V}} = V + \bar{V}$, we have $V + \bar{V} = F + \sqrt{-1}\ F$ where $F = (V + \bar{V}) \cap E$.[1] Then $F \neq (0)$ is an invariant subspace of E. Hence we have $F = E$ and $V + \bar{V} = E^C$. Similarly we have $V \cap \bar{V} = (0)$. Thus (2) is proved. Now V is irreducible. In fact, if V contains an invariant complex subspace U such that $U \neq V$, $U \neq (0)$, then $U + \bar{U} \neq E^C$ which contradicts to (2). Similarly $\bar{V}$ is irreducible. The irreducible representations induced by d^C on V, $\bar{V}$ are, as is seen easily, conjugate to each other. Thus, we have $\dim_R E = 2 \dim_C V$, i.e. if $[d] \in R_n^{II}(\mathfrak{g})$, then n must be an even integer.

Let us associate to $[d] \in R_n^{II}(\mathfrak{g})$ the irreducible complex representation class $[\rho] \in C_{n/2}(\mathfrak{g})$, where ρ is the representation induced by d^C on V or on $\bar{V}$ as above. $[\rho]$ is determined up to conjugate representation class. Let us introduce an equivalence relation $\approx$ in the set $C_{n/2}(\mathfrak{g})$ by

$$[\rho_1] \approx [\rho_2] \text{ if } [\rho_1] = [\rho_2] \text{ or } [\bar{\rho}_1] = [\rho_2]$$

and denote by $\hat{C}_{n/2}(\mathfrak{g})$ the set of all equivalence class in $C_{n/2}(\mathfrak{g})$ with respect to the equivalence relation $\approx$. Then by the above mapping

$$R_n^{II}(\mathfrak{g}) \ni [d] \to [\rho] \in C_{n/2}(\mathfrak{g})$$

there is introduced a mapping

$$\Psi_2: [d] \to (\approx)\text{-equivalence class of } [\rho]$$

from $R_n^{II}(\mathfrak{g})$ into $\hat{C}_{n/2}(\mathfrak{g})$.

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[1] In general, a complex subspace W of E^C has a form $W = F + \sqrt{-1}\ F$ (where F is a real subspace of E), if and only if $W = \bar{W}$. Moreover, if $W = \bar{W}$, then F is given by $F = W \cap E$.

Now let us explain other mappings Ψ_3, Ψ_4. Let (ρ, V) be an irreducible complex representation of first class. Then (ρ_R, V_R) is reducible. Let E be an invariant (real) subspace of V_R such that $E \neq V_R$, $E \neq (0)$. Then $E+\sqrt{-1}\,E$ and $E \cap \sqrt{-1}\,E$ are invariant (complex) subspaces of V and we have $E+\sqrt{-1}\,E \neq (0)$, $E \cap \sqrt{-1}\,E \neq V$. Hence we have

$$V_R = E+\sqrt{-1}\,E,\ E\cap\sqrt{-1}\,E = 0. \tag{3}$$

Now E is irreducible. In fact if E contains an invariant (real) subspace F such that $F \neq E$, $F \neq (0)$, then we have $V \neq F+\sqrt{-1}\,F$ which contradicts to (3). Similarly $\sqrt{-1}\,E$ is irreducible. Moreover the irreducible real representations induced by ρ_R on E and $\sqrt{-1}\,E$ are equivalent to each other. In fact, the one-to-one linear mapping $x \to \sqrt{-1}\,x$ from E onto $\sqrt{-1}\,E$ gives the equivalence of E and $\sqrt{-1}\,E$. Let d be the irreducible real representation induced by ρ_R on E or on $\sqrt{-1}\,E$, then we have a mapping

$$\Psi_3:\ [\rho] \to [d]$$

from $C'_n(\mathfrak{g})$ into $R_n(\mathfrak{g})$.

Now let (ρ, V) be an irreducible complex representation of second class. Then (ρ_R, V_R) is an irreducible real representation of degree $2n$. Moreover, as is remarked in §4, ρ and $\bar{\rho}$ give equivalent real representations ρ_R, $\bar{\rho}_R$. Hence there is induced a mapping

$$\Psi_4:\ (\approx)\text{-equivalence class of } [\rho] \to [\rho_R]$$

from $\hat{C}_n(\mathfrak{g})$ into $R_{2n}(\mathfrak{g})$. We denote by $\hat{C}''_n(\mathfrak{g})$ the subset of $\hat{C}_n(\mathfrak{g})$ consisting of $(\approx)$-equivalenct classes containing an irreducible complex representation of second class.

Now under these preparations, we can state the fundamental theorem of E. Cartan as follows:

THEOREM 1. *(i) Ψ_1 is a one-to-one mapping from $R'_n(\mathfrak{g})$ onto $C'_n(\mathfrak{g})$. Ψ_3 is a one-to-one mapping from $C'_n(\mathfrak{g})$ onto $R'_n(\mathfrak{g})$. Ψ_1 and Ψ_3 are the inverse mappings of each other. (ii) Ψ_2 is a one-to-one mapping from $R''_{2n}(\mathfrak{g})$ onto $\hat{C}''_n(\mathfrak{g})$. Ψ_4 is a one-to-one mapping from $\hat{C}''_n(\mathfrak{g})$ onto $R''_{2n}(\mathfrak{g})$. Ψ_2 and Ψ_4 are the inverse mappings of each other.*

Proof. (i) Let $[d] \in R'_n(\mathfrak{g})$. Then $\Psi_1([d]) = [d^c]$. Let E be the representation space of the representation d. Then E^C is the representation space of

d^c. Then, putting $d^C = \rho$, $E^c = V$, let us show that ρ_R is reducible. In fact, we have $V_R = E + \sqrt{-1}\ E$, $E \cap \sqrt{-1}\ E = (0)$, and E is an invariant subspace of V_R. Thus d^c belongs to $C_n^I(\mathfrak{g})$ and $\Psi_1(R_n^I(\mathfrak{g})) \subset C_n^I(\mathfrak{g})$. Moreover, since ρ_R induces an irreducible real representation d on E, we have

$$\Psi_3\Psi_1([d]) = [d]$$

for every $[d] \in R_n^I(\mathfrak{g})$.

Next let $[\rho] \in C_n^I(\mathfrak{g})$. Let V be the representation space of ρ. Since V_R is reducible, there is an invariant subspace E of V_R such that $E \neq V_R$, $E \neq (0)$. Then we have $V_R = E + \sqrt{-1}\ E$, $E \cap \sqrt{-1}\ E = (0)$ by (3). Then V can be regarded as E^c. Denoting by d the irreducible real representation induced by ρ_R on E, we have then $d^C = \rho$. Then we have $[d] \in R_n^I(\mathfrak{g})$. Thus we have shown that $\Psi_3(C_n^I(\mathfrak{g})) \subset R_n^I(\mathfrak{g})$ and

$$\Psi_1\Psi_3([\rho]) = [\rho]$$

for every $[\rho] \in C_n^I(\mathfrak{g})$. Thus (i) is proved. (ii) Let $[d] \in R_{2n}^{II}(\mathfrak{g})$. Let E be the representation space of the representation d. Then $E^C = V$ contains an irreducible, invariant subspace U such that $V = U + \overline{U}$, $U \cap \overline{U} = (0)$ ($\overline{U}$ is the complex conjugate of U with respect to the complex conjugation of E^c with respect to E). Let ρ be the irreducible representation induced by d^C on U. Let us show that (ρ_R, U_R) is an irreducible real representation. In fact, if U_R contains an invariant subspace F such that $F \neq U_R$, $F \neq (0)$, we have $F + \overline{F} = F_0 + \sqrt{-1}\ F_0$ where $F_0 = (F + \overline{F}) \cap E$. Then F_0 is an invariant subspace of E such that $F_0 \neq E$, $F_0 \neq (0)$. This contradicts to the fact that E is irreducible. Thus we have $[\rho] \in C_n^{II}(\mathfrak{g})$. Let us show moreover that $\rho_R \sim d$. In fact, let us associate to a vector $u \in U$ a vector $\varphi(u) = u + \overline{u} \in E^c$. Then $\varphi(u) \in E$. The mapping $\varphi: U \to E$ thus defined induces a linear mapping $\varphi: U_R \to E$. Since every element $x \in E$ is expressible uniquely as $x = u + \overline{u} (u \in U)$, φ is a linear isomorphism from U_R onto E. Now let X be any element of the Lie algebra $\mathfrak{g}$. Then we have

$$\varphi \circ \rho_R(X) = d(X) \circ \varphi$$

since U and $\overline{U}$ are invariant subspaces and $d(X)$ commute with the mapping $x \to \overline{x}$. Thus we have $\rho_R \sim d$, and we have proved that $\Psi_2(R_{2n}^{II}(\mathfrak{g})) \subset \hat{C}_n^{II}\ (\mathfrak{g})$ and

$$\Psi_1\Psi_2([d]) = [d]$$

for every $[d]\in R_{2n}^{II}(\mathfrak{g})$.

Next, let $[\rho]\in C_n^{II}(\mathfrak{g})$. Let V be the representation space of the representation ρ. Put $E=V_R$ and $d=\rho_R$, then (d, E) is an irreducible real representation of $\mathfrak{g}$. Let us denote the linear automorphism $x\rightarrow\sqrt{-1}\,x$ of the real vector space E by Φ. Then $\Phi^2=-I$ (I means the identity operator of E). Let U_+, U_- be the eigen space of the linear automorphism Φ^C of the complex vector space E^C associated to eigen values $\sqrt{-1}$, $-\sqrt{-1}$ respectively:

$$U_+=\{x\in E^C\ ;\ \Phi^C(x)=\sqrt{-1}\ x\},\ U_-=\{x\in E^C\ ;\ \Phi^C(x)=-\sqrt{-1}\ x\}.$$

Then we have $E^C=U_++U_-$, $U_+\cap U_-=(0)$. Let us denote by $x\rightarrow\bar{x}$ the complex conjugation of E^C with respect to E. Then since $\Phi^C(\bar{x})=\overline{\Phi^C(x)}$ we have

$$\bar{U}_+=U_-.$$

Moreover U_+, U_- are invariant subspaces because Φ commutes with every $d(X)=\rho_R(X)$, $X\in\mathfrak{g}$. Thus we have $[d]\in R_{2n}^{II}(\mathfrak{g})$, and then U_+ and U_- are irreducible invariant subspaces of E^C. Let us denote by ρ_1 the irreducible representation induced by d^C on U_+. Then $\rho\sim\rho_1$. In fact, an element $u=x+\sqrt{-1}\,y\in E^C$ $(x, y\in E)$ is in U_+ if and only if $x=\Phi(y)$. Let us associate to an element y of V (we note that as a set $V=V_R=E$) the element $\Phi(y)+\sqrt{-1}\ y$ of U_+. Then we have a mapping $\varphi:\ y\rightarrow\Phi(y)+\sqrt{-1}\ y$ from V onto U_+. Obviously φ is linear over R. Moreover φ is linear over C, becuase we have $\varphi(\sqrt{-1}\ y)=\varphi(\Phi y)=\Phi^2y+\sqrt{-1}\ \Phi(y)=-y+\sqrt{-1}\ \Phi(y)=\sqrt{-1}\ \varphi(y)$.

Moreover, φ is an isomorphism. In fact, if $\varphi(y)=0$ we have $\Phi(y)=0$, $y=0$, Thus φ is a complex linear isomorphism from V onto U_+. Now let X be any element of $\mathfrak{g}$. Then, since $d=\rho_R$, we have $\varphi\circ\rho(X)=\rho_1(X)\circ\varphi$. Thus we have shown that $\Psi_4(\hat{C}_n^{II}(\mathfrak{g}))\subset R_{2n}^{II}(\mathfrak{g})$ and that

$$\Psi_2\Psi_4((\approx)\text{-equivalence class of }[\rho])=(\approx)\text{-equivalence class of }[\rho]$$

for every $[\rho]\in C_n^{II}(\mathfrak{g})$. Thus (ii) is proved.

Remark. Theorem 1 is also valid for associative algebras and Jordan algebras etc. over R.

§ 6. Reduction of the Problem (A) to the complex irreducible representations

By theorem 1 we have to consider only complex irreducible representations exclusively. In the following we treat only complex representation, so we say simply representation instead of complex representation.

Now Problem A is thus reduced to the following problems:

Problem (A_1): *Find all irreducible (complex) representations of a given real Lie algebra* $\mathfrak{g}$.

Problem (A_2): *Let* (ρ, V) *be an irreducible (complex) representation of* $\mathfrak{g}$. *Decide whether* ρ *is of first class or of second class.*

Now among these problems, Problem (A_1) is equivalent to find all irreducible representations of the complex form $\mathfrak{g}^C$ of $\mathfrak{g}$. It is well-known that the problem of finding all irreducible representation of a given complex Lie algebra is reduced to the case of simple Lie algebras (cf. § 7, 8 below). We shall explain in the following that Problem (A_2) is also reduced to the case of simple Lie algebras.

§ 7. Reduction of the Problem (A) to the reductive case

Let $\mathfrak{g}$ be a Lie algebra over R and $\mathfrak{r}$ the radical of $\mathfrak{g}$. If (d, E) is a completely reducible real representation of $\mathfrak{g}$ over the finite dimensional real vector space E, then, as is well-known,[2] every element of the ideal $[\mathfrak{r}, \mathfrak{g}]$ is mapped by d to zero. Thus every completely reducible representation of $\mathfrak{g}$ is that of $\mathfrak{g}/[\mathfrak{r}, \mathfrak{g}]$. Now $\bar{\mathfrak{g}} = \mathfrak{g}/[\mathfrak{r}, \mathfrak{g}]$ is a reductive Lie algebra, i.e. the radical $\bar{\mathfrak{r}} = \mathfrak{r}/[\mathfrak{r}, \mathfrak{g}]$ of $\bar{\mathfrak{g}}$ coincides with the center of $\bar{\mathfrak{g}}$. Hence we may assume without loss of generality, in dealing with the Problem A, that $\mathfrak{g}$ is a reductive Lie algebra. Let $\mathfrak{z}$ be the center of $\mathfrak{g}$. Then a representation (d, E) of $\mathfrak{g}$ is a completely reducible representation of $\mathfrak{g}$, if and only if for every element $Z \in \mathfrak{z}$, $d(Z)$ is a semi-simple linear operator of E.[3]

Now let $\mathfrak{a}$ be any ideal of a reductive Lie algebra $\mathfrak{g}$. Then since there is an ideal $\mathfrak{b}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$, $\mathfrak{a} \cap \mathfrak{b} = (0)$, the center of $\mathfrak{a}$ is contained in the center $\mathfrak{z}$ of $\mathfrak{g}$. Hence *every completely reducible representation of* $\mathfrak{g}$ *induces also a completely reducible representation of* $\mathfrak{a}$.

§ 8. Induced irreducible representations on ideals

Let $\mathfrak{g}$ be a reductive Lie algebra over R and $\mathfrak{a}$ be an ideal of $\mathfrak{g}$. Then there

2) cf. for example, C. Chevalley, Algebraic Lie Algebras, Ann. of Math. vol. 48 (1946).
3) cf. C. Chevalley, Théorie des groupes de Lie, III (1955), Chap. IV, § 4, n^0 1.

is an ideal $\mathfrak{b}$ of $\mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{a} + \mathfrak{b}, \ \mathfrak{a} \cap \mathfrak{b} = (0).$$

Now let (ρ, V) be a completely reducible representation of $\mathfrak{g}$. Then (ρ, V) induces a representation of $\mathfrak{a}$ over V which is also completely reducible (cf. § 7). Hence V can be decomposed into a direct sum of $\mathfrak{a}$-invariant subspaces:

$$V = V_1 + V_2 + \dots + V_r, \tag{4}$$

where every V_i is a minimal $\mathfrak{a}$-invariant subspace, i.e. the representation of $\mathfrak{a}$ induced by ρ on V_i is irreducible. Let B be any element in $\mathfrak{b}$. Let us consider a linear mapping φ_B from V_i into V_j defined as follows: For $x \in V_i$, let $\varphi_B(x)$ be the V_j-component of $\rho(B)x$ i.e. if we write

$$\rho(B)x = y_1 + \dots + y_r, \ y_k \in V_k (k = 1, \dots, r)$$

then $\varphi_B(x) = y_j$. Since every V_k is $\mathfrak{a}$-invariant, we have $\varphi_B \circ \rho(x) = \rho(X) \circ \varphi_B$ for every $X \in \mathfrak{a}$. Then, if V_i and V_j are not equivalent as the representation spaces of $\mathfrak{a}$ we have $\varphi_B = 0$ by Schur's lemma. In other words, let $V_{k_1}, \dots, V_{k_p}$ be the system of all subspaces V_k in (4) which is equivalent to V_i as representation spaces of $\mathfrak{a}$, then $U = V_{k_1} + \dots + V_{k_p}$ is $\mathfrak{b}$-invariant. Hence U is also $\mathfrak{g}$-invariant. If (ρ, V) is irreducible with respect to $\mathfrak{g}$, then $U = V$. Thus we have the following lemma by means of Jordan-Hölder's theorem.

LEMMA 2. *Let (ρ, V) be an irreducible representation of a reductive Lie algebra $\mathfrak{g}$ and $\mathfrak{a}$ be an ideal of $\mathfrak{g}$. Then every minimal $\mathfrak{a}$-invariant subspaces of V are equivalent to each other as representation spaces of $\mathfrak{a}$ with respect to the representation of $\mathfrak{a}$ induced by ρ.*

In this case we shall denote by $V_\mathfrak{a}$ one of the minimal $\mathfrak{a}$-invariant subspaces of V, and by $\rho_\mathfrak{a}$ the irreducible representation of $\mathfrak{a}$ induced by ρ on $V_\mathfrak{a}$. The representation $(\rho_\mathfrak{a}, V_\mathfrak{a})$ is determined up to an equivalence. We shall call this irreducible representation $(\rho_\mathfrak{a}, V_\mathfrak{a})$ of $\mathfrak{a}$ *the induced irreducible representation of $\mathfrak{a}$ by the irreducible representation (ρ, V) of $\mathfrak{g}$.*

Now let (ρ, V) be an irreducible representation of $\mathfrak{g}$ and (4) be a decomposition of V into a direct sum of irreducible $\mathfrak{a}$-invariant subspaces $V_1, \dots, V_r$. We may take V_1 as $V_\mathfrak{a}$. Since V_1 and V_i are equivalent, we can choose equivalence mappings $\varphi_1^i: V_1 \to V_i$ with $\varphi_1^1 =$ identity. We put $\varphi_1^i \circ (\varphi_1^j)^{-1} = \varphi_j^i$, then $\varphi_j^i: V_j \to V_i$ is an equivalence mapping as representation spaces of $\mathfrak{a}$.

Let us fix the system $\{\varphi_j^i\}$ of equivalence mappings. Note that $\varphi_j^i \circ \varphi_k^j = \varphi_k^i$. Now let C^r be the Cartesian space with r complex components. Let us construct a representation of $\mathfrak{b}$ on C^r. Let $B \in \mathfrak{b}$. Denoting by π_j the projection from V onto V_j with respect to the decomposition (4), we have a linear endomorphism $\varphi_j^i \circ \pi_j \circ \rho(B)$ of V which is commutative with every $\rho(X)$, $X \in \mathfrak{a}$. Then, by Schur's lemma, $\varphi_j^i \circ \pi_j \circ \rho(B)$ is a scalar operator on V_i. Denote this scalar by $\sigma_j^i(B)$, then we obtain $\rho(B)\varphi_1^i(x) = \sum_j \sigma_j^i(B)\varphi_i^j \varphi_1^i(x)$ for $x \in V_1$. Denote by $\sigma(B)$ the $r \times r$ matrix $\sigma(B) = (\sigma_j^i(B))$. $\sigma(B)$ is a linear endomorphism of C^r. Now $B \to \sigma(B)$ is a representation of $\mathfrak{b}$ on C^r. To show this, let us consider a bilinear mapping from $V_\mathfrak{a} \times C^r$ into V defined as follows: let $x \in V_\mathfrak{a}$ $(= V_1)$, $\lambda \in C^r$. Then we write $[x, \lambda] = \sum_{i=1}^{r} \lambda_i \varphi_1^i(x)$ where $\lambda = (\lambda_1, \ldots, \lambda_r) \in C^r$. Then $(x, \lambda) \to [x, \lambda]$ is a bilinear mapping $V_\mathfrak{a} \times C^r \to V$ and obviously any element of V can be expressed as a finite sum of elements of a form $[x, \lambda]$, $x \in V_\mathfrak{a}$, $\lambda \in C^r$. Then we obtain an onto linear mapping $V_\mathfrak{a} \otimes C^r \to V$ such that $x \otimes \lambda \to [x, \lambda]$. Since $\dim V = \dim (V_\mathfrak{a} \otimes C^r)$, this linear mapping is a linear isomorphism of V with $V_\mathfrak{a} \otimes C^r$. So we identify V with $V_\mathfrak{a} \otimes C^r$ and write $x \otimes \lambda$ instead of $[x, \lambda]$. Now we have

$$(5) \qquad \rho(A)\ (x \otimes \lambda) = \sum \lambda_i \rho(A)\varphi_1^i(x) = \sum \lambda_i \varphi_1^i(\rho_\mathfrak{a}(A)x) = \rho_\mathfrak{a}(A)x \otimes \lambda$$

for every $A \in \mathfrak{a}$, and by $\rho(B)\varphi_1^i(x) = \sum_j \sigma_j^i(B)\varphi_i^j \varphi_1^i(x) = \sum_j \sigma_j^i(B)\varphi_1^j(x)$, we have for any $B \in \mathfrak{b}$

$$(6) \qquad \rho(B)\ (x \otimes \lambda) = \sum_i \lambda_i \rho(B)\varphi_1^i(x) = \sum_{ij} \lambda_i \sigma_j^i(B)\varphi_1^j(x) = x \otimes \sigma(B)\lambda$$

Then for $B_1, B_2 \in \mathfrak{b}$ we obtain by

$$\begin{aligned} \rho([B_1, B_2])\ (x \otimes \lambda) &= \rho(B_1)\rho(B_2)\ (x \otimes \lambda) - \rho(B_2)\rho(B_1)\ (x \otimes \lambda) \\ &= x \otimes \sigma(B_1)\sigma(B_2)\lambda - x \otimes \sigma(B_2)\sigma(B_1)\lambda \\ &= x \otimes [\sigma(B_1), \sigma(B_2)]\lambda \end{aligned}$$

that $\sigma([B_1, B_2]) = [\sigma(B_1), \sigma(B_2)]$ i.e., $B \to \sigma(B)$ is a representation of $\mathfrak{b}$ on C^r.

Now let $\pi_\mathfrak{a}$ and $\pi_\mathfrak{b}$ be the projections from $\mathfrak{g}$ onto $\mathfrak{a}$ and $\mathfrak{b}$ respectively with respect to the decomposition $\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$. Then $\rho_\mathfrak{a} \circ \pi_\mathfrak{a}$ and $\sigma \circ \pi_\mathfrak{b}$ are representations of $\mathfrak{g}$. From (5), (6) we also see that the representation (ρ, V) of $\mathfrak{g}$ is equivalent to the tensor product of two representation $(\rho_\mathfrak{a} \circ \pi_\mathfrak{a}, V_\mathfrak{a})$, $(\sigma \circ \pi_\mathfrak{b}, C^r)$:

$\rho=\rho_{\mathfrak{a}}\circ\pi_{\mathfrak{a}}\dot{\otimes}\sigma\circ\pi_{\mathfrak{b}}$,[4)] $V=V_{\mathfrak{a}}\otimes C^r$. The representation (σ, C^r) of $\mathfrak{b}$ is irreducible. In fact if C^r contains a non-trivial $\mathfrak{g}$-invariant subspace U, then $V_{\mathfrak{a}}\otimes U$ is obviously a non-trivial $\mathfrak{g}$-invariant subspace of $V=V_{\mathfrak{a}}\otimes C^r$ by (5), (6).

Now let us show that (σ, C^r) is equivalent to the induced irreducible representation of $\mathfrak{b}$ by the irreducible representation (ρ, V) of $\mathfrak{g}$. In fact, let $e_1, \ldots, e_s$ be a base of $V_{\mathfrak{a}}$. Then $V=\sum_i e_i\otimes C^r$ is a direct sum of $\mathfrak{b}$-invariant subspaces $e_i\otimes C^r$, and since every $e_i\otimes C^r$ is $\mathfrak{b}$-irreducible, $e_i\otimes C^r$ is a minimal $\mathfrak{b}$-invariant subspace of V. Hence $\rho_{\mathfrak{b}}\sim\sigma$ by (6). Thus we have the following

LEMMA 3. *Let $\mathfrak{g}$ be a reductive Lie algebra over R and $\mathfrak{g}=\mathfrak{a}+\mathfrak{b}$ be a decomposition of $\mathfrak{g}$ into ideals $\mathfrak{a}$, $\mathfrak{b}$ of $\mathfrak{g}$. Then every irreducible representation (ρ, V) of $\mathfrak{g}$ is equivalent to the tensor product of two irreducible representations $(\rho_{\mathfrak{a}}\circ\pi_{\mathfrak{a}}, V_{\mathfrak{a}})$ and $(\rho_{\mathfrak{b}}\circ\pi_{\mathfrak{b}}, V_{\mathfrak{b}})$, where $\pi_{\mathfrak{a}}$ and $\pi_{\mathfrak{b}}$ are projections of $\mathfrak{g}$ onto $\mathfrak{a}$ and $\mathfrak{b}$ respectively.*

Conversely, if (ρ_1, U_1) and (ρ_2, U_2) are arbitrary irreducible representations of $\mathfrak{a}$ and $\mathfrak{b}$ respectively, then $(\rho_1\circ\pi_{\mathfrak{a}}\dot{\otimes}\rho_2\circ\pi_{\mathfrak{b}}, U_1\otimes U_2)$ is an irreducible representation of $\mathfrak{g}$. To show this, let $e_1, \ldots, e_r$ be any base of U_2 $(r=\dim U_2)$. Then we have $U_1\otimes U_2=\sum_j U_1\otimes e_j$ (direct sum), and every $U_1\otimes e_j$ is an $\mathfrak{a}$-invariant subspace of $U_1\otimes U_2$ which is $\mathfrak{a}$-irreducible. Then by Jordan-Hölder's theorem, every minimal $\mathfrak{a}$-invariant subspace of $U_1\otimes U_2$ are equivalent to each other and are equivalent to U_1. Analogously, every minimal $\mathfrak{b}$-invariant subspace of $U_1\otimes U_2$ are equivalent to each other and are equivalent to U_2. Now let V be any minimal $\mathfrak{g}$-invariant subspace of $U_1\otimes U_2$, and let ρ be the irreducible representation of $\mathfrak{g}$ induced by $\rho_1\circ\pi_{\mathfrak{a}}\dot{\otimes}\rho_2\circ\pi_{\mathfrak{b}}$ on V. Then, from what we remarked above, we have $\rho_{\mathfrak{a}}\sim\rho_1$, $\rho_{\mathfrak{b}}\sim\rho_2$. Then $\dim V=\dim U_1\cdot\dim U_2$. Hence $V=U_1\otimes U_2$. Thus $U_1\otimes U_2$ is irreducible.

Thus in order to find all irreducible representation of $\mathfrak{g}$, it is sufficient to find all irreducible representations of $\mathfrak{a}$ and $\mathfrak{b}$. We note here that *for two*

4) In general, the tensor product of two representations (ρ_1, V_1), (ρ_2, V_2) of a Lie algebra $\mathfrak{g}$ is defined as the following representation (ρ, V) of $\mathfrak{g}$: the representation space V is the tensor product of V_1, V_2, i.e. $V=V_1\otimes V_2$, and for $X\in\mathfrak{g}$, $\rho(X)$ is an endomorphism of V given by

$$\rho(X)=\rho_1(X)\otimes I_2+I_1\otimes\rho_2(X),\ \text{i.e.}\ \rho(X)\ (x\otimes y)=\rho_1(X)x\otimes y+x\otimes\rho_2(X)y,$$

where I_1, I_2 denote the identical operators of V_1, V_2 respectively. This representation ρ is denoted by $\rho=\rho_1\dot{\otimes}\rho_2$ (ρ is also called the tensor sum of ρ_1, ρ_2).

irreducible representations (ρ, V), (σ, U) *of* $\mathfrak{g}$, *we have* $\rho\sim\sigma$ *if and only if* $\rho_{\mathfrak{a}}\sim\sigma_{\mathfrak{a}}$ *and* $\rho_{\mathfrak{b}}\sim\sigma_{\mathfrak{b}}$. These facts are easily extended to the case of the decomposition of $\mathfrak{g}$ into many ideals: $\mathfrak{g}=\mathfrak{a}+\mathfrak{b}+\ldots+\mathfrak{c}$. If we take in particular the decomposition of $\mathfrak{g}$ into simple ideals:

$$\mathfrak{g}=\mathfrak{g}_1+\mathfrak{g}_2+\ldots+\mathfrak{g}_r,$$

then, Problem (A_1) is reduced to the case of simple Lie algebras.

§ 9. Criterions of self-conjugateness

Let (ρ, V) be an irreducible representation of a (reductive) Lie algebra $\mathfrak{g}$ over R. Let us consider the condition for ρ to be self-conjugate. If $\rho\sim\bar{\rho}$, then there exists an anti-linear automorphism J of V such that $J\circ\rho(X)=\rho(X)\circ J$ for every $X\in\mathfrak{g}$ (cf. § 4). Then J^2 is a linear automorphism of V which is commutative with every $\rho(X)$, $X\in\mathfrak{g}$. Then by Schur's lemma, J^2 is a scalar operator of V: $J^2=cI$ $(c\in C)$. Now let us call (after E. Cartan) an anti-linear automorphism J of a complex vector space V an ***anti-involution*** if J^2 is a scalar operator of V. If J is an anti-involution of V and $J^2=cI$, then c is a real number. In fact, let $e_1, \ldots, e_n$ be any base of V. Then, putting $Je_i=\sum_j\alpha_i^je_j$ $(\alpha_i^j\in C)$, we have $J^2e_i=\sum_{j,k}\bar{\alpha}_i^j\alpha_j^ke_k$. Hence, if we denote by A the complex matrix (α_i^j), we have

$$A\bar{A}=cI.$$

Then by $c\neq 0$, we have $A\bar{A}=\bar{A}A$ and so $c=\bar{c}$. Hence c is real. If $c>0$ $(c<0)$ then J is called an anti-involution of the *first* (*second*) *kind*. We also say that the index of J is $+1$ (-1) if J is an anti-involution of the first (second) kind. We remark that if J is an anti-involution of index ε $(\varepsilon=\pm1)$, then for any complex number $\gamma\neq 0$, γJ is also an anti-involution of index ε (Note that $(\gamma J)^2=|\gamma|^2J^2$).

We have seen in the above that if (ρ, V) is a self-conjugate, irreducible representation, then there is an anti-involution J which is invariant by ρ. Now let us note that such an anti-involution is unique up to scalar multiples. In fact, if J and J' are invariant anti-involutions, then $J'J^{-1}$ is a linear automorphism of V which is commutative with every $\rho(X)$, $X\in\mathfrak{g}$. Hence $J'=\gamma J$ for some $\gamma\in C$ by Shur's lemma. Thus the index of J is independent on the choice of J. This index is called the *index* of a self-conjugate, irreducible representation

(ρ, V).

LEMMA 4. *Let (ρ, V) be an irreducible representation of $\mathfrak{g}$. Then ρ is of the first class if and only if ρ is self-conjugate and of index* 1.

Proof. Let (ρ, V) be of the first class. Then V_R contains a ρ_R-invariant (real) subspace E such that

$$V = E + \sqrt{-1}\, E,\ E \cap \sqrt{-1}\, E = (0).$$

Let J be the complex conjugate operation of V with respect to E: $J(x + \sqrt{-1}\, y) = x - \sqrt{-1}\, y$ $(x, y \in E)$. Then $J^2 = I$ and J is invariant by ρ since E is ρ_R-invariant.

Conversely let ρ be self-conjugate and of index 1. Then there is an anti-involution J of V which is invariant by ρ and $J^2 = I$. Let $E = \{x \in V;\ Jx = x\}$. Then E is a real subspace, i.e. E is a subspace of V_R and moreover E is invariant by ρ_R. Now every element $x \in V$ can be expressed as $x = \frac{1}{2}(x + Jx) + \frac{1}{2}(x - Jx)$, where we have $x + Jx \in E$ and $x - Jx \in \{x \in V;\ Jx = -x\} = \sqrt{-1}\, E$. Thus we have $V = E + \sqrt{-1}\, E$, $E \cap \sqrt{-1}\, E = (0)$. Then E is a non-trivial ρ_R-invariant subspace of V_R. Thus (ρ_R, V_R) is reducible and ρ is of the first class. Thus lemma 4 is proved.

Thus Problem (A_2) is reduced to decide the self-conjugateness and the index of an irreducible representation. We note here a necessary condition for a representation (ρ, V) to be self-conjugate.

LEMMA 5. *If a representation (ρ, V) of a real Lie algebra $\mathfrak{g}$ is self-conjugate, then $\rho(\mathfrak{g}) \cap \sqrt{-1}\, \rho(\mathfrak{g}) = (0)$.*

Proof. Let J be a anti-linear automorphism of V which is invariant by ρ. If $\rho(A) = \sqrt{-1}\, \rho(B) \in \rho(\mathfrak{g}) \cap \sqrt{-1}\, \rho(\mathfrak{g})$, $(A, B \in \mathfrak{g})$, then we have $J\rho(A)J^{-1} = \rho(A)$, $J(\sqrt{-1}\, \rho(B))J^{-1} = \sqrt{-1}\, \rho(B)$. On the other hand, $J(\sqrt{-1}\, \rho(B))J^{-1} = -\sqrt{-1}\, J\rho(B)J^{-1} = -\sqrt{-1}\, \rho(B)$ Therefore we have $\rho(A) = \sqrt{-1}\, \rho(B) = 0$ and $\rho(\mathfrak{g}) \cap \sqrt{-1}\, \rho(\mathfrak{g}) = (0)$, Q.E.D.

Now let $\mathfrak{g}$ be a reductive Lie algebra over R and let

$$\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2 + \dots + \mathfrak{g}_r$$

be the decomposition of $\mathfrak{g}$ into simple ideals $\mathfrak{g}_1, \dots, \mathfrak{g}_r$. We shall denote by π_i the projection of $\mathfrak{g}$ onto $\mathfrak{g}_i$ with respect to the above decomposition. Let

(ρ, V) be an irreducible representation of $\mathfrak{g}$ and ρ_i $(i=1, \ldots, r)$ be the induced irreducible representations of $\mathfrak{g}_i$ by the irreducible representation ρ of $\mathfrak{g}$. Under these notations we have the following

LEMMA 6. *$\rho \sim \bar{\rho}$ if and only if $\rho_i \sim \bar{\rho}_i$ for $i=1, \ldots, r$. In this case the index ε of ρ is given by $\varepsilon = \varepsilon_1 \varepsilon_2 \ldots \varepsilon_r$ where ε_i is the index of ρ_i $(i=1, \ldots, r)$.*

Proof. Assume $\rho \sim \bar{\rho}$. Then there is an anti-involution J which is invariant by ρ. Let V_1 be a minimal $\mathfrak{g}_1$-invariant subspace of V. Then JV_1 is also a minimal $\mathfrak{g}_1$-invariant subspace of V as is seen easily. Then V_1 and JV_1 are equivalent as representation spaces of $\mathfrak{g}_1$ by lemma 2. Hence we have $\rho_1 \sim \bar{\rho}_1$. Analogously $\rho_i \sim \bar{\rho}_i$ $(i=1, \ldots, r)$. Conversely assume that $\rho_i \sim \bar{\rho}_i$ $(i=1, \ldots, r)$. Let $V_1, \ldots, V_r$ be the representation spaces of $\rho_1, \ldots, \rho_r$ respectively. Then we may assume that $V = V_1 \otimes \ldots \otimes V_r$ and $\rho = \rho_1 \circ \pi_1 \oplus \ldots \oplus \rho_r \circ \pi_r$. Let J_i $(i=1, \ldots, r)$ be an anti-involution of V_i which is invariant by ρ_i and $J_i^2 = \varepsilon_i I$. Then $J = J_1 \otimes \ldots \otimes J_r$ is an anti-involution of V and $J^2 = J_1^2 \otimes \ldots \otimes J_r^2 = \varepsilon_1 \ldots \varepsilon_r I$. Moreover J is invariant by ρ, since for $X = X_1 + \ldots + X_r \in \mathfrak{g}$ $(X_i \in \mathfrak{g}_i, i=1, \ldots, r)$ we have $J\rho(X) = (J_1 \otimes \ldots \otimes J_r)(\rho_1(X_1) \otimes I \otimes \ldots \otimes I + \ldots + I \otimes \ldots \otimes I \otimes \rho_r(X_r)) = \rho(X)J$. Thus we have completed the proof.

Thus Problem (A_2) is reduced to the case of real simple Lie algebras by lemmas 4, 6. In the following we shall consider this case.

§ 10. Irreducible representation of real simple Lie algebras

Let $\mathfrak{g}$ be a real simple Lie algebra. Then the following three cases are possible:

a) $\mathfrak{g}$ is 1-dimensional abelian Lie algebra,

b) $\mathfrak{g}$ is simple, non-abelian Lie algebra and $\mathfrak{g}^C$ is not simple,

c) $\mathfrak{g}$ is simple, non-abelian Lie algebra and $\mathfrak{g}^C$ is simple.

Let $\mathfrak{g}$ be an abelian Lie algebra of dimension 1. Then an irreducible representation (ρ, V) of $\mathfrak{g}$ is of degree 1. Obviously ρ is self-conjugate if and only if every element of $\rho(\mathfrak{g})$ is a real multiple of the identity. Moreover, if $\rho \sim \bar{\rho}$ then clearly the index of ρ is equal to 1. Next let us consider the cases b), c) simultaneously. For this purpose we consider a real semi-simple Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. We denote by l the dimension of $\mathfrak{h}$. Then $\mathfrak{h}^C$ is a Cartan subalgebra of $\mathfrak{g}^C$. We denote by $Z \rightarrow \bar{Z}$ the complex conjugate operation of $\mathfrak{g}^C$ with respect to $\mathfrak{g}$. Then we have $\overline{\alpha X+\beta Y}=\bar{\alpha}\overline{X}+\bar{\beta}\overline{Y}$ and $\overline{[X, Y]}=[\overline{X}, \overline{Y}]$ for every $X, Y\in\mathfrak{g}^C$, $\alpha, \beta\in C$. Let $\Delta\ni\alpha, \beta, \ldots$, be the root system of $\mathfrak{g}^C$ with respect to the Cartan subalgebra $\mathfrak{h}^C$. Let Λ be a linear form on $\mathfrak{h}^C$, then we denote by $\overline{\Lambda}$ the linear form on $\mathfrak{h}^C$ defined by $\overline{\Lambda}(H)=\overline{\Lambda(\overline{H})}$ for every $H\in\mathfrak{h}^C$. Then the mapping $\Lambda\to\overline{\Lambda}$ is an anti-linear involution of the dual vector space $(\mathfrak{h}^C)^*$ of $\mathfrak{h}^C$. We have clearly $\overline{\overline{\Lambda}}=\Lambda$

LEMMA 7. *$\overline{\Delta}=\Delta$, i.e. the mapping $\Lambda\to\overline{\Lambda}$ induces a permutation of Δ.*

Proof. For $\alpha\in\Delta$, take a root vector $E_\alpha\neq 0$ in $\mathfrak{g}^C$. Then $[H, E_\alpha]=\alpha(H)E_\alpha$ for every $H\in\mathfrak{h}^C$. Hence we have $[\overline{H}, \overline{E}_\alpha]=\overline{\alpha(H)}\overline{E}_\alpha$, i.e. $[H, \overline{E}_\alpha]=\bar{\alpha}(H)\overline{E}_\alpha$. Consequently we have $\bar{\alpha}\in\Delta$, Q.E.D.

Let R_l be the real subspace of $(\mathfrak{h}^C)^*$ consisting of all linear combinations of roots with real coefficients. Then the canonical inner product[5] (Λ_1, Λ_2) on $(\mathfrak{h}^C)^*$ is positive definite on R_l, and R_l is an Euclidean space with respect to this inner product (Λ_1, Λ_2). The anti-linear involution $\Lambda\to\overline{\Lambda}$ leaves R_l invariant. Then by lemma 7, we have[6]

$$(7) \qquad (\Lambda_1, \Lambda_2)=(\overline{\Lambda}_1, \overline{\Lambda}_2)$$

for every $\Lambda_1, \Lambda_2\in R_l$.

Let $\Pi=\{\alpha_1, \ldots, \alpha_l\}$ be a fundamental root system[7] in Δ. Then by lemma 7, $\overline{\Pi}=\{\bar{\alpha}_1, \ldots, \bar{\alpha}_l\}$ is also a fundamental root system in Δ. Hence there is an element S_0 in the Weyl group $\mathbf{W}$ of $\mathfrak{g}^C$ with respect to $\mathfrak{h}^C$ such that $S_0(\Pi)=\overline{\Pi}$[8].

Now let (ρ, V) be a representation of $\mathfrak{g}$. Then ρ can be uniquely extended to a representation of $\mathfrak{g}^C$ on V which we also denote by (ρ, V). Let Λ be a weight of $\mathfrak{g}^C$ (with respect to $\mathfrak{h}^C$) in the representation (ρ, V). Then $\overline{\Lambda}$ is a weight of $\mathfrak{g}^C$ in the representation $(\bar{\rho}, \overline{V})$. In fact, let $x\neq 0$ a vector in V such that $\rho(H)x=\Lambda(H)x$ for every $H\in\mathfrak{h}^C$. Then $\overline{\rho(H)}\bar{x}=\overline{\Lambda(H)}\bar{x}$. Now we have

[5] Let φ be the Killing form of $\mathfrak{g}^C$. Then for any $\Lambda\in(\mathfrak{h}^C)^*$, there coresponds uniquely an element $H_\Lambda\in\mathfrak{h}^C$ such that $\varphi(H_\Lambda, H)=\Lambda(H)$ for every H in $\mathfrak{h}^C$. Then the canonical inner product of $\Lambda_1, \Lambda_2\in(\mathfrak{h}^C)^*$ is given by $(\Lambda_1, \Lambda_2)=\varphi(H_{\Lambda_1}, H_{\Lambda_2})$.

[6] cf. [2], Exposé n^0 11 et 12, Théorème 1.

[7] I.e. every $\alpha\in\Delta$ is expressible uniquely as $\alpha=\sum m_i\alpha_i$ with integral coefficients m_i such that $m_1\geqq 0, \ldots, m_l\geqq 0$ or $m_1\leqq 0, \ldots, m_l\leqq 0$.

[8] cf. [2], Exposé n^0 16, Théorème 1.

$\bar{\rho}(H) = \overline{\rho(\bar{H})}$, because if $H = H_1 + \sqrt{-1}\, H_2$ $(H_1, H_2 \in \mathfrak{h})$, then $\bar{\rho}(H) = \bar{\rho}(H_1) + \sqrt{-1}\ \bar{\rho}(H_2) = \overline{\rho(H_1)} + \sqrt{-1}\ \overline{\rho(H_2)} = \overline{\rho(H_1 - \sqrt{-1}\ H_2)}$. Hence $\bar{\rho}(H)\bar{x} = \bar{\Lambda}(H)\bar{x}$ for every $H \in \mathfrak{h}^C$. Consequently $\bar{\Lambda}$ is a weight of $\mathfrak{g}^C$ in the representation $(\bar{\rho}, \bar{V})$.

Thus, if we denote by $W(\rho)$ the set of all weights in the representation (ρ, V) then we have

$$(8) \qquad \overline{W(\rho)} = W(\bar{\rho}).$$

A weight Λ in the representation (ρ, V) is called *extreme* if we have for any root α, $\Lambda + \alpha \notin W(\rho)$ or $\Lambda - \alpha \notin W(\rho)$. Then we have by (8) and lemma 7 the following

LEMMA 8. *If Λ is an extreme weight in (ρ, V), then $\bar{\Lambda}$ is an extreme weight in $(\bar{\rho}, \bar{V})$.*

Now let us introduce a lexicographical linear order in R_l such that Π becomes the set of all simple roots[9] in Δ with respect to this linear order. Then we can speak of the highest weight in the representation (ρ, V). The following lemma is well-known.

LEMMA 9. *If Λ_0 is the highest weight in (ρ, V) and Λ_1 is an extreme weight in the irreducible representation (ρ, V). Then there is an element S in the Weyl group W such that $S(\Lambda_1) = \Lambda_0$.*

Proof. Let Λ_2 be the highest weight among the set of weights $\{S(\Lambda_1);\ S \in W\}$. Replacing Λ_1 by Λ_2 if necessary, we may assume that $\Lambda_1 = \Lambda_2$. Then we have $S_\alpha(\Lambda_1) = \Lambda_1 - \dfrac{2(\Lambda_1, \alpha)}{(\alpha, \alpha)}\alpha \leqq \Lambda_1$. Hence we have $\Lambda_1 - \alpha \in W(\rho)$ for every positive root α such that $(\Lambda_1, \alpha) \neq 0$, then $\Lambda_1 + \alpha$ is not a weight in (ρ, V). In other words, if we denote by E_α a root vector belonging to the root α, then we have $\rho(E_\alpha) V_{\Lambda_1} = (0)$ for $\alpha > 0$, where $V_{\Lambda_1} = \{x \in V;\ \rho(H)x = \Lambda_1(H)x$ for every $H \in \mathfrak{h}^C\}$. Then easy induction shows that every subspace of the following form

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[9] A simple root is a positive root which not expressible as a sum of two positive roots. cf. [2], Exposé n^0 10. Now, a lexicographical linear order in R_l is defined as follows: let $\xi = \sum \xi_i \alpha_i$, $\eta = \sum \eta_i \alpha_i$ be in R_l. Then we define $\xi > \eta$ if $\xi_1 = \eta_1, \dots, \xi_{r-1} = \eta_{r-1}$, $\xi_r > \eta_r$ for some r, $1 \leqq r \leqq l$. Then the set of all simple roots in Δ with respect to this linear order coincides with $\alpha_1, \dots, \alpha_l$.

$$V_{\Lambda_1},\ \rho(E_{\beta_1})\ldots\rho(E_{\beta_t})V_{\Lambda_1},\ (\beta_i\in\Delta,\ i=1,\ldots,t)$$

coincides with a subspace of the following form

$$V_{\Lambda_1},\ \rho(E_{\gamma_1})\ldots\rho(E_{\gamma_s})V_{\Lambda_1},\ (\gamma_j\in\Delta,\ j=1,\ldots,s,\ \gamma_j<0).$$

Then by virtue of the irreducibility of V, we see that

$$V=V_{\Lambda_1}+\sum_{\beta_i\in\Delta}\rho(E_{\beta_1})\ldots\rho(E_{\beta_t})V_{\Lambda_1}=V_{\Lambda_1}+\sum_{\gamma_j\in\Delta,\ \gamma_j<0}\rho(E_{\gamma_1})\ldots\rho(E_{\gamma_s})V_{\Lambda_1}.$$

Thus, Λ_1 is the highest weight in (ρ, V), Q.E.D.

Now let $\Lambda_1,\ldots,\Lambda_l$ be the fundamental weight system of $\mathfrak{g}^C$ determined by Π, i.e. $\Lambda_1,\ldots,\Lambda_l$ be the elements in R_l such that

$$\left(\Lambda_i,\ \frac{2\alpha_j}{(\alpha_i,\alpha_i)}\right)=\delta_{ij},\ \ (1\leqq i,\ j\leqq l).$$

Then, by (7), $\overline{\Lambda}_1,\ldots,\overline{\Lambda}_l$ are the fundamental weight system of $\mathfrak{g}^C$ determined by $\overline{\Pi}$. On the other hand, since $S_0(\Pi)=\overline{\Pi}$, we have $S_0\{\Lambda_1,\ldots,\Lambda_l\}=\{\overline{\Lambda}_1,\ldots,\overline{\Lambda}_l\}$. i.e. $S_0(\Lambda_i)=\overline{\Lambda}_{\sigma(i)}$ $(i=1,\ldots,l)$ for some permutation σ of $\{1,\ldots,l\}$.

Now let $(\rho_i,\ V_i)$ $(i=1,\ldots,l)$ be the irreducible representation which has Λ_i as the highest weight. $\rho_1,\ldots,\rho_l$ are called the fundamental representations determined by Π. Then the highest weight Λ_i' of the irreducible representation $\bar{\rho}_i$ is expressible in the form $\Lambda_i'=S(\overline{\Lambda}_i)$, where S is an element in the Weyl group by lemmas 8, 9. Then we have

$$\Lambda_i'=SS_0(\Lambda_{\sigma^{-1}(i)}).$$

Then we have $\Lambda_i'=\Lambda_{\sigma^{-1}(i)}$ since Λ_i' and $\Lambda_{\sigma^{-1}(i)}$ are both dominant weights.[10] Consequently, we have

(9) $$\bar{\rho}_i\sim\rho_{\sigma^{-1}(i)}\ \ (i=1,\ldots,l)$$

Now we see that $\sigma^2=1$ by (9). Then arranging the order $\alpha_1,\ldots,\alpha_l$ if necessary, we may and shall assume that $\sigma(1)=2$, $\sigma(3)=4,\ldots,\sigma(2k-1)=2k$, $\sigma(2k+1)=2k+1,\ldots,\sigma(l)=l$.

Let $(\rho,\ V)$ be an irreducible representation and Λ be the highest weight of ρ. Then we have

$$\Lambda=m_1\Lambda_1+\ldots+m_l\Lambda_l$$

where $m_1,\ldots,m_l$ are non-negative integers. Then we have $\overline{\Lambda}=\Sigma m_i\overline{\Lambda}_i$

[10] A weight Λ is called dominant if $S\Lambda\leqq\Lambda$ for any element S in the Weyl group W.

$= S_0(\Sigma m_i \Lambda_{\sigma^{-1}(i)})$. Consequently $\Sigma m_i \Lambda_{\sigma^{-1}(i)}$ is conjugate to the highest weight Λ' of $\bar{\rho}$ under the Weyl group. On the other hand, since $\Sigma m_i \Lambda_{\sigma^{-1}(i)}$ is a dominant weight, $\Sigma m_i \Lambda_{\sigma^{-1}(i)}$ must coincide with Λ': $\Lambda' = \Sigma m_i \Lambda_{\sigma^{-1}(i)} = \Sigma m_{\sigma(i)} \Lambda_i$.

Then we have $\rho \sim \bar{\rho}$ if and only if $\Lambda = \Lambda'$, in other words, we have $\rho \sim \bar{\rho}$ if and only if

(10) $$m_1 = m_2,\ m_3 = m_4,\ \ldots,\ m_{2k-1} = m_{2k}.$$

Now let us consider the index ε of ρ when $\rho \sim \bar{\rho}$. Let ε_{2k+j} be the index of ρ_{2k+j} $(j = 1, \ldots, l-2k)$. We assert that

(11) $$\varepsilon = \varepsilon_{2k+1}^{m_{2k+1}} \cdots \varepsilon_l^{m_l}.$$

To prove (11), we shall recall the definition of the *Cartan composite*:

Let (ρ, V), (σ, U) be two irreducible representations of $\mathfrak{g}$. Let Λ, Λ' be the highest weight of ρ, σ respectively. Let W be the minimal invariant subspace of $V \otimes U$ generated by $V_\Lambda \otimes U_{\Lambda'}$,[11] and τ be the induced irreducible representation by $\rho \oplus \sigma$ on W. Then the irreducible representation (τ, W) is called the Cartan composite of ρ and σ, which we denote by $\tau = \rho * \sigma$, $W = V * U$. Then the highest weight of τ is $\Lambda + \Lambda'$. The operation $*$ is associative and $\rho * \sigma \sim \sigma * \rho$. By the criterion (10), if $\rho \sim \bar{\rho}$ and $\sigma \sim \bar{\sigma}$ then we have $\tau \sim \bar{\tau}$. Now

LEMMA 10. *Let (ρ, V) (σ, U) be irreducible, self-conjugate representations of indices ε, ε' respectively. Then the index of $\tau = \rho * \sigma$ is $\varepsilon\varepsilon'$.*

Proof. Let J, J' be the anti-involutions on V, U which are invariant by ρ, σ respectively and $J^2 = \varepsilon I$, $J'^2 = \varepsilon' I$. Then $J \otimes J'$ is an anti-involution on $V \otimes U$ invariant by $\rho \otimes \sigma$. We have $(J \otimes J')^2 = \varepsilon\varepsilon' \cdot I$. Now put $W = V * U$ and decompose $V \otimes U$ into the direct sum of irreducible subspaces:

$$V \otimes U = W_1 + \ldots + W_r,\quad (W_1 = W).$$

Let us denote by π_i the projection from $V \otimes U$ onto W_i with respect to the above decomposition. Then $\varphi_i = \pi_i \circ (J \otimes J')$ is an anti-linear mapping from W_1 into W_i, and we have

$$\varphi_i \circ \tau(X) = \tau(X) \circ \varphi_i \qquad \text{for every } X \in \mathfrak{g}.$$

Since every W_i is irreducible, we have then, $\overline{W}_1 \sim W_i$ if $\varphi_i \neq 0$. However we

11) V_Λ and $U_{\Lambda'}$ mean the eigen-spaces of Λ, Λ' respectively.

have $\overline{W}_1 \sim W_1 + W_i$ for every $i > 1$, hence we must have $\varphi_i = 0$ for every $i > 1$. In other words, $(J \otimes J')(W_1) = W_1$. Thus, $J \otimes J'$ induces an anti-involution on W_1 of index $\varepsilon\varepsilon'$. $J \otimes J'$ is clearly invariant by τ, Q.E.D.

LEMMA 11. *Let (ρ, V) be any irreducible representation of $\mathfrak{g}$. Then $\rho * \overline{\rho}$ is self-conjugate and of index* 1.

Proof. Let J: $V \otimes \overline{V} \to V \otimes \overline{V}$ be a mapping defined by

$$J(x \otimes \overline{y}) = y \otimes \overline{x}, \qquad (x, y \in V).$$

Then J is an anti-involution of index 1. J is invariant by $\rho \oplus \overline{\rho}$:

$$(\rho \oplus \overline{\rho})(X) \circ J(x \otimes \overline{y}) = \rho(X) y \otimes \overline{x} + y \otimes \overline{\rho}(X)\overline{x} = J(x \otimes \overline{\rho}(X)\overline{y} + \rho(X)x \otimes \overline{y})$$
$$= J \circ (\rho \oplus \overline{\rho})(X)(x \otimes \overline{y}).$$

Now let $\Lambda = \Sigma m_i \Lambda_i$ be the highest weight of ρ. Then the highest weight of $\rho * \overline{\rho}$ is given by $(m_1 + m_2)\Lambda_1 + (m_1 + m_2)\Lambda_2 + \ldots + (m_{2k-1} + m_{2k})\Lambda_{2k} + 2m_{2k+1}\Lambda_{2k+1} + \ldots + 2m_l \Lambda_l$. Hence $\rho * \overline{\rho}$ is self-conjugate by the criterion (10). Then analogously as in the proof of lemma 10, we have $J(V * \overline{V}) = V * \overline{V}$. Thus $\rho * \overline{\rho}$ is of index 1, Q.E.D.

Now let us prove (11). Let us express the highest Λ of the irreducible representation (ρ, V) as follows: $\Lambda = m_1 \Lambda_1 + \ldots + m_l \Lambda_l$. Then, we have

$$\rho = \overbrace{\rho_1 * \ldots * \rho_1}^{m_1\text{-times}} * \ldots * \overbrace{\rho_l * \ldots * \rho_l}^{m_l\text{-times}}.$$

Consequently, by lemmas 10, 11, we have (11). (Note that $\overline{\rho}_1 \sim \rho_2$, $\overline{\rho}_3 \sim \rho_4$, $\ldots$, $\overline{\rho}_{2k-1} \sim \rho_{2k}$). Thus we have the following

THEOREM 2. *Let $\mathfrak{g}$ be a real semi-simple Lie algebra and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Let $\alpha_1, \ldots, \alpha_l$ be any fundamental root system of $\mathfrak{g}^C$ with respect to the Catyan subalgebra $\mathfrak{h}^C$ of $\mathfrak{g}^C$. Let $\Lambda_1, \ldots, \Lambda_l$ be the fundamental weights of $\mathfrak{g}^C$ determined by $\alpha_1, \ldots, \alpha_l$. Let $\rho_1, \ldots, \rho_l$ be the irreducible representations of $\mathfrak{g}^C$ whose highest weights are $\Lambda_1, \ldots, \Lambda_l$ respectively. (The linear order between weights is determined by $\alpha_1, \ldots, \alpha_l$).*

(i) Then there is a permutation σ of $1, \ldots, l$ such that

$$\overline{\rho}_{\sigma(i)} \sim \rho_i \qquad (i = 1, \ldots, l),$$

and $\sigma^2 = 1$.

(ii) Let us arrange the order of $\alpha_1, \ldots, \alpha_l$ so that $\overline{\rho}_1 \sim \rho_2$, $\overline{\rho}_3 \sim \rho_4$, $\ldots$, $\overline{\rho}_{2k-1}$

$\sim\rho_{2k}$, $\bar{\rho}_{2k+1}\sim\rho_{2k+1}$, ..., $\bar{\rho}_l\sim\rho_l$ *in* (*i*). *Let* ε_{2k+j} *be the index of* ρ_{2k+j} ($j=1$, ..., $l-2k$). *Let* (ρ, V) *be an irreducible representation of* $\mathfrak{g}$ *with the highest weight*

$$\Lambda = m_1\Lambda_1 + m_2\Lambda_2 + \dots + m_l\Lambda_l,$$

then the highest weight of $\bar{\rho}$ *is given by*

$$m_2\Lambda_1 + m_1\Lambda_2 + m_4\Lambda_3 + m_3\Lambda_4 + \dots + m_{2k}\Lambda_{2k-1} + m_{2k-1}\Lambda_{2k} + m_{2k+1}\Lambda_{2k+1} + \dots + m_l\Lambda_l,$$

and ρ *is self-conjugate if and only if*

$$m_1 = m_2,\ m_3 = m_4,\ \dots,\ m_{2k-1} = m_{2k},$$

and then the index ε *of* ρ *is given by*

$$\varepsilon = \varepsilon_{2k+1}^{m_{2k+1}} \dots \varepsilon_l^{m_l}.$$

§ 11. A Criterion in Case (*b*)

As an application of Theorem 2, let us consider the case where $\mathfrak{g}$ is a real simple Lie algebra such that $\mathfrak{g}^C$ is not simple. In this case $\mathfrak{g}^C$ is a direct sum of two (complex) simple ideals[12]:

$$\mathfrak{g}^C = \mathfrak{a} + \bar{\mathfrak{a}}$$

where bar means the complex conjugate operation of $\mathfrak{g}^C$ with respect to $\mathfrak{g}$. Then the scalar restriction $\mathfrak{a}_R$ is isomorphic with $\mathfrak{g}$ under the mapping $x \to X + \bar{X}(X\in\mathfrak{a}_R)$. Let $\mathfrak{b}$ be any Cartan subalgebra of $\mathfrak{a}$. Then $\mathfrak{h} = \{X+\bar{X};\ X\in\mathfrak{b}\}$ is a Cartan subalgebra of $\mathfrak{g}$ as is seen easily. Further $\bar{\mathfrak{b}}$ is a Cartan subalgebra of $\bar{\mathfrak{a}}$, and we have $\mathfrak{h}^C = \mathfrak{b} + \bar{\mathfrak{b}}$. Let Δ_1 be the root system of $\mathfrak{a}$ with respect to $\mathfrak{b}$. Then every $\alpha\in\Delta_1$ is extended to a linear form on $\mathfrak{h}^C$ (which we also denote by α) putting $\alpha(X)=0$ for every $X\in\bar{\mathfrak{b}}$. Then α becomes a root of $\mathfrak{g}^C$. Thus we can regard that Δ_1 is a subset of the root system Δ of $\mathfrak{g}^C$ with respect to the Cartan subalgebra $\mathfrak{h}^C$. Then, $\bar{\Delta}_1$ is the root system of $\bar{\mathfrak{a}}$ with respect to $\bar{\mathfrak{b}}$. Let

$$\Pi_1 = \{\alpha_1, \dots, \alpha_k\},\ \bar{\Pi}_1 = \{\bar{\alpha}_1, \dots, \bar{\alpha}_k\},$$

be fundamental root systems of $\mathfrak{a}$, $\bar{\mathfrak{a}}$ respectively. Then

[12] This is seen analogously as in the formula (2).

$$\Pi = \{\alpha_1, \ldots, \alpha_k, \bar{\alpha}_1, \ldots, \bar{\alpha}_k\}.$$

is a fundamental root system of $\mathfrak{g}'$.
Now let $\{\Lambda_1, \ldots, \Lambda_k\}$ be the fundamental weight system of $\mathfrak{a}$ determined by $\{\alpha_1, \ldots, \alpha_k\}$. Then $\{\bar{\Lambda}_1, \ldots, \bar{\Lambda}_k\}$ is the fundamental weight system of $\bar{\mathfrak{a}}$ determined by $\{\bar{\alpha}_1, \ldots, \bar{\alpha}_k\}$. Then $\{\Lambda_1, \ldots, \Lambda_k, \bar{\Lambda}_1, \ldots, \bar{\Lambda}_k\}$ the fundamental weight system of $\mathfrak{g}^c$ determined by $\{\alpha_1, \ldots, \alpha_k, \bar{\alpha}_1, \ldots, \bar{\alpha}_k\}$. Now let $\rho_1, \ldots, \rho_k$ be the fundamental irreducible representations of $\mathfrak{a}$ with highest weights $\Lambda_1, \ldots, \Lambda_k$ respectively. Then $\bar{\rho}_1, \ldots, \bar{\rho}_k$ are the fundamental irreducible representations of $\bar{\mathfrak{a}}$ with highest weight $\bar{\Lambda}_1, \ldots, \bar{\Lambda}_k$ respectively. Let us regard $\rho_1, \ldots, \rho_k, \bar{\rho}_1, \ldots, \bar{\rho}_k$ as representation of $\mathfrak{g}^c$. Then $\rho_1, \ldots, \rho_k, \bar{\rho}_1, \ldots, \bar{\rho}_k$ are the fundamental irreducible representation of $\mathfrak{g}^c$ determined by $\alpha_1, \ldots, \alpha_k, \bar{\alpha}_1, \ldots, \bar{\alpha}_k$.

Now let (ρ, V) be an irreducible representation of $\mathfrak{g}$ with highest weight Λ. Put $\Lambda = \sum_{i=1}^{k} m_i \Lambda_i + \sum_{i=1}^{k} m_i' \bar{\Lambda}_i$. Then, by Theorem 2 we see that ρ is self-conjugate if ane only if $m_i = m_i'$ $(i = 1, \ldots, k)$. Moreover, if ρ is self-conjugate, then the index of ρ is necessarily equal to 1. Now let us extend the representation (ρ, V) to the representation of $\mathfrak{g}'$ (this representation is also denoted by (ρ, V)). Let σ_1, σ_2 be the induced irreducible representation of $\mathfrak{a}$, $\bar{\mathfrak{a}}$ respectively by ρ. Then the highest weight of σ_1, σ_2 are $\Sigma m_i \Lambda_i$, $\Sigma m_i' \bar{\Lambda}_i$ respectively. In fact, let $x \in V_\Lambda$, $x \neq 0$. Then we have $\rho(H)x = \Lambda(H)x$, $(H \in \mathfrak{h}')$. If we put $H = B_1 + \bar{B}_2$, $(B_1, B_2 \in \mathfrak{b})$, then

$$\rho(H)x = (\Sigma m_i \Lambda_i(B_1) + \Sigma m_i' \bar{\Lambda}_i(\bar{B}_2))x.$$

If $H \in \mathfrak{b}$, then $B_2 = 0$, and we have

$$\rho(H)x = (\Sigma m_i \Lambda_i(H))x$$

Thus $\Sigma m_i \Lambda_i$, $\Sigma m_i' \bar{\Lambda}_i$ are weights of σ_1, σ_2. If σ_1 has a weight Λ' higher than $\Sigma m_i \Lambda_i$, then $\Lambda' + \Sigma m_i' \bar{\Lambda}_i$ is a weight of ρ higher than Λ. This is a contradiction. Hence $\Sigma m_i \Lambda_i$, $\Sigma m_i' \bar{\Lambda}_i$ are highest weights.

Now $\mathfrak{a}_R \cong \bar{\mathfrak{a}}_R$ by the canonical isomorphism $X \to \bar{X}$ $(X \in \mathfrak{a}_R)$. If we identify $\mathfrak{a}_R$ and $\bar{\mathfrak{a}}_R$ under this isomorphism, then σ_1, σ_2 can be regarded as the representations of $\mathfrak{a}_R$. Then we have $m_i = m_i'$ $(i = 1, \ldots, k)$ if and only if $\bar{\sigma}_1 \sim \sigma_2$ as the representation of $\mathfrak{a}_R$. Thus we have the following

THEOREM 3. *Let* $\mathfrak{g}$ *be a simple Lie algebra over R such that* $\mathfrak{g}^C$ *is not simple. Let* $\mathfrak{g}^C=\mathfrak{a}+\bar{\mathfrak{a}}$ *be the decomposition of* $\mathfrak{g}^C$ *into simple ideals. Let* ρ *be an irreducible representation of* $\mathfrak{g}$, *and* σ_1, σ_2 *be the induced irreducible representation of* $\mathfrak{a}$, $\bar{\mathfrak{a}}$ *by the extension of* ρ *to* $\mathfrak{g}^C$. *If we identify* $\mathfrak{a}_R$, $\bar{\mathfrak{a}}_R$ *under the isomorphism* $X\to\overline{X}\,(X\in\mathfrak{a})$, *we can regard* σ_1, σ_2 *as representations of* $\mathfrak{a}_R$. *Then* $\rho\sim\bar{\rho}$ *if and only if* $\bar{\sigma}_1\sim\sigma_2$ *as representations of* $\mathfrak{a}_R$. *If* $\rho\sim\bar{\rho}$, *then the index of* ρ *is* 1.

§ 12. An application to self-contragradient representations

Let $\tilde{\mathfrak{g}}$ be a semi-simple Lie algebra over C. Let (ρ, V) be a representation of $\tilde{\mathfrak{g}}$. Let us denote by (ρ^*, V^*) the contragradient representation of (ρ, V), i.e. V^* is the dual vector space of V and ρ^* is given by $\rho^*(X)=-{}^t\rho(X)$ for any $X\in\tilde{\mathfrak{g}}$. (ρ, V) is called self-contragradient if $\rho\sim\rho^*$.

Now let $\mathfrak{g}$ be a compact real form of $\tilde{\mathfrak{g}}$. Let us denote by $\rho|\mathfrak{g}$ the restriction of a representation ρ to $\mathfrak{g}$. Then, since any continuous representation of a compact group is equivalent to a representation by unitary matrices, we have $(\rho|\mathfrak{g})^*\sim\overline{(\rho|\mathfrak{g})}$ for any representation ρ of $\mathfrak{g}$. Moreover, two representations of $\tilde{\mathfrak{g}}$ are equivalent if and only if their restrictions to $\mathfrak{g}$ are equivalent. Since we have $\rho^*|\mathfrak{g}\sim(\rho|\mathfrak{g})^*$, the problem of the self-contragradience of a representation ρ of $\tilde{\mathfrak{g}}$ is reduced to that of the self-conjugateness of $\rho|\mathfrak{g}$, i.e. we have $\rho\sim\rho^*$ if and only if $\overline{(\rho|\mathfrak{g})}\sim\rho|\mathfrak{g}$. Then we can apply theorem 2. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Let $\alpha_1, \ldots, \alpha_l$ be any fundamental root system of $\tilde{\mathfrak{g}}$ with respect to $\mathfrak{h}^C$, and $\Lambda_1, \ldots, \Lambda_l$ be the fundamental weight system determined by $\alpha_1, \ldots, \alpha_l$, and $\rho_1, \ldots, \rho_l$ be the irreducible representations of $\tilde{\mathfrak{g}}$ whose highest weights are $\Lambda_1, \ldots, \Lambda_l$ respectively.

Then, by theorem 2, there exists a involutive permutation σ of $1, \ldots, l$ such that $\rho^*_{\sigma(i)}\sim\rho_i$ $(i=1, \ldots, l)$.

Now, let ρ be an irreducible representation of $\tilde{\mathfrak{g}}$ with the highest weight $\Lambda=\sum_{i=1}^{l}m_i\Lambda_i$. Then the highest weight of ρ^* is given by $\sum_{i=1}^{l}m_{\sigma(i)}\Lambda_i$. Hence ρ is self-contragradient if and only if $m_i=m_{\sigma(i)}$ $(i=1, \ldots, l)$.

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