ON THE EXISTENCE OF UNRAMIFIED SEPARABLE INFINITE SOLVABLE EXTENSIONS OF FUNCTION FIELDS OVER FINITE FIELDS*

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In the present note, using the results in the previous paper,[1] we shall prove the following existence theorem:

THEOREM. *Let k be a finite field with q elements and K/k be a regular extension of dimension one over k. Then, if $q \geqq 11$ and the genus g_K of K/k is greater than one, there exists an unramified separable infinite solvable extension of K which is regular over k.*[2]

§ 1. The results in [1]

1.1. Let k be a finite field with q elements and K/k be a regular extension of dimension one over k. Let L/k be an unramified separable normal extension of K which is also regular over k. We denote by $G(L/K)$ the galois group of L/K. We denote by C_L and C_K non-singular complete models of K/k and L/k, respectively, and denote by $\hat{\pi}_{L/K}$ the trace mapping of C_L onto C_K. We denote by $J_L(J_K)$ and $\varphi_L(\varphi_K)$ the jacobian variety of $C_L(C_K)$ and a canonical mapping of $C_L(C_K)$ into $J_L(J_K)$, respectively, where we may assume that $J_L(J_K)$ and $\varphi_L(\varphi_K)$ are also defined over k. We denote by $\pi_{L/K}$ the extension of $\hat{\pi}_{L/K}$ which is a homomorphism of J_L onto J_K such that $\pi_{L/K} \circ \varphi_L = \varphi_K \circ \hat{\pi}_{L/K} + c$ with a constant point c. After a suitable translation of φ_K, we assume that

$$\pi_{L/K} \circ \varphi_L = \varphi_K \circ \hat{\pi}_{L/K}. \tag{1}$$

We denote by $A(\ \ , k)$ the subgroup of k-rational points of a commutative group variety A.

Each element ε_ν of $G(L/K)$ induces an automorphism $\{\eta_L(\varepsilon_\nu)\}$ of J_L and

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1) We shall refer this paper with [1].

2) We mean by an infinite solvable extension a solvable extension of infinite degree.

a translation $\{a_L(\varepsilon_\nu)\}$ on J_L, both defined over k, such that

$$\varphi_L(P^{\varepsilon_\nu^{-1}}) = \eta_L(\varepsilon_\nu)\varphi_L(P) + a_L(\varepsilon_\nu) \qquad (\varepsilon_\nu \in G(L/K),\ P \in C_L),$$

where $\eta_L(\varepsilon_\nu)$ and $a_L(\varepsilon_\nu)$ $(\varepsilon_\nu \in G(L/K))$ are defined over k and they are independent on the choice of P. From the definition we have

$$a_L(\varepsilon_\nu\varepsilon_\mu) = \eta_L(\varepsilon_\nu)a_L(\varepsilon_\mu) + a_L(\varepsilon_\nu). \tag{2}$$

If we put $\varphi'_L = \varphi_L + c$ and $\varphi'_L(P^{\varepsilon_\nu^{-1}}) = \eta_L(\varepsilon_\nu)\varphi'_L(P) + a'_L(\varepsilon_\nu)$, then we have

$$a'_L(\varepsilon_\nu) = a_L(\varepsilon_\nu) + (\delta_{J_L} - \eta_L(\varepsilon_\nu))c. \tag{3}$$

1. 2. Let x be a generic point of J_L over k. Let $A_{L/K}$ and $B_{L/K}$ be respectively the loci of

$$\Big(\sum_{\varepsilon_\nu\in G(L/K)} \eta_L(\varepsilon_\nu)\Big)x \quad \text{and} \quad \Big(\sum_{\varepsilon_\nu\in G(L/K)} (\delta_{J_L} - \eta_L(\varepsilon_\nu))\Big)x$$ [3]

over k. Let $\bar{A}_{L/K}$ be the quotient abelian variety of J_L by $B_{L/K}$ and $\alpha_{L/K}$ be the natural mapping of J_L onto $\bar{A}_{L/K}$. Then $A_{L/K}$, $B_{L/K}$, $\bar{A}_{L/K}$ and $\alpha_{L/K}$ are defined over k. $A_{L/K}$ and $B_{L/K}$ generate J_L and the intersection $A_{L/K} \cap B_{L/K}$ is a finite group. Moreover $B_{L/K}$ is the irreducible component of $\pi_{L/K}^{-1}(0)$. Let $\bar{\pi}_{L/K}$ be the homomorphism of $\bar{A}_{L/K}$ onto J_K such that

$$\pi_{L/K} = \bar{\pi}_{L/K}\alpha_{L/K}. \tag{4}$$

In [1] we have proved the following facts:

THEOREM A. *$\bar{\pi}_{L/K}$ is separable and $\bar{\pi}_{L/K}^{-1}(0) = \{\alpha_{L/K}a_L(\varepsilon_\nu) \mid \varepsilon_\nu \in G(L/K)\}$. If L/K is an unramified separable abelian extension, then the mapping $\varepsilon_\nu \to \alpha_{L/K}a_L(\varepsilon_\nu)$ is an isomorphism of $G(L/K)$ onto $\bar{\pi}_{L/K}^{-1}(0)$.*

THEOREM B. *If L/K is an unramified separable abelian extension, then*

$$J_K(\ ,k)/\pi_{L/K}(J_L(\ ,k)) \cong G(L/K).$$

THEOREM C. *Let $\mathfrak{g}$ be any subgroup of $J_K(\ ,k)$. Then there exists an unramified separable abelian extension $K(\mathfrak{g})$ of K such that* i) *$K(\mathfrak{g})$ is regular over k and* ii) $\pi_{K(\mathfrak{g})/K}J_{L(\mathfrak{g})}(\ ,k) = \mathfrak{g}$.

The field $K(\mathfrak{g})$ in theorem C is given as follows:

Let A be the quotient abelian variety of J_K by $\mathfrak{g}$ and μ be the natural homomorphism of J_K onto A. Let $\mathfrak{p}_{J_K}$ be the endomorphism of J_K which is

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[3] δ_J means the identity endomorphism of J.

induced by the automorphism $\xi \to \xi^q$ of the universal domain and λ be the homomorphism of A onto J_K such that $\lambda\mu = \delta_{J_K} - \mathfrak{p}_{J_K}$. Then λ and μ are separable homomorphisms defined over k. Let y be the point of A such that $k(\lambda y) = K$ and $\lambda y = \varphi_K(P)$ with a point P of C_K. Then the field $K(\mathfrak{g})$ is $k(y)$ and the galois automorphisms of $K(\mathfrak{g})/K$ are induced by the translations $y \to y + t$ $(t \in \lambda^{-1}(0))$.

§ 2. The proof of the theorem

2. 1. To prove the theorem, it is sufficient to prove the following two lemmas:

LEMMA 1. *Let L/K be an unramified separable normal extension which is also regular over k and $\mathfrak{g}$ be a subgroup of $J_L(\ ,k)$. Then $L(\mathfrak{g})$ is normal over K if and only if the following conditions are satisfied for every $\varepsilon_\nu \in G(L/K)$:*

i) $\eta_L(\varepsilon_\nu)(\mathfrak{g}) = \mathfrak{g}$,

ii) $a_L(\varepsilon_\nu) \in \mathfrak{g}$.

LEMMA 2. *Let L/K be an unramified separable normal extension which is also regular over k and l be a prime number. Let $\mathfrak{g}$ be a subgroup of $J_L(\ ,k)$ such that $L(\mathfrak{g})/K$ is normal and $[L(\mathfrak{g}) : L] = l$. Then if $q \geqq 11$ and the genus g_K of K/k is greater than one there exists a subgroup $\mathfrak{g}_1$ of $J_{L(\mathfrak{g})}(\ ,k)$ such that* i) $\mathfrak{g}_1 \neq J_{L(\mathfrak{g})}(\ ,k)$ *and* ii) $(L(\mathfrak{g}))(\mathfrak{g}_1)$ *is normal over K.*

2. 2. The proof of lemma 1.

First we assume that $L(\mathfrak{g})/K$ is normal and denote by $[\varepsilon_\nu]$ a representative of $\varepsilon_\nu \in G(L/K)$ in $G(L(\mathfrak{g})/K)$. Then we have i) $\eta_L(\varepsilon_\nu)\pi_{L(\mathfrak{g})/L} = \pi_{L(\mathfrak{g})/L}\eta_{L(\mathfrak{g})}([\varepsilon_\nu])$ and ii) $a_L(\varepsilon_\nu) = \pi_{L(\mathfrak{g})/L}a_{L(\mathfrak{g})}([\varepsilon_\nu])$ $(\varepsilon_\nu \in G(L/K))$. Hence, by virtue of theorem C, we have

$$\text{i) } a_L(\varepsilon_\nu) = \pi_{L(\mathfrak{g})/L}a_{L(\mathfrak{g})}([\varepsilon_\nu]) \text{ and ii) } \eta_L(\varepsilon_\nu)(\mathfrak{g}) = \eta_L(\varepsilon_\nu)(\pi_{L(\mathfrak{g})/L}J_{L(\mathfrak{g})}(\ ,k)) \\ = \pi_{L(\mathfrak{g})/L}\eta_{L(\mathfrak{g})}([\varepsilon_\nu])J_{L(\mathfrak{g})}(\ ,k) = \pi_{L(\mathfrak{g})/L}J_{L(\mathfrak{g})}(\ ,k) = \mathfrak{g} \quad (\varepsilon_\nu \in G(L/K)).$$

Conversely we assume that $\mathfrak{g}$ satisfies the conditions of the lemma. Let y be a point of $\bar{A}_{L(\mathfrak{g})/L}$ such that $k(\bar{\pi}_{L(\mathfrak{g})/L}y) = L$ and $\bar{\pi}_{L(\mathfrak{g})/L}\, y$ is a point of $\varphi_L(C_L)$. By virtue of theorem C, $\bar{A}_{L(\mathfrak{g})/L}$ is the quotient variety of J_L by $\mathfrak{g}$ and μ is the natural mapping of J_L onto $\bar{A}_{L(\mathfrak{g})/L}$, where μ is the homomorphism of J_L onto $\bar{A}_{L(\mathfrak{g})/L}$ such that $\pi_{L(\mathfrak{g})/L}\mu = \delta_{J_K} - \mathfrak{p}_{J_K}$. Namely $k(y) = L(\mathfrak{g})$. Since $a_L(\varepsilon_\nu) \in \mathfrak{g}$, there exist points $b([\varepsilon_\nu])$ in $\bar{A}_{L(\mathfrak{g})/L}(\ ,k)$ such that $\bar{\pi}_{L(\mathfrak{g})/L}b([\varepsilon_\nu]) = a_L(\varepsilon_\nu)$ $(\varepsilon_\nu \in G(L/K))$. From the condition i) there exist automorphisms $\eta([\varepsilon_\nu])$ of $\bar{A}_{L(\mathfrak{g})/L}$ such that $\bar{\pi}_{L(\mathfrak{g})/L}\eta([\varepsilon_\nu]) = \eta_L(\varepsilon_\nu)\bar{\pi}_{L(\mathfrak{g})/L}$ $(\varepsilon_\nu \in G(L/K))$. Let $\tilde{C}$ be the locus of y over k. Then $\eta([\varepsilon_\nu])y + b([\varepsilon_\nu]) + t$ $(\varepsilon_\nu \in G(L/K),\ t \in \bar{\pi}^{-1}_{L(\mathfrak{g})/L}(0))$ are also points on $\tilde{C}$ and they are conjugates of y over K. This proves that $L(\mathfrak{g})$ is normal over K.

2.3. The proof of lemma 2

We denote by ε the generator of $G(L(\mathfrak{g})/L)$ and denote by $[\varepsilon_\nu]$ a representative of ε_ν in $G(L(\mathfrak{g})/K)$. Since (ε) is normal in $G(L(\mathfrak{g})/K)$, there exists an integer s_ν such that $[\varepsilon_\nu]\varepsilon[\varepsilon_\nu]^{-1} = \varepsilon^{s_\nu}$ $(\varepsilon_\nu \in G(L/K))$.

Since $\sum_{\nu=1}^{l}(\delta_{J_{L(\mathfrak{g})}} - \eta_{L(\mathfrak{g})}(\varepsilon^\nu)) = \delta_{J_{L(\mathfrak{g})}} - \eta_{L(\mathfrak{g})}(\varepsilon))\ (\sum_{\nu=1}^{l}(\delta_{J_{L(\mathfrak{g})}} + \eta_{L(\mathfrak{g})}(\varepsilon) + \dots + \eta_{L(\mathfrak{g})}(\varepsilon^{\nu-1}))$, we observe that $B_{L(\mathfrak{g})/L} \subseteqq (\delta_{J_{L(\mathfrak{g})}} - \eta_{L(\mathfrak{g})}(\varepsilon))(J_{L(\mathfrak{g})})$. On the other hand $J_{L(\mathfrak{g})}$ is generated by $B_{L(\mathfrak{g})/L}$ and $A_{L(\mathfrak{g})/L}$ and $(\delta_{J_{L(\mathfrak{g})}} - \eta_{L(\mathfrak{g})}(\varepsilon))(A_{L(\mathfrak{g})/L}) = 0$, hence $B_{L(\mathfrak{g})/L} = (\delta_{J_{L(\mathfrak{g})}} - \eta_{L(\mathfrak{g})}(\varepsilon))(J_{L(\mathfrak{g})})$. Therefore, by virtue of (3), if we translate $\varphi_{L(\mathfrak{g})}$ suitably, we can assume that $a_{L(\mathfrak{g})}(\varepsilon)$ belongs to $A_{L(\mathfrak{g})/L}$. From (2) we have

$$\begin{aligned}
a_{L(\mathfrak{g})}(\varepsilon^{s_\nu}) &= a_{L(\mathfrak{g})}([\varepsilon_\nu]\varepsilon[\varepsilon_\nu]^{-1}) \\
&= \eta_{L(\mathfrak{g})}([\varepsilon_\nu])\eta_{L(\mathfrak{g})}(\varepsilon)a_{L(\mathfrak{g})}([\varepsilon_\nu]^{-1}) + \eta_{L(\mathfrak{g})}([\varepsilon_\nu])a_{L(\mathfrak{g})}(\varepsilon) \\
&\quad + a_{L(\mathfrak{g})}([\varepsilon_\nu]) \\
&= \eta_{L(\mathfrak{g})}([\varepsilon_\nu])(\eta_{L(\mathfrak{g})}(\varepsilon) - \delta_{J_{L(\mathfrak{g})}})a_{L(\mathfrak{g})}([\varepsilon_\nu]^{-1}) \\
&\quad + (\eta_{L(\mathfrak{g})}([\varepsilon_\nu])a_{L(\mathfrak{g})}([\varepsilon]^{-1}) + a_{L(\mathfrak{g})}([\varepsilon_\nu])) \\
&\quad + \eta_{L(\mathfrak{g})}([\varepsilon_\nu])a_{L(\mathfrak{g})}(\varepsilon) \\
&= \eta_{L(\mathfrak{g})}([\varepsilon_\nu])(\eta_{L(\mathfrak{g})}(\varepsilon) - \delta_{J_{L(\mathfrak{g})}})a_{L(\mathfrak{g})}([\varepsilon_\nu]^{-1}) \\
&\quad + \eta_{L(\mathfrak{g})}([\varepsilon_\nu])a_{L(\mathfrak{g})}(\varepsilon).
\end{aligned}$$

On the other hand, since $\eta_{L(\mathfrak{g})}([\varepsilon_\nu])(\delta_{J_{L(\mathfrak{g})}} + \eta_{L(\mathfrak{g})}(\varepsilon) + \dots + \eta_{L(\mathfrak{g})}(\varepsilon^{l-1}))\eta_{L(\mathfrak{g})}([\varepsilon_\nu]^{-1})$ $= \delta_{J_{L(\mathfrak{g})}} + \eta_{L(\mathfrak{g})}(\varepsilon) + \dots + \eta_{L(\mathfrak{g})}(\varepsilon^{l-1})$, we have $\eta_{L(\mathfrak{g})}([\varepsilon_\nu])(A_{L(\mathfrak{g})/L}) = A_{L(\mathfrak{g})/L}$. Therefore

$$(4) \qquad (\eta_{L(\mathfrak{g})}(\varepsilon) - \delta_{J_{L(\mathfrak{g})}})a_{L(\mathfrak{g})}([\varepsilon_\nu]^{-1}) \in A_{L(\mathfrak{g})/L} \qquad (\varepsilon_\nu \in G(L/K)).$$

We denote by $\eta_{L(\mathfrak{g})}(\varepsilon)_{B_{L(\mathfrak{g})/L}}$ the restriction of $\eta_{L(\mathfrak{g})}(\varepsilon)$ of $B_{L(\mathfrak{g})/L}$. Let r be any prime except p. Then r-addic representation $\{M_r(\eta_{L(\mathfrak{g})}(\varepsilon^\nu)_{B_{L(\mathfrak{g})/L}})\}$ of $\{\eta_{L(\mathfrak{g})}(\varepsilon^\nu)_{B_{L(\mathfrak{g})/L}}\}$ is equivalent to

$$\begin{pmatrix} \zeta^{\nu} E_{2(g_L-1)} & & & \\ & \zeta^{2\nu} E_{2(g_L-1)} & & \\ & & \ddots & \\ & & & \zeta^{(l-1)\nu} E_{2(g_L-1)} \end{pmatrix},$$

where g_L is the genus of L/k, ζ is an l-th root of unity and $E_{2(g_L-1)}$ is the identity matrix of degree $2(g_L-1)$. This shows that

$$(5) \qquad \det M_r(\delta_{B_{L(\mathfrak{g})/L}} - \eta_{L(\mathfrak{g})}(\varepsilon)_{B_{L(\mathfrak{g})/L}}) = (\prod_{\nu=1}^{l-1}(1-\zeta^{\nu}))^{2(g_L-1)} = l^{2(g_L-1)}.$$

We denote by $\mathfrak{h}$ the group of points t of $J_{L(\mathfrak{g})}$ such that

$$(\delta_{J_{L(\mathfrak{g})}} - \eta_{L(\mathfrak{g})}(\varepsilon))t \in A_{L(\mathfrak{g})/L}.$$

Then we have

$$\begin{aligned} I_1 = [\mathfrak{h} : A_{L(\mathfrak{g})/L}] &= [(\delta_{B_{L(\mathfrak{g})/L}} - \eta_{L(\mathfrak{g})}(\varepsilon)_{B_{L(\mathfrak{g})/L}})^{-1}(0) : \{0\}] \\ &\leqq \det M_r((\delta_{B_{L(\mathfrak{g})/L}} - \eta_{L(\mathfrak{g})}(\varepsilon)_{B_{L(\mathfrak{g})/L}})) \\ &= l^{2(g_L-1)}. \end{aligned}$$

On the other hand, by virtue of Riemann's conjecture of congrunce ζ-functions, the absolute values of characteristic roots of $M_r(\mathfrak{p}_{B_{L(\mathfrak{g})/L}})$[4] are all $\sqrt{q}$. Hence the absolute values of characteristic roots of $M_r(\delta_{B_{L(\mathfrak{g})/L}} - \mathfrak{p}_{B_{L(\mathfrak{g})/L}})$ are not less than $\sqrt{q-2\sqrt{q}+1}$. This shows that

$$|\det M_r(\delta_{B_{L(\mathfrak{g})/L}} - \mathfrak{p}_{B_{L(\mathfrak{g})/L}})| \geqq (q-2\sqrt{q}+1)^{(l-1)(g_L-1)}.$$

Since $J_{L(\mathfrak{g})}(\ ,k) = (\delta_{J_{L(\mathfrak{g})}} - \mathfrak{p}_{J_{L(\mathfrak{g})}})^{-1}(0)$, $A_{L(\mathfrak{g})/L}(\ ,k) = (\delta_{A_{L(\mathfrak{g})/L}} - \mathfrak{p}_{A_{L(\mathfrak{g})/L}})^{-1}(0)$, $B_{L(\mathfrak{g})/L}(\ ,k) = \delta_{B_{L(\mathfrak{g})/L}} - \mathfrak{p}_{B_{L(\mathfrak{g})/L}})^{-1}(0)$ and $\delta_* - \mathfrak{p}_*$ are separable, we have

$$\begin{aligned} I_2 = [J_{L(\mathfrak{g})}(\ ,k) : A_{L(\mathfrak{g})/K}(\ ,k)] &= \det M_r(\delta_{J_{L(\mathfrak{g})}} - \mathfrak{p}_{J_{L(\mathfrak{g})}}) / \det M_r(\delta_{A_{L(\mathfrak{g})/L}} - \mathfrak{p}_{A_{L(\mathfrak{g})/L}}) \\ &= \det M_r(\delta_{B_{L(\mathfrak{g})/L}} - \mathfrak{p}_{B_{L(\mathfrak{g})/L}}) \geqq (q+1-2\sqrt{q})(g_L-1)(l-1). \end{aligned}$$

From $q \geqq 11$ we have $(q+1-2\sqrt{q}) > 5$. On the other hand $\log_{10} 5 > \frac{2}{3}$, hence $(l-1)\log_{10} 5 > \frac{2}{3}(l-1) > 2\log_{10} l$ for $l>1$. This shows that $(q+1-2\sqrt{q})^{(l-1)} > l^2$. By virtue of $g_L > g_K > 1$, $I_2 \geqq (q+1-2\sqrt{q})^{(l-1)(g_L-1)} > l^{2(g_L-1)} \geqq I_1$ for $l>1$. This proves that $\mathfrak{g}_1 = \mathfrak{h} \cap J_{L(\mathfrak{g})}(\ ,k)$ is a proper subgroup of $J_{L(\mathfrak{g})}(\ ,k)$. From (4) all $a_{L(\mathfrak{g})}(\varepsilon')$ $(\varepsilon' \in G(L(\mathfrak{g})/K))$ belong to $\mathfrak{g}_1$. Hence, by

[4] $\mathfrak{p}_{B_{L(\mathfrak{g})/L}}$ means the endomorphism of $B_{L(\mathfrak{g})/L}$ induced by the automorphism $\xi \to \xi^q$ of the universal domain.

virtue of lemma 1, it is sufficient to prove $\mathfrak{g}_1 = \eta_{L(\mathfrak{g})}([\varepsilon_\nu]\varepsilon^{\mu})(\mathfrak{g}_1)$ $(\varepsilon_\nu \in G(L/K))$. Since ζ is a primitive l-th root of unity, $1+\zeta+\ldots+\zeta^{\nu-1}$ $(\nu=1, 2, \ldots, l-1)$ are units in $Q(\zeta)$, where Q means the field of rational numbers. This shows that

$$(\delta_{B_{L(\mathfrak{g})/L}} + \eta_{L(\mathfrak{g})}(\varepsilon)_{B_{L(\mathfrak{g})/L}} + \ldots + \eta_{L(\mathfrak{g})}(\varepsilon^{\nu-1})_{B_{L(\mathfrak{g})/L}}) \qquad (\nu=1, 2, \ldots, l-1)$$

are automorphisms of $B_{L(\mathfrak{g})/L}$. On the other hand $\mathfrak{h}$ is generated by $A_{L(\mathfrak{g})/L}$ and $\mathfrak{h}_{B_{L(\mathfrak{g})/L}} = \{t \mid t \in B_{L(\mathfrak{g})/L},\ (\delta_{B_{L(\mathfrak{g})/L}} - \eta_{L(\mathfrak{g})}(\varepsilon))t \in A_{L(\mathfrak{g})/L}\}$. Moreover we observe that $\eta_{L(\mathfrak{g})}([\varepsilon_\nu])(\delta_{B_{L(\mathfrak{g})/L}} - \eta_{L(\mathfrak{g})}(\varepsilon)_{B_{L(\mathfrak{g})/L}}) = (\delta_{B_{L(\mathfrak{g})/L}} - \eta_{L(\mathfrak{g})}(\varepsilon^{s_\nu})_{B_{L(\mathfrak{g})/L}})\eta_{L(\mathfrak{g})}([\varepsilon_\nu])$ and $\eta_{L(\mathfrak{g})}([\varepsilon_\nu]^{-1})B_{L(\mathfrak{g})/L} = B_{L(\mathfrak{g})/L}$.

This shows that $\eta_{L(\mathfrak{g})}([\varepsilon_\nu])(\mathfrak{h}_{B_{L(\mathfrak{g})/L}}) = \mathfrak{h}_{B_{L(\mathfrak{g})/L}}$, namely $\mathfrak{h} = \eta_{L(\mathfrak{g})}([\varepsilon_\nu])(\mathfrak{h})$. Hence $\eta_{L(\mathfrak{g})}([\varepsilon_\nu])(\mathfrak{g}_1) = \eta_{L(\mathfrak{g})}([\varepsilon_\nu])\ (\mathfrak{h} \cap J_{L(\mathfrak{g})}(\ ,k)) = \mathfrak{h} \cap J_{L(\mathfrak{g})}(\ ,k) = \mathfrak{g}_1$.

Reference

[1] H. Morikawa, Generalized jacobian varieties and separable abelian extension of function fields, Nagoya Journal, **12** (1957), pp. 231–254.

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