

ON AUTOMORPHISMS OF A KÄHLERIAN STRUCTURE

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Is every isometry, or more generally, every affine transformation of a Kählerian manifold a complex analytic transformation? The answer is certainly negative in the case of a complex Euclidean space. This question has been recently studied by Lichnerowicz [8] and Schouten-Yano [11] from the infinitesimal point of view; they have found some conditions in order that every infinitesimal motion of a Kählerian manifold preserve the complex structure. (As a matter of fact, [11] has dealt with the case of a pseudo-Kählerian manifold, which does not differ essentially from a Kählerian manifold as far as the question at hand is concerned.)

In the present paper, we generalize their results by a different approach. In order to explain our main idea, we shall first give a few definitions (1 and 2) and state our main results (3). The proofs are given in the subsequent sections.

1. Kählerian structures

Let M be a complex analytic manifold of complex dimension n . Its complex structure is defined by a real analytic tensor field I of type $(1, 1)$ with $I^2 = -\mathbf{1}$ ¹⁾ on the underlying $2n$ -dimensional real analytic manifold which satisfies the condition of integrability $I[X, Y] - [IX, Y] - [X, IY] - I[IX, IY] = 0$ for all real vector fields X and Y (for example, [1]). A differentiable transformation f of M is said to preserve the complex structure I if $\delta f \circ I = I \circ \delta f$, where δf denotes the differential of f . This is equivalent to saying that f is a complex analytic transformation. If $\delta f \circ I = -I \circ \delta f$, we say that f maps I into the conjugate complex structure $-I$; f is then a conjugate analytic transformation.

A real analytic Riemannian metric g on a complex analytic manifold M is called Kählerian if it is hermitian, that is, $g(IX, IY) = g(X, Y)$ for all real vector fields X and Y , and if I is a parallel tensor field with respect to the

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¹⁾ Throughout the present note, $\mathbf{1}$ denotes the identity transformation.

Riemannian connection arising from g . Likewise, given a real analytic manifold M with Riemannian metric g , we shall say that a complex structure I on M is Kählerian if g is Kählerian with respect to I in the above sense. Such a pair (I, g) defines a Kählerian structure on M . By an isometry (resp. affine transformation) of a Kählerian manifold M , we understand of course an isometry (resp. affine transformation) of the underlying Riemannian manifold. By an automorphism of M , we mean an isometry which preserves the complex structure.

A Kählerian manifold M (with complex structure I and Riemannian metric g) will be called *non-degenerate* if the restricted homogeneous holonomy group σ_x of the underlying Riemannian manifold contains the endomorphism I_x of the tangent space T_x at $x \in M$, where x is an arbitrary reference point for the holonomy group and I_x is the value at x of the tensor field I . Note that the condition $I_x \in \sigma_x$ is independent of the choice of a reference point x . For any point y , let τ denote the parallel displacement: $T_x \rightarrow T_y$ along an arbitrary curve from x to y . Since I is a parallel tensor field, we have $I_y = \tau \cdot I_x \cdot \tau^{-1}$. On the other hand, we have $\sigma_y = \tau \cdot \sigma_x \cdot \tau^{-1}$.

Finally, we shall say that a Riemannian manifold M of dimension > 1 is irreducible if the restricted homogeneous holonomy group is irreducible, that is, if it does not admit any non-trivial invariant subspace as a group of linear transformations on the (real) tangent vector space.

2. Complex and quaternionian structures on a real vector space

In this section, we shall indicate an intrinsic way of defining real representations of $GL(n, C)$, $U(n)$, $SU(n)$ or $GL(n, Q)$, $Sp(l)$, etc. such as described in C. Chevalley: *Theory of Lie Groups I*, Chapter I. The exposition is quite elementary but important for our purpose.

A complex structure I on a real m -dimensional vector space T is, by definition, an endomorphism of T such that $I^2 = -\mathbf{1}$. It allows us to define the set T as an n -dimensional vector space over the field of complex numbers C , where $m = 2n$. More precisely, we define

$$(a + bi)X = aX + bIX$$

for $a, b \in R$ (field of real numbers) and $X \in T$. We denote by T^* the vector space over C thus obtained.

If g is a positive definite inner product on T such that $g(IX, IY) = g(X, Y)$

for all $X, Y \in T$, then we may define a positive definite hermitian inner product g^* on T^* by

$$g^*(X, Y) = g(X, Y) - ig(IX, Y)$$

for $X, Y \in T^*$, where X and Y are considered as elements of T on the right hand side of the above equation.

If τ is an endomorphism of T which commutes with I , then it may be considered as an endomorphism τ^* of T^* , as is clear from $\tau(iX) = \tau(IX) = I(\tau X) = i\tau X$. If furthermore τ leaves g invariant, then τ^* leaves g^* invariant.

We shall say that a group of linear transformations of a real vector space T of dimension $m = 2n$ is *contained in a real representation of $U(n)$* if T admits a complex structure I which commutes with every $\tau \in G$ and a positive definite inner product g which is invariant by I and every $\tau \in G$. In this case, G is isomorphic with the subgroup $G^* = \{\tau^*; \tau \in G\}$ (in the above notation) of the unitary group on T^* with respect to g^* . If furthermore $\det \tau^* = 1$ for every $\tau \in G$, then we say that G is *contained in a real representation of $SU(n)$* .

By a *quaternionian structure* on a complex vector space T^* of dimension n , we shall mean a conjugate linear transformation J of V with $J^2 = -1$, that is, a $1-1$ map of T^* onto itself such that

$$J(X + Y) = JX + JY, \quad J(aX) = \bar{a}JX \quad \text{and} \quad J^2X = -X$$

for all $X, Y \in T^*$ and $a \in \mathbb{C}$, where \bar{a} denotes the complex conjugate of a . It allows us to consider T^* as a vector space \tilde{T} over the field of quaternions \mathbb{Q} in the following fashion. We represent every quaternion q in the form $q = a + bj$ (j being an element of \mathbb{Q} such that $j^2 = -1$ and $ij = -ji$) and define the scalar multiplication by

$$q \cdot X = aX + bJX$$

for all $X \in T^*$. As a vector space over \mathbb{Q} , \tilde{T} is of dimension l where $n = 2l$.

If g^* is a positive definite hermitian inner product on T^* such that $g^*(JX, JY) = g^*(Y, X)(= \overline{g^*(X, Y)})$, then we can define a positive definite symplectic inner product \tilde{g} in the vector space \tilde{T} over \mathbb{Q} in the following fashion:

$$\tilde{g}(X, Y) = g^*(X, Y) + g^*(X, JY)j$$

for all $X, Y \in \tilde{T}$. Namely, \tilde{g} is \mathbb{Q} -valued and satisfies the following conditions:

- 1) $\tilde{g}(Y, X)$ is the symplectic conjugate of $\tilde{g}(X, Y)$;
- 2) $\tilde{g}(X+X', Y) = \tilde{g}(X, Y) + \tilde{g}(X', Y)$;
- 3) $\tilde{g}(X, X) \geq 0$ for every X and it is 0 if and only if $X=0$.

If τ^* is an endomorphism of T^* over C which commutes with J , then it may be regarded as an endomorphism $\tilde{\tau}$ of \tilde{T} over Q . If furthermore τ^* leaves g^* invariant, then $\tilde{\tau}$ leaves \tilde{g} invariant.

We say that a group G^* of linear transformations of a complex vector space T^* of dimension $n=2l$ is *contained in a complex representation* of $S_p(l)$ if T^* admits a quaternionian structure J and a positive definite hermitian inner product g^* such that

$$g^*(JX, JY) = g^*(Y, X), \quad \tau^* \cdot J = J \cdot \tau^* \quad \text{and} \quad g^*(\tau^*X, \tau^*Y) = g^*(X, Y)$$

for all $X, Y \in T^*$ and $\tau^* \in G^*$. In this case, G^* is isomorphic with a subgroup of $S_p(l)$ on the vector space \tilde{T} over Q with respect to the symplectic inner product \tilde{g} .

Finally, we define a *quaternionian structure on a real vector space* T . It is a pair of endomorphisms I and J of T such that $I^2 = J^2 = -1$ and $IJ = -JI$. Such a structure makes it possible to regard T as a vector space \tilde{T} over Q , the scalar multiplication being defined by

$$(a + bi + cj + dk)X = aX + bIX + cJX + d(IJ)X$$

for $a, b, c, d \in R$ and $X \in T$. Another way of seeing this is to consider, first, T with a complex structure I as a vector space T^* over C and then consider the given endomorphism J as a quaternionian structure on T^* , which is obviously possible.

This being said, we are able to use the following expression. We say that a group of linear transformations G on a real m -dimensional vector space T is *contained in a real representation* of $S_p(l)$, with $m=4l$, if T admits a quaternionian structure (I, J) and a positive definite inner product g which are both invariant by I, J and every element of G . It is now easy to see that, in this case, G is isomorphic with a subgroup of $S_p(l)$ on the l -dimensional vector space \tilde{T} over Q with a suitable symplectic inner product.

By using the fact that $S_p(l)$ is connected, we can prove that if G is contained in a real representation of $S_p(l)$, then it is contained in a real representation of $SU(n)$, where $m=2n$ and $n=2l$. We omit the detail of the proof.

3. Main results

THEOREM 1. *Every simply connected and complete Kählerian manifold M is a direct product $M_0 \times M_1 \times \dots \times M_k$, where M_0 is a complex Euclidean space of dimension ≥ 0 and M_1, \dots, M_k are irreducible Kählerian manifolds. If M is non-degenerate, M_0 does not appear and M_1, \dots, M_k are all non-degenerate.*

THEOREM 2.²⁾ *Let M be an irreducible Kählerian manifold whose restricted homogeneous holonomy group is not contained in a real representation of $S_p(1)$, where $\dim M = 4l$. Then every affine transformation of M preserves the complex structure I or maps I into the conjugate complex structure. The largest connected group of affine transformations $A^0(M)$ preserves the complex structure.*

THEOREM 3. *If M is a complete non-degenerate Kählerian manifold, then the largest connected group of affine transformations $A^0(M)$ consists of automorphisms.*

If M is a pseudo-Kählerian manifold, we can still define the notion of non-degeneracy. If we replace "Kählerian" by "pseudo-Kählerian" and "complex structure" by "almost complex structure" respectively, then all the results stated above remain true.

It is of some interest to compare our problem with the following: is every affine transformation of a Riemannian manifold an isometry? This question has been settled by Hano and one of the authors as follows. Every simply connected and complete Riemannian manifold M is a direct product of a Euclidean space M_0 and irreducible Riemannian manifolds M_1, \dots, M_k (the so-called de Rham decomposition) [10]. The largest connected group of affine transformations $A^0(M)$ is naturally isomorphic with $A^0(M_0) \times A^0(M_1) \times \dots \times A^0(M_k)$ [2]. On the other hand, every affine transformation of a complete irreducible Riemannian manifold is an isometry [5]. It follows that, *if M is a complete Riemannian manifold whose restricted homogeneous holonomy group does not leave any non-zero vector invariant, then $A^0(M)$ consists of isometries.*

Our Theorem 1 corresponds to the de Rham decomposition of a Riemannian manifold. By using the above result of Hano, our problem is reduced to the case of an irreducible Kählerian manifold, to which Theorem 2 is an answer. Here we do not need the condition of completeness but require an assumption

²⁾ A similar result has been obtained also by M. Obata.

on the holonomy group. Now, what is a condition which assures that every component of the de Rham decomposition of a given Kählerian manifold satisfies the assumption of Theorem 2? The non-degeneracy is such a condition. The following theorem shows the relationship of this notion to the known facts on Ricci curvature, thus giving a heuristic interpretation of the results of Lichnerowicz [7], [8].

THEOREM 4. *Let M be a Kählerian manifold of complex dimension n .*

- 1) *If M is irreducible and the Ricci curvature is not zero, then M is non-degenerate.*
- 2) *If M is non-degenerate and n is not divisible by 4, then the Ricci curvature of M is not zero.*
- 3) *If the Ricci curvature of M is non-singular at some point of M , then M is non-degenerate.*

Finally we add

COROLLARY. *Let M be a $2n$ -dimensional simply connected real analytic Riemannian manifold which is irreducible. Then the following three cases are possible:*

- 1) *If the restricted homogeneous holonomy group σ is not contained in a real representation of $U(n)$, there exists no Kählerian structure at all on M .*
- 2) *If σ is contained in a real representation of $U(n)$ but not of $S_p(1)$, $n = 2l$, then there exist exactly two Kählerian structures on M which are mutually conjugate.*
- 3) *If σ is contained in a real representation of $S_p(1)$, $n = 2l$, then there exist continuously many distinct Kählerian structures on M .*

4. Proof of Theorem 1

The underlying Riemannian manifold of M admits the de Rham decomposition $M = M_0 \times M_1 \times \dots \times M_k$. It is not difficult (see [3]) to see that every component M_i has a Kählerian structure induced from that of M and that M is the direct product of M_0, M_1, \dots, M_k as Kählerian manifolds. The homogeneous holonomy group $\sigma(M)$ of M is decomposed into the direct product of the homogeneous holonomy groups $\sigma(M_i)$ of M_i , $i = 0, 1, \dots, k$, where $\sigma(M_0)$ consists of the identity only. It follows that if M is non-degenerate, the Euclidean

part M_0 does not exist and the irreducible components $M_i, i = 1, 2, \dots, k$, are all non-degenerate.

5. Proof of Theorem 2

Let f be an affine transformation of M and δf its differential. Then $I^f = \delta f^{-1} \cdot I \cdot \delta f$ is a tensor field of type $(1, 1)$ which clearly satisfies the condition $I^f \cdot I^f = -1$. We show that it is a parallel tensor field. Let c be an arbitrary curve from x to y and let τ be the linear mapping of T_x onto T_y defined by parallel displacement along c . Let c^* be the image curve of c by f and let τ^* be the linear mapping of $T_{f(x)}$ onto $T_{f(y)}$ defined by parallel displacement along c^* . Since f is an affine transformation, we have $\delta f \cdot \tau = \tau^* \cdot \delta f$ on T_x [9]. On the other hand, we have $I_{f(y)} \cdot \tau^* = \tau^* \cdot I_{f(x)}$ since I is a parallel tensor field. Therefore we get

$$\begin{aligned} I_y^f \cdot \tau &= \delta f^{-1} \cdot I_{f(y)} \cdot \delta f \cdot \tau = \delta f^{-1} \cdot I_{f(y)} \cdot \tau^* \cdot \delta f \\ &= \delta f^{-1} \cdot \tau^* \cdot I_{f(x)} \cdot \delta f = \tau \cdot \delta f^{-1} \cdot I_{f(x)} \cdot \delta f = \tau \cdot I_x^f, \end{aligned}$$

which proves our assertion. In particular, I_x^f commutes with every element of σ_x .

Now let A be the algebra (over the field of real numbers R) formed by all endomorphisms of T_x which commute with every element of σ_x . Since σ_x is irreducible, every non-zero element of A has an inverse, that is, A is a division algebra. By a well known theorem in algebra, A is isomorphic either with the field of real numbers R , or the field of complex numbers C , or else the field of quaternions Q . Since A contains an element I_x with $I_x^2 = -1$, it cannot be isomorphic with R . If A were isomorphic with Q , it would follow that σ_x is contained in a real representation of $S_p(l)$, with $n = 2l$; indeed, again by the irreducibility of σ_x , we see that the inner product g_x of T_x induced from the Kählerian metric of M is invariant by the elements I and J of A which correspond to the units i and j of Q . Hence A is isomorphic with C . The only complex numbers whose square are -1 are i and $-i$. Since I_x^f is in A and $I_x^f \cdot I_x^f = -1$, we have either $I_x^f = I_x$ or $I_x^f = -I_x$. Since I and I^f are parallel tensor fields, we have $I^f = I$ or $I^f = -I$. This concludes the proof of the first part of Theorem 2.

In order to prove the second part, let f be an element of $A^0(M)$. We take a continuous 1-parameter family f_t of affine transformations such that $f_0 = \text{identi-}$

ty transformation and $f_1 = f$. If we form $I^t = \delta f_t^{-1} \cdot I \cdot \delta f_t$, then we have $I_x^t = I_x$ or $-I_x$ from what we have seen. Since I_x^t is a continuous 1-parameter family of endomorphisms of T_x such that $I_x^t = I_x$ for $t = 0$, I_x^t must coincide with I_x for every t . In particular, we have $I_x^f = I_x$, that is, $I^f = I$. This proves that f preserves the complex structure I .

6. Proof of Theorem 3

Let \tilde{M} be the universal covering manifold of M provided with a naturally induced Kählerian structure. It is easy to see that \tilde{M} is also complete and non-degenerate. By the argument in [2], it is sufficient to prove Theorem 3 for \tilde{M} . By Theorem 1, $\tilde{M} = M_1 \times \dots \times M_k$ where each M_i is irreducible and non-degenerate. The homogeneous holonomy group $\sigma(M_i)$ is not contained in a real representation of $S_p(l)$, $n = 2l$. In fact, we show that the division algebra A considered in the proof of Theorem 1 cannot be isomorphic with \mathbb{Q} . If it were so, the elements I, J and K of A corresponding to i, j and $k \in \mathbb{Q}$ must commute with the element $I_0, I_0^2 = -1$, of A determined by the given complex structure of M_i , which is contained in A since M_i is non-degenerate. This is a contradiction. Hence Theorem 2 shows that $A^0(M_i)$ preserves the complex structure of M_i . Since $A^0(\tilde{M})$ is the direct product of $A^0(M_i)$, $i = 1, 2, \dots, k$, we see that $A^0(\tilde{M})$ preserves the complex structure of \tilde{M} . On the other hand, we already know ([2] and [5]) that $A^0(\tilde{M})$ consists of isometries. Hence $A^0(\tilde{M})$ consists of automorphisms of \tilde{M} .

7. Proof of Theorem 4

1) The complex structure I_x of the tangent space T_x is invariant by the restricted homogeneous holonomy group σ_x operating on T_x , we may consider σ_x as a subgroup of $U(n)$ as indicated in 2. Since the Ricci curvature is not zero, σ_x is not a subgroup of $SU(n)$ ([7], see also [4] and [6]) which means that σ_x has a non-discrete center. σ_x being irreducible, the center must be of dimension 1 by Schur's lemma. Hence σ_x contains the 1-parameter subgroup $\{e^{2\pi i r} \cdot \mathbf{1}; r \text{ reals}\}$ of $U(n)$, in particular, the transformation $i \cdot \mathbf{1}$. In real representation, this means that σ_x contains the endomorphism I_x , that is, M is non-degenerate.

2) If we consider σ_x as a subgroup of $U(n)$, then σ_x contains the transformation $i \cdot \mathbf{1}$ whose determinant, the n -th power of i , is not equal to 1 since n is

not divisible by 4. Hence σ_x is not a subgroup of $SU(n)$ and the Ricci curvature is not zero.

3) Let x be a point of M where the Ricci curvature is non-singular. Let U be a properly chosen neighborhood of x which is isometric to the direct product $U_0 \times U_1 \times \dots \times U_k$, where U_0 is locally a complex Euclidean space and U_1, \dots, U_k are irreducible Kählerian manifolds (the argument is similar to that in the proof of Theorem 1). We may consider U_0, U_1, \dots, U_k as submanifolds of U passing through x . Then the Ricci curvature of U at x is the direct sum of the Ricci curvatures of U_0, U_1, \dots, U_k . Therefore the Ricci curvature of each U_i is non-singular at x . It follows that there does not exist U_0 and that U_1, \dots, U_k are non-degenerate by 1) of Theorem 4. Since $\sigma_x(M)$ contains $\sigma_x(U) = \sigma_x(U_1) \times \dots \times \sigma_x(U_k)$, we see that $\sigma_x(M)$ contains the endomorphism I_x .

8. Proof of Corollary

At any reference point x , we consider the division algebra A formed by all endomorphisms of T_x commuting with every element of σ_x . If A is isomorphic with R , then there is no element $I \in A$ with $I^2 = -1$; there is no Kählerian structure on M . If A is isomorphic with C , then let I_x be an element of A with $I_x^2 = -1$. By parallel displacement of I_x , we get a parallel tensor field I of type $(1, 1)$ such that $I^2 = -1$ and $g(IX, IY) = g(X, Y)$. It is known that an almost complex structure which is parallel with respect to a Riemannian connection (or, more generally, an affine connection without torsion) is integrable [1]. Hence I is a Kählerian structure. It is clear that I and its complex conjugate structure $-I$ are the only Kählerian structures on M .

If A is isomorphic with Q , then it contains continuously many elements S with $S^2 = -1$. Namely, if I, J and K are the elements of A which correspond to i, j and k of Q respectively, then we may take $S = bI + cJ + dK$, where b, c and d are real numbers such that $b^2 + c^2 + d^2 = 1$. For any such element S of A , we get a Kählerian structure on M by the same argument as before. Hence M has continuously many distinct Kählerian structures.

9. Remarks

In the case of a compact Kählerian manifold M , the largest connected group of affine transformations $A^0(M)$ consists of isometries (theorem of Yano,

which has been generalized in [7]) and preserves the complex structure of M , as is remarked in [8]. Indeed, the form F associated to the Kählerian structure of $M : F(X, Y) = g(IX, Y)$ is harmonic and invariant by every 1-parameter group of isometries. It follows that I is also invariant by the 1-parameter group.

The above statement is no longer true for the total group $A(M)$ of affine transformations. For example, in a complex projective space P_n with usual Fubini-Study metric, the transformation defined by $(z^0, z^1, \dots, z^n) \rightarrow (\bar{z}^0, \bar{z}^1, \dots, \bar{z}^n)$ in terms of homogeneous coordinates z^0, \dots, z^n is isometric but not complex analytic.

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