CORRECTIONS TO MY PAPER "ON KRULL'S CONJECTURE CONCERNING VALUATION RINGS"

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The proof of Theorem 1 in the paper "On Krull's conjecture concerning valuation rings" (vol. 4 (1952) of this journal) is not correct.[1] We want to give here a corrected proof of the theorem: From p. 30, *l.* 14 to p. 31, *l.* 7 should be changed as follows.

Further we observe that if $w(a-b)>2\alpha$, then $(x+a)/(x+b)$ is unit in $\mathfrak{o}$. Hence we may assume that $w(a_i-b_j)<2\alpha$ for any (i, j).

Next, we will show two lemmas concerning the valuations w_λ and w_e:

LEMMA A. *Set* $d=\Pi_1^{n'}(x+a_i)/\Pi_1^{m'}(x+b_j)$ *and assume that* $w(a_i)=w(b_j)=\sigma$ $(a<\sigma<2\alpha)$ *for any* i *and* j. *Let* e *be any element of* K *such that* $w(e)=\sigma$. *Then either* $w_e(d)\geqq w_\sigma(d)$ *or there exists one* b_j *such that* $w_e(d)\geqq w_{b_j}(d)$.

Proof. We may use the induction argument on $m'+n'$. Obviously $w_e(x+a_i)=\min(w(a_i-e), 2\alpha)$, $w_e(x+b_j)=\min(w(b_j-e), 2\alpha)$: Let σ' be the maximum of these values. We renumber a_i and b_j so that $w_e(x+a_i)=w_e(x+b_j)=\sigma'$ if and only if $i\leqq r$, $j\leqq s$. Now it must be observed that $w_e(x+a_i)=w(a_j-a_1)$ or $w(a_i-b_1)$ for $i>r$, according to $r\neq 0$ or $s\neq 0$, and that similar fact holds for b_j.

1) When $r=n'$, $s=m'$ and $r\geqq s$, we have obviously $w_e(d)\geqq w_\sigma(d)$.

2) When $r<s$ and $r+s\neq m'+n'$: Set $d'=\Pi_1^r(x+a_i)/\Pi_1^s(x+b_j)$. Then $w_e(d')>w_\sigma(d')$ and therefore there exists on b_j $(j\leqq s)$ such that $w_e(d')\geqq w_{b_j}(d')$. Since the values of factors of d other than those of d' are invariant under the replacement of w_e by w_{b_j}, we have $w_e(d)\geqq w_{b_j}(d)$.

3) When $r=n'$, $s=m'$ and $r<s$: Let $\sigma*$ be the minimum of values $w(a_i-a_{i'})$, $w(a_i-b_j)$ and $w(b_j-b_{j'})$ and let $e*$ be an element of K such that $w(a_i-e*)=w(b_j-e*)=\sigma*$ for any i, j.[2] Then since $w_e(d)\geqq w_{e*}(d)$, we

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1) Prof. P. Ribenboim has communicated to the writer that the proof is not correct. The writer is grateful to him for his kind communication.

2) Such elements e^*, e'' and so on exist because K is algebraically closed and therefore the residue class field of the valuation ring of w is algebraically closed (and contains infinitely many elements).

may replace e by e^*. Next, let $\sigma^{**} > \sigma^*$ be the next smallest value among $w(a_i - a_{i'})$, $w(a_i - b_j)$ and $w(b_j - b_{j'})$ if they are not all equal; otherwise, we have obviously $w_{b_j}(d) \leqq w_e(d)$ for any b_j and we have nothing to prove in this case.[3] We separate a_i's and b_j's to equivalent classes modulo the ideal of the valuation ring $\mathfrak{v}$ of w generated by an element e^{**} of K such that $w(e^{**}) = \sigma^{**}$. Since $r < s$, there exists a class $C = \{a_{i_1}, \ldots, a_{i_t}, b_{j_1}, \ldots, b_{j_u}\}$ such that $t < u$. Let e'' be an element of K such that $w(a_{i_k} - e'') = w(b_{j_l} - e'') = \sigma^{**}$ $(k \leqq t, l \leqq u)$.[2] Then for other a_i's, $w(a_i - e'') = \sigma^*$; for other b_j's, $w(b_j - e'') = \sigma^*$. Hence we have $w_{e''}(d) < w_e(d)$. Applying the observation in 2) to $w_{e''}$, we have the required result.

4) Now we have only to treat the case when $r + s \neq m' + n'$ and $r \geqq s$. Let σ'' be the maximum of values $w_e(x + a_i)$ $(i > r)$ and $w_e(x + b_j)$ $(j > s)$ and renumber a_i and b_j so that $w_e(x + a_i) = w_e(x + b_j) = \sigma''$ if and only if $r < i \leqq r'$, $s \leqq j \leqq s'$. Further let e' be an element of K such that $w(a_i - e') = w(b_j - e') = \sigma''$ for any $i \leqq r'$, $j \leqq s'$.[2] Since $r \geqq s$, we have $w_{e'}(d) \leqq w_e(d)$ and we may replace e by e'. If we are still in the case 4) with $w_{e'}$, we repeat the similar process and we reach after a finite number of steps to one of the cases 1), 2), 3). Thus the lemma is proved completely.

LEMMA B. *Assume, in Lemma A, further that $m' \geqq n'$ and $m' \neq 0$. Then there exists one b_j such that $w_{b_j}(d) < w_\sigma(d)$.*

Proof. Let e be an element of K such that $w(e) = w(a_i - e) = w(b_j - e) = \sigma$ for any i and j.[2] Then we have $w_e(d) = w_\sigma(d)$. By virtue of Lemma A, we have only to show that there exists an element e''' $(w(e''') = \sigma)$ such that $w_{e'''}(d) < w_e(d)$. If $m' > n'$, then by the same process in 3) above, we see the existence of e'''. Assume that $m' = n'$ and we will make use of induction argument on m'. We apply the same process in 3) above. Then either there exists one class C as above, which contains more b_j's than a_i's, or any such classes have the same number of a_i's and b_j's. In the former case, take the element e'' as above (with respect to this class C). Then $w_{e''}(d) < w_e(d)$ and the assertion is proved in this case. On the other hand, let, say, $C = \{a_i, b_i \ (i \leqq r'')\}$ be an equivalent class in the latter case. Then since $r'' < m'$, we see the

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3) If we take σ^{**}, in this case, to be any number in G which is greater than σ^*, then we see also the proof by the same way as below.

existence of an element e''' of K sucht that $w_{e'''}(d'') < w_e(d'')$, where $d'' = \Pi_1^{r''}(x + a_i)/\Pi_1^{r''}(x + b_i)$. Since there exists one b_j such that $w(b_j - e''')$ is greater than some $w(a_i - e''')$ $(i, j \leqq r'')$, we see that $w(a_{i'} - e''')$ and $w(b_{j'} - e''')$ are all equal for i', $j' > r''$. Therefore we have $w_{e'''}(d) < w_e(d)$ and the assertion is proved.

Now we will return to the proof of the theorem.

First we assume that $w_{\lambda_0}(c) = 0$ for some λ_0 $(\alpha \leqq \lambda_0 \leqq 2\alpha)$. Let i_0, r, j_0 and s be such that $w(a_i) = \lambda_0$ if and only if $i_0 < i \leqq i_0 + r$, $w(b_j) = \lambda_0$ if and only if $j_0 < j \leqq j_0 + s$. Set $\lambda_1 = \max(\alpha, w(a_{i_0}), w(b_{j_0}))$, $\lambda_2 = \min(2\alpha, w(a_{i_0+r+1}), w(b_{j_0+s+1}))$. Then

$$\begin{aligned} w_{\lambda_1}(c) &= w(c_0) + \sum_{i \leqq i_0} w(a_i) - \sum_{j \leqq j_0} w(b_j) + (n - i_0)\lambda_1 - (m - j_0)\lambda_1 \geqq 0, \\ w_{\lambda_0}(c) &= w(c_0) + \sum_{i \leqq i_0} w(a_i) - \sum_{j \leqq j_0} w(b_j) + (n - i_0)\lambda_0 - (m - j_0)\lambda_0 = 0, \\ w_{\lambda_2}(c) &= w(c_0) + \sum_{i \leqq i_0} w(a_i) - \sum_{j \leqq j_0} w(b_j) + r\lambda_0 + (n - r - i_0)\lambda_2 - s\lambda_0 \\ &\qquad - (m - s - j_0)\lambda_1 \leqq 0. \end{aligned}$$

Hence we have

$$w_{\lambda_1}(c) = w_{\lambda 1}(c) - w_{\lambda_0}(c) = (n - i_0)(\lambda_1 - \lambda_0) - (m - j_0)(\lambda_1 - \lambda_0) \geqq 0.$$

Hence, if $\lambda_0 \neq \alpha$, we have $\lambda_1 < \lambda_0$ and $n - i_0 \leqq m - j_0$.

Similarly we have

$$w_{\lambda_2}(c) = w_{\lambda_2}(c) - w_{\lambda_0}(c) = (n - r - i_0)(\lambda_2 - \lambda_0) - (m - s - j_0)(\lambda_2 - \lambda_0) \geqq 0.$$

Hence, if $\lambda_0 \neq 2\alpha$, we have $n - r - i_0 \geqq m - s - j_0$. Thus in the case when λ_0 is equal to neither α nor 2α, we first have $r \leqq s$. If $s \neq 0$, then Lemma B shows that there exists one b_j $(j_0 < j \leqq j_0 + s)$ such that $w_{\lambda_0}(c) > w_{b_j}(c)$, which is a contradiction. Hence $r = s = 0$. Therefore we have further that $n - i_0 = m - j_0$. In the case when $\lambda_0 = \alpha$ or $\lambda_0 = 2\alpha$, we see easily that $r = s = 0$ and $n - i_0 = m - j_0$ because $\alpha \notin G$. If $\lambda_1 \neq \alpha$, then there exists one a_i or b_j such that $w(a_i)$ or $w(b_j)$ is equal to λ_1, which is a contradiction because $w_{\lambda_1}(c) = 0$ by the above equality. Hence $\lambda_1 = \alpha$. Similarly we have $\lambda_2 = 2\alpha$. From $\lambda_1 = \alpha$, we have $i_0 = j_0 = 0$, whence $m = n$; from $\lambda_2 = 2\alpha$, we have $a_i = b_j = 0$ for all i and j. Hence we have $c = c_0 \in K$ and $w_\lambda(c) = 0$ for any λ. This proves (1). Next we assume that $w_\alpha(c) > 0$. Let us consider $w_\lambda(c)$ as a function of variable λ $(\alpha \leqq \lambda \leqq 2\alpha)$; it is obviously a continuous function and it takes the smallest and the largest values ε_1 and δ_1 in $\alpha \leqq \lambda \leqq 2\alpha$. By virtue of (1), we see that ε_1 is positive. Then (2) follows easily from the fact that $w_e(c) \neq w_{w(e)}(c)$ occurs only when $w(e)$ is one of $w(a_i)$ or $w(b_j)$; by the symmetricity of the assertion in Lemma A, we see that these values $w_e(c)$ are bounded by the maximum and minimum of values $w_{w(e)}(c)$, $w_{a_i}(c)$ and $w_{b_j}(c)$.

Since $w_{b_j}(c) \notin G$, the minimum is not zero and (2) is proved.

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