ON THE DIMENSION OF MODULES AND ALGEBRAS (III)

GLOBAL DIMENSION[1]

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Let Λ be a ring with unit. If A is a left Λ-module, the *dimension* of A (notation: $\mathrm{l.dim}_\Lambda A$) is defined to be the least integer n for which there exists an exact sequence

$$0 \longrightarrow X_n \longrightarrow \ldots \longrightarrow X_0 \longrightarrow A \longrightarrow 0$$

where the left Λ-modules $X_0, \ldots, X_n$ are projective. If no such sequence exists for any n, then $\mathrm{l.dim}_\Lambda A = \infty$. The *left global dimension* of Λ is

$$\mathrm{l.gl.dim}\,\Lambda = \sup\ \mathrm{l.dim}_\Lambda A$$

where A ranges over all left Λ-modules. The condition $\mathrm{l.dim}_\Lambda A < n$ is equivalent with $\mathrm{Ext}^n_\Lambda(A, C) = 0$ for all left Λ-modules C. The condition $\mathrm{l.gl.dim}\,\Lambda < n$ is equivalent with $\mathrm{Ext}^n_\Lambda = 0$. Similar definitions and theorems hold for right Λ-modules.

In the first section of this paper it is shown that the global dimension of Λ is completely determined by the dimensions of the cyclic modules over Λ, i.e., the modules generated by a single element. In the next section the notion of *weak global dimension* of Λ (notation: $\mathrm{w.gl.dim}\,\Lambda$) is introduced, and using the previous result it is proven that if Λ is both left and right Noetherian, then $\mathrm{l.gl.dim}\,\Lambda = \mathrm{w.gl.dim}\,\Lambda = \mathrm{r.gl.dim}\,\Lambda$.

The rest of the paper, which is independent of the first two sections, is devoted to a study of the global dimension of semi-primary rings. The principal result here is that $\mathrm{l.dim}_\Lambda \Gamma = \mathrm{l.gl.dim}\,\Lambda = \mathrm{w.gl.dim}\,\Lambda = \mathrm{r.gl.dim}\,\Lambda = \mathrm{r.dim}_\Lambda \Gamma$, where $\Gamma = \Lambda/N$, N being the radical of Λ.

The definitions and notations employed in this paper are based on those

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introduced by H. Cartan and S. Eilenberg in [1].

§ 1. Global dimension and ideals

THEOREM 1. For each ring Λ we have

(a) $$\text{l. gl. dim } \Lambda = \sup_B \text{ l. dim}_\Lambda B$$

(b) $$= \sup_I \text{ l. dim}_\Lambda \Lambda/I$$

where B ranges over all left Λ-modules generated by a single element and I ranges over all left ideals of Λ.

If further Λ is not semi-simple (i.e., l. gl. dim $\Lambda > 0$), then

(c) $$\text{l. gl. dim } \Lambda = 1 + \sup_I \text{ l. dim}_\Lambda I.$$

The equivalence of (a) and (b) is obvious. From [1; I, 4.2] we deduce that Λ is semi-simple if and only if Λ/I is projective for all left ideals I of Λ. It follows from this and [1; IV, 2.3], which we state below without proof as Proposition 2, that (b) implies (c).

PROPOSITION 2. If $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ is an exact sequence of left Λ-modules with A projective and A'' not projective, then $\text{l. dim}_\Lambda A'' = 1 + \text{l. dim}_\Lambda A'$.

Therefore in order to prove Theorem 1, it suffices to prove statement (a) of Theorem 1. This proof is based on

PROPSITION 3. Let A be a left Λ-module, I a non-empty well-ordered set and $(A_i)_{i\in I}$ a family of submodules of A such that if $i, j \in I$ and $i \leqq j$, then $A_i \subseteqq A_j$. If $\bigcup_{i\in I} A_i = A$ and $\text{l. dim}_\Lambda (A_i/A_i') \leqq n$ for all $i \in I$ where $A_i' = \bigcup_{j<i} A_j$, then $\text{l. dim}_\Lambda A \leqq n$.

Proof. The proof is by induction on n. If $n = 0$, then for all $i \in I$ we have $\text{l. dim} (A_i/A_i') \leqq 0$. Therefore each A_i/A_i' is projective. This implies that each of the exact sequences

$$0 \longrightarrow A_i' \longrightarrow A_i \longrightarrow A_i/A_i' \longrightarrow 0$$

splits. Thus there exist submodules C_i of A_i such that

(i) $A_i = A_i' + C_i$ (direct sum),

(ii) each C_i is isomorphic to A_i/A_i' and therefore is projective.

From (i) and the hypothesis that $A = \bigcup_{i\in I} A_i$, it follows that $A = \sum_{i\in I} C_i$ (direct sum).

From (ii) we have that A is projective, since by [1; I, 2.1] the direct sum of projective modules is projective. Therefore $\mathrm{l.dim}_\Lambda A = 0$ and the proposition is established in the case $n = 0$.

Suppose $n > 0$ and the proposition has been established for $n-1$. Also, suppose $\mathrm{l.dim}_\Lambda (A_i/A'_i) \leqq n$ for all $i \in I$. Let F be the free Λ-module generated by the elements of A and F_i (respectively F'_i) the free Λ-module generated by the elements of A_i (respectively A'_i). Further, let $R = \mathrm{Ker}\,(F \longrightarrow A)$ and define $R_i = F_i \cap R$, $R'_i = F'_i \cap R$.

From the relations $A_i \supseteqq A'_i$, $F_i \supseteqq F'_i$, $R_i \supseteqq R'_i$ and the exact sequences

$$0 \longrightarrow R_i \longrightarrow F_i \longrightarrow A_i \longrightarrow 0$$
$$0 \longrightarrow R'_i \longrightarrow F'_i \longrightarrow A'_i \longrightarrow 0,$$

it follows that the sequences

$$0 \longrightarrow R_i/R'_i \longrightarrow F_i/F'_i \longrightarrow A_i/A'_i \longrightarrow 0$$

are exact for all $i \in I$. Each F_i/F'_i is a free Λ-module and therefore projective, since each F'_i is generated by a subset of a basis for F_i. Therefore by Proposition 2 we have $\mathrm{l.dim}_\Lambda (R_i/R'_i) \leqq n-1$. It can easily be established that the family $(R_i)_{i \in I}$ has the properties that i, $j \in I$ and $i \leqq j$ implies that $R_i \subseteqq R_j$, $R = \bigcup_{i \in I} R_i$ and $R'_i = \bigcup_{j < i} R_j$. Thus by the induction hypothesis $\mathrm{l.dim}\ R \leqq n-1$. Since the sequence

$$0 \longrightarrow R \longrightarrow F \longrightarrow A \longrightarrow 0$$

is exact, it follows from Proposition 2 that $\mathrm{l.dim}\ A_\Lambda \leqq 1 + \mathrm{l.dim}_\Lambda R \leqq n$.

We now prove (a) of Theorem 1.

Let A be an arbitrary Λ-module. Well order the elements x_i of A and denote by A_i (respectively A'_i) the submodule of A generated by x_j for $j \leqq i$ (respectively $j < i$). Then A_i/A'_i is either 0 or generated by the single element x_i. Therefore $\mathrm{l.dim}\,(A_i/A'_i) \leqq n$, where $n = \sup_B \mathrm{l.dim}\, B$, B ranging over all left Λ-modules generated by a single element. Since the family $(A_i)_{i \in I}$ of submodules satisfies the hypothesis of Proposition 3, it follows that $\mathrm{l.dim}\ A \leqq n$. Therefore $\mathrm{l.gl.dim}\ \Lambda \leqq n$. But by definition $\mathrm{l.gl.dim}\ \Lambda \geqq n$. Therefore $\mathrm{l.gl.dim}\ \Lambda = n$, which completes the proof of Theorem 1.

§ 2. Global dimension of Noetherian rings

Let A be a left Λ-module. In addition to $\mathrm{l.dim}_\Lambda A$ we introduce (see [1;

VI, Exer. 3]) the *weak left dimension* of A as follows:

$$-1 \leqq \text{w. l. dim}_\Lambda A \leqq \infty,$$

where $\text{w. l. dim}_\Lambda A < n$ if and only if $\text{Tor}_n^\Lambda (C, A) = 0$ for all right Λ-modules C. For a right Λ-module C we define $\text{w. r. dim}_\Lambda C$ similarly.

We introduce the *weak global dimension* of Λ as follows:

$$0 \leqq \text{w. gl. dim}\, \Lambda \leqq \infty,$$

where $\text{w. gl. dim}\, \Lambda < n$ if and only if $\text{Tor}_n^\Lambda = 0$.

For the weak global dimension there is no distinction between "left" and "right" dimension. Indeed, we have

$$\begin{aligned} \text{w. gl. dim}\, \Lambda &= \sup_A \text{w. l. dim}_\Lambda A \\ &= \sup_C \text{w. r. dim}_\Lambda C \end{aligned}$$

where A ranges over all left Λ-modules while C ranges over all right Λ-modules. Since the functors Tor_n^Λ commute with direct limits, we may restrict A (respectively C) to range over finitely generated left (respectively right) Λ-modules.

THEOREM 4. If the ring Λ is left Noetherian, then

$$\text{l. gl. dim}\, \Lambda = \text{w. gl. dim}\, \Lambda.$$

Similarly, if Λ is right Noetherian, then

$$\text{r. gl. dim}\, \Lambda = \text{w. gl. dim}\, \Lambda.$$

Proof. By Theorem 1 we have

$$\text{l. gl. dim}\, \Lambda = \sup_A \text{l. dim}_\Lambda A$$

where A ranges over all finitely generated left Λ-modules. Since Λ is left Noetherian, we have by [1; VI, Exer. 3] that

$$\text{l. dim}_\Lambda A = \text{w. l. dim}_\Lambda A$$

for each finitely generated left Λ-module A. This yields the conclusion.

COROLLARY 5. If Λ is both left and right Noetherian, then

$$\text{l. gl. dim}\, \Lambda = \text{w. gl. dim}\, \Lambda = \text{r. gl. dim}\, \Lambda.$$

This common value will be denoted by $\text{gl. dim}\, \Lambda$.

§ 3. Semi-primary rings

Before discussing semi-primary rings, we prove the following general lemma.

LEMMA 6.[2] Let Λ be an arbitrary ring, N a nilpotent left ideal in Λ, and T a (covariant or contravariant) half exact functor defined for all left Λ-modules. If $T(A)=0$ for each left Λ-module such that $NA=0$, then $T=0$.

Proof. Suppose the lemma is false. Then there is a left Λ-module A such that $T(A)\neq 0$. Since N is nilpotent, there is a maximal index $k\geqq 0$ such that $T(N^kA)\neq 0$ (where $N^0=\Lambda$). Consider the exact sequence

$$0\longrightarrow N^{k+1}A\longrightarrow N^kA\longrightarrow N^kA/N^{k+1}A\longrightarrow 0.$$

Since $T(N^{k+1}A)=0=T(N^kA/N^{k+1}A)$ and T is half exact, we have $T(N^kA)=0$. This contradiction proves the lemma.

Let Λ be a ring (with unit). We say Λ is *semi-primary* if there is a two-sided nilpotent ideal N of Λ, which we call the *radical* of Λ, such that $\Gamma=\Lambda/N$ is semi-simple. It is clear that if Λ is semi-primary, its radical is unique.

PROPOSITION 7. Let Λ be semi-primary, with radical N, and let $\Gamma=\Lambda/N$. Then for each left Λ-module A the following conditions are equivalent:

(a) $\mathrm{Tor}_n^\Lambda(\Gamma, A)=0$

(b) $\mathrm{Tor}_n^\Lambda(C, A)=0$ for every simple right Λ-module C

(c) $\mathrm{w.l.dim}_\Lambda A<n$

(d) $\mathrm{Ext}_\Lambda^n(A, \Gamma)=0$

(e) $\mathrm{Ext}_\Lambda^n(A, C)=0$ for every simple left Λ-module C

(f) $\mathrm{l.dim}_\Lambda A<n$.

Proof. (a)$\Rightarrow$(b). If C is a simple right Λ-module, then C is a direct summand of Γ. Since Tor commutes with direct sums in either variable, $\mathrm{Tor}_n^\Lambda(C, A)$ is a direct summand of $\mathrm{Tor}_n^\Lambda(\Lambda, A)$. Therefore if $\mathrm{Tor}_n^\Lambda(\Gamma, A)=0$, then $\mathrm{Tor}_n^\Lambda(C, A)=0$.

(b)$\Rightarrow$(c). Consider a right Λ-module B such that $BN=0$. Since B can be considered a right module over the semi-simple ring Γ, we have that B is semi-simple, i.e., B is the direct sum of simple right Λ-modules. Now, Tor commutes with direct sum and (b) states that $\mathrm{Tor}_n^\Lambda(C, A)=0$ for all simple right Λ-modules C. Thus $\mathrm{Tor}_n^\Lambda(B, A)=0$ for any B such that $BN=0$. Since

[2] This is a generalization of [2, Proposition 3].

$\operatorname{Tor}_n^{\Lambda}(\ \ , A)$ is a half exact functor, we deduce from Lemma 6 that $\operatorname{Tor}_n^{\Lambda}(B, A) = 0$ for all right Λ-modules B.

(c)$\Longrightarrow$(d). By [1; VI, 5.1] we have

$$\operatorname{Ext}_{\Lambda}^n(A, \operatorname{Hom}_Z(B, T)) \approx \operatorname{Hom}_Z(\operatorname{Tor}_n^{\Lambda}(B, A), T)$$

where B is an arbitrary right Λ-module and $T = R/Z$, the additive group of real numbers reduced modulo the integers. Since (c) implies $\operatorname{Tor}_n^{\Lambda}(B, A) = 0$, we have $\operatorname{Ext}_{\Lambda}^n(A, \operatorname{Hom}_Z(B, T)) = 0$.

Now choose $B = \operatorname{Hom}_Z(\Gamma, T)$, which we consider a right Γ-module (and therefore a right Λ-module) by defining $(f\gamma_1)(\gamma_2) = f(\gamma_1\gamma_2)$ for all $f \in \operatorname{Hom}_Z(\Gamma, T)$ and $\gamma_1, \gamma_2 \in \Gamma$. Since Γ is semi-simple, the left Γ-module $\operatorname{Hom}_Z(B, T)$ is semi-simple and thus every submodule of $\operatorname{Hom}_Z(B, T)$ is a direct summand. The Γ-monomorphism $\varphi : \Gamma \longrightarrow \operatorname{Hom}_Z(B, T)$, defined by $\varphi(\gamma)f = f(\gamma)$ for all $f \in B$, $\gamma \in \Gamma$, shows that Γ is isomorphic to a submodule and therefore to a dirct summand of $\operatorname{Hom}_Z(B, T)$. Since Ext commutes with finite direct sum on the second variable and $\operatorname{Ext}_{\Lambda}^n(A, \operatorname{Hom}_Z(B, T)) = 0$, we have $\operatorname{Ext}_{\Lambda}^n(A, \Gamma) = 0$.

(d)$\Longrightarrow$(e). Same argument as that used to prove (a)$\Longrightarrow$(b) with the functor $\operatorname{Ext}_{\Lambda}^n(A, \ \ )$ substituted for $\operatorname{Tor}_n^{\Lambda}(\ \ , A)$.

(e)$\Longrightarrow$(f). Consider a left Λ-module B such that $NB = 0$. Then B can be considered a left Γ-module. Since Γ is semi-simple, B is semi-simple, i.e., $B \approx \sum C_i$, direct sum of simple Γ-modules C_i. Now $\sum C_i$ is a submodule and therefore a direct summand of the Γ-module ΠC_i, the direct product of the C_i. Thus $\operatorname{Ext}_{\Lambda}^n(A, \sum C_i)$ is a direct summand of $\operatorname{Ext}_{\Lambda}^n(A, \Pi C_i)$. But $\operatorname{Ext}_{\Lambda}^n(A, \Pi C_i) = \Pi \operatorname{Ext}_{\Lambda}^n(A, C_i) = 0$. Thus $\operatorname{Ext}_{\Lambda}^n(A, B) = 0$ for all B such that $NB = 0$. Since $\operatorname{Ext}_{\Lambda}^n(A, \ \ )$ is a half exact functor, we deduce from Lemma 6 that $\operatorname{Ext}_{\Lambda}^n(A, B) = 0$ for all left Λ-modules B, i.e., $\operatorname{l.dim} A < n$.

(f)$\Longrightarrow$(a). This follows immediately from the general proposition that $\operatorname{l.dim}_{\Lambda} A \geqq \operatorname{w.l.dim}_{\Lambda} A$ (see [1; VI, Exer. 3]).

As an immediate consequence of this proposition we have.

Corollary 8. If Λ is a semi-primary ring and A is s left Λ-module, then

$$\operatorname{w.l.dim}_{\Lambda} A = \operatorname{l.dim}_{\Lambda} A.$$

Similarly, if A is a right Λ-module, then

$$\operatorname{w.r.dim}_{\Lambda} A = \operatorname{r.dim}_{\Lambda} A.$$

From Corollary 8 we conclude.

COROLLARY 9. If Λ is a semi-primary ring, then

$$\text{l. gl. dim } \Lambda = \text{w. gl. dim } \Lambda = \text{r. gl. dim } \Lambda.$$

This common value we designate by gl. dim Λ.

PROPOSITION 10. For each left Λ-module A, the following conditions are equivalent:

(a) $\text{Ext}_\Lambda^n(\Gamma, A) = 0$
(b) $\text{Ext}_\Lambda^n(C, A) = 0$ for every simple left Λ-module C
(c) l. inj. $\dim_\Lambda A < n$.

Proof. (a)$\Rightarrow$(b). The same argument can be employed as was used in Proposition 7 to prove that (d) + (e), but applied to the first variable instead of the second variable.

(b)$\Rightarrow$(c). Consider a left Λ-module B such that $NB = 0$. Then, as we have seen before, $B \approx \sum C_i$, the direct sum of simple left Λ-modules C_i. Now $\text{Ext}_\Lambda^n(\sum C_i, A) \approx \Pi \text{Ext}_\Lambda^n(C_i, A)$. Therefore, since (b) states that $\text{Ext}_\Lambda^n(C_i, A) = 0$ for all i, we have $\text{Ext}_\Lambda^n(B, A) = 0$ for all B such that $NB = 0$. Since $\text{Ext}_\Lambda^n(\quad, A)$ is a half exact functor, we deduce from Lemma 6 that $\text{Ext}_\Lambda^n(B, A) = 0$ for all left Λ-modules B, i.e., l. inj. $\dim_\Lambda A < n$.

(c)$\Rightarrow$(a). This follows from the definition of the left injective dimension of a module. (See [1; VI, 2. la].)

COROLLARY 11. If Λ is a semi-primary ring with radical N and $\Gamma = \Lambda/N$, then

(a) gl. dim Λ = l. inj. $\dim_\Lambda \Gamma$
(b) $= \text{l.} \dim_\Lambda \Gamma$
(c) $= 1 + \text{l.} \dim_\Lambda N$
(d) $= \sup_C \text{l.} \dim_\Lambda C$
(e) $= \sup_C \text{l. inj.} \dim_\Lambda C$

where C ranges over all simple left Λ-modules. Also, (a) – (c) hold with "left" replaced by "right."

Proof. (a). If A is a left Λ-module we have by Proposition 7, (d) and (f), that $\text{l. dim}_\Lambda A \leqq \text{l. inj. dim}_\Lambda \Gamma$. Thus $\text{gl. dim}\, \Lambda \geqq \text{l. inj. dim}_\Lambda \Gamma$. But by [1; VI, 2.6] we have $\text{gl. dim}\, \Lambda \geqq \text{l. inj. dim}_\Lambda \Gamma$. Therefore $\text{gl. dim}\, \Lambda = \text{l. inj. dim}_\Lambda \Gamma$.

(b). If A is a left Λ-module we have by Proposition 10, (a) and (c), that $\text{l. inj. dim}_\Lambda A \leqq \text{l. dim}_\Lambda \Gamma$. Therefore $\text{gl. dim}\, \Lambda \leqq \text{l. dim}_\Lambda \Gamma$. But by definition $\text{gl. dim}\, \Lambda \geqq \text{l. dim}_\Lambda \Gamma$. Thus $\text{gl. dim}\, \Lambda = \text{l. dim}_\Lambda \Gamma$.

(c). If $N \neq 0$, then Λ is not semi-simple and thus $\text{gl. dim}\, \Lambda = \text{l. dim}_\Lambda \Gamma > 0$. Therefore, by Proposition 2, we deduce from the exact sequence

$$0 \longrightarrow N \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow 0$$

that $\text{gl. dim}\, \Lambda = \text{l. dim}_\Lambda \Gamma = 1 + \text{l. dim}_\Lambda N$. If $N = 0$, then Λ is semi-simple, i.e., $\text{gl. dim}\, \Lambda = 0$. Since $\text{l. dim}_\Lambda 0 = -1$, $\text{gl. dim}\, \Lambda = 1 + \text{l. dim}_\Lambda N$.

(d). Since Γ is semi-simple, $\Gamma \approx \sum C_i$, finite direct sum of simple left Λ-modules, where the C_i have the property that if C is a simple left Λ-module, then $C \approx C_i$ for some i. Therefore $\sup_C \text{l. dim}_\Lambda C = \text{l. dim}_\Lambda \Gamma = \text{gl. dim}\, \Lambda$, where C ranges over all simple left Λ-modules.

(e). This is proved in an analogous fashion to (d).

Corollary 12. If Λ is a semi-primary ring, then the following are equivalent:

(a) $\text{gl. dim}\, \Lambda < n$

(b) $\text{Ext}^n_\Lambda(\Gamma, \Gamma) = 0$, both Γ's considered as left Λ-modules

(c) $\text{Ext}^n_\Lambda(\Gamma, \Gamma) = 0$, both Γ's considered as right Λ-modules

(d) $\text{Tor}^\Lambda_n(\Gamma, \Gamma) = 0$, first Γ considered a right Λ-module, second Γ considered a left Λ-module

Proof. Since by Corollary 9 we have

$$\text{gl. dim}\, \Lambda = \text{l. dim}_\Lambda \Gamma = \text{w. dim}_\Lambda \Gamma = \text{r. dim}_\Lambda \Gamma,$$

it follows from the definitions of these various terms that (a) implies (b), (c), (d). The proofs that (b), (c), (d) each imply (a) are all similar. Consequently we will prove only that (b) implies (a) and leave the others to the reader.

By Proposition 7(d), we have that if $\text{Ext}^n_\Lambda(\Gamma, \Gamma) = 0$, then $\text{l. dim}_\Lambda \Gamma < n$. But we have by Corollary 11(b) that $\text{l. dim}_\Lambda \Gamma = \text{gl. dim}\, \Lambda$. Therefore if $\text{Ext}^n_\Lambda(\Gamma, \Gamma) = 0$, then $\text{gl. dim}\, \Lambda < n$.

Proposition 13. If Λ satisfies the left minimum condition and $\text{gl. dim}\, \Lambda > 0$, then there is an indecomposable left ideal J in Λ such that $J^2=0$ and $\text{gl.dim}\,\Lambda = 1 + \text{l.dim}_\Lambda J$.

Proof. Let J be a left ideal contained in the radical N, minimal with respect to the property that $\text{l.dim}_\Lambda J = \text{l.dim}_\Lambda N$ (ideals of this type exist since N is such an ideal). If $J = A + B$ (direct sum), then $\text{l.dim}_\Lambda J = \sup(\text{l.dim}_\Lambda A, \text{l.dim}_\Lambda B)$. By the minimal property of J, either A or B must be the trivial ideal. Thus J is indecomposable.

Suppose $J^2 \neq 0$. Then there is an element $\lambda^* \in J$ such that $J\lambda^* \neq 0$. Consider the exact sequence

$$(*)\ 0 \longrightarrow K \longrightarrow J \stackrel{f}{\longrightarrow} J\lambda^* \longrightarrow 0$$

where $f(\lambda) = \lambda\lambda^*$ and $K = \text{Ker } f$. Since J is nilpotent, $J\lambda^* \neq J$ and f is not a monomorphism. Thus $0 \neq K \neq J$. Therefore $J\lambda^*$ and K are proper ideals in J. Consequently we have $\sup(\text{l.dim}_\Lambda J\lambda^*, \text{l.dim}_\Lambda K) < \text{l.dim}_\Lambda J$. But in view of the exact sequence $(*)$ and [1; VI, 2.3] we have $\text{l.dim}\, J \leqq \sup(\text{l.dim}_\Lambda J\lambda^*, \text{l.dim}_\Lambda K)$. This contradiction proves that $J^2 = 0$. Since by definition $\text{l.dim}_\Lambda J = \text{l.dim}_\Lambda N$, we have by Corollary 11(c) $\text{gl.dim}\,\Lambda = 1 + \text{l.dim}_\Lambda J$. Q.E.D.

§ 4. Applications

Proposition 14. Let Λ be a semi-primary ring such that each simple left Λ-module is isomorphic to a left ideal in Λ, then $\text{gl.dim}\,\Lambda = 0, \infty$.

Proof. Suppose $\text{gl.dim}\,\Lambda = n$, $0 < n < \infty$. By Corollary 11(d) we have $\text{gl.dim}\,\Lambda = \text{l.dim}_\Lambda C$, where C is a simple left Λ-module. By hypothesis, $C \approx I$, where I is an ideal in Λ. Thus $\text{l.dim}\, I = \text{gl.dim}\,\Lambda = n$. But since $n > 0$, Λ/I is not projective. Therefore Proposition 2 applied to the exact sequence

$$0 \longrightarrow I \longrightarrow \Lambda \longrightarrow \Lambda/I \longrightarrow 0$$

gives $\text{l.dim}\,\Lambda/I = 1 + \text{l.dim}\, I = 1 + n$. However, $\text{l.dim}_\Lambda \Lambda/I \leqq \text{gl.dim}\,\Lambda = n$. This contradiction proves the proposition.

Proposition 15. The hypothesis of Proposition 14 is satisfied in each of the following cases:

(a) Λ is a direct sum of a finite number of primary rings (a semi-primary ring Λ is primary if $\Gamma = \Lambda/N$ is a simple ring).

(b) Λ is a semi-primary commutative ring.

(c) Λ satisfies both the left and right minimum conditions and every two-sided ideal in Λ is a principal right ideal and a principal left ideal.

(d) Λ is a quasi-Frobenius ring.

Proof. (a). Suppose Λ is a primary ring. If $N=0$, we are finished. Assume $N\neq 0$. Let k be the maximum index such that $N^k\neq 0$. Since $NN^k=0$, N^k is a Γ-module. Thus N^k is semi-simple and therefore contains at least one simple left ideal of Λ. But Γ is a simple ring. Thus there is only one isomorphism class of simple left Λ-modules. Therefore each simple left Λ-module is isomorphic to a simple ideal in Λ. The rest of (a) is obvious.

(b). Since N is nilpotent, every set of orthogonal idempotents in Γ can be "lifted" to an orthogonal set of idempotents in Λ. From this and the commutativity of Λ, it follows that Λ is a finite direct sum of primary rings. Thus (d) is reduced to (a).

(c). By [3; Chapter 4, Theorem 37] we have that Λ is a direct sum of primary rings. Therefore (c) is also reduced to (a).

(d). This is an immediate consequence of the definition of a quasi-Frobenius ring as given in [4].

§ 5. Tensor products of semi-primary algebras

THEOREM 16. If Λ_1 and Λ_2 are algebras over a field K, then

$$\text{w. gl. dim}\,(\Lambda_1\otimes\Lambda_2)\geqq \text{w. gl. dim}\,\Lambda_1+\text{w. gl. dim}\,\Lambda_2.$$

If further Λ_1 and Λ_2 are semi-primary algebras and $\Gamma_1\otimes\Gamma_2$ is semi-simple, then $\Lambda_1\otimes\Lambda_2$ is a semi-primary algebra and

$$\text{gl. dim}\,(\Lambda_1\otimes\Lambda_2)=\text{gl. dim}\,\Lambda_1+\text{gl. dim}\,\Lambda_2.$$

Proof. By [1; XI, 3.1] we have

$$\sum_{p+q=n}\operatorname{Tor}_p^{\Lambda_1}(C_1,\ A_1)\otimes\operatorname{Tor}_q^{\Lambda_2}(C_2,\ A_2)\approx\operatorname{Tor}_n^{\Lambda_1\otimes\Lambda_2}(C_1\otimes C_2,\ A_1\otimes A_2)$$

for all $n\geqq 0$, where C_1 and C_2 are right Λ_1 and Λ_2-modules and A_1 and A_2 are left Λ_1 and Λ_2-modules. Since K is a field, $\operatorname{Tor}_p^{\Lambda_1}(C_1,\ A_1)\neq 0$ and $\operatorname{Tor}_q^{\Lambda_2}(C_2,\ A_2)\neq 0$ implies that $\operatorname{Tor}_{p+q}^{\Lambda_1\otimes\Lambda_2}(C_1\otimes C_2,\ A_1\otimes A_2)\neq 0$. Thus

$$\text{w. gl. dim}(\Lambda_1\otimes\Lambda_2)\geqq\text{w. gl. dim}\,\Lambda_1+\text{w. gl. dim}\,\Lambda_2.$$

From the exact sequence

$$N_1\otimes\Lambda_2+\Lambda_1\otimes N_2\longrightarrow\Lambda_1\otimes\Lambda_2\xrightarrow{j}\Gamma_1\otimes\Gamma_2\longrightarrow 0$$

we deduce that the Ker f is nilpotent. If we assume that $\Gamma_1\otimes\Gamma_2$ is semi-simple, we have that $\Lambda_1 \otimes \Lambda_2$ is semi-primary with radical the Ker f. Now we have

$$\sum_{p+q=n} \operatorname{Tor}_p^{\Lambda_1}(\Gamma_1, \Gamma_1) \otimes \operatorname{Tor}_p^{\Lambda_2}(\Gamma_2, \Gamma_2) \approx \operatorname{Tor}_n^{\Lambda_1 \otimes \Lambda_2}(\Gamma_1 \otimes \Gamma_2, \Gamma_1 \otimes \Gamma_2)$$

for all $n \geqq 0$. Since K is a field, we deduce from Corollary 12 and these isomorphisms that

$$\text{gl. dim}\,(\Lambda_1 \otimes \Lambda_2) = \text{gl. dim}\,\Lambda_1 + \text{gl. dim}\,\Lambda_2.$$

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