NOTE ON THE GROUP OF AFFINE TRANSFORMATIONS OF AN AFFINELY CONNECTED MANIFOLD

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The purpose of the present note is to reform Mr. K. Nomizu's result[1] on the group of all affine transformations of an affinely connected manifold. We shall prove the following.

THEOREM. *The group of all affine transformations of an affinely connected manifold is a Lie group.*

Mr. K. Nomizu proves the theorem when the affinely connected manifold is complete.[2] And he gives out a question whether this assumption of completeness is really necessary. We shall show it is possible to prove the theorem without any assumption by considering a Riemannian metric in the bundle of frames of the manifold, which is naturally defined by the affine connection.

After preparing this note we heard from Mr. Nomizu that he has also proved the same theorem and using this result Mr. S. Kobayashi[3] has proved similar results on transformation groups of manifolds with certain connections.

In section 1 we resume the definitions and properties about affine connections, geodesic curves and regular neighbourhoods, which are given in Mr. K. Nomizu's paper. The definition of the group of affine transformations is given in section 2. The proof of the theorem is given in the last four sections.

1. Let M be a connected differentiable manifold[4] of dimension n with an

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1) K. Nomizu; On the group of affine transformations of an affinely connected manifold, Proc. Amer. Math. Soc. vol. 4 (1953).

2) For the definition of "completeness" see.[1]

3) S. Kobayasi; Groupe de transformations qui laissent invariante une connexion infinitesimale, Comptes rendus, 238 (1954).

4) The term "differentiable" will always mean "of class C^∞". As for the definitions and the notations of manifold, tangent vector, differential form, etc. we follow C. Chevalley; Theory of Lie groups, Princeton University Press, 1946. A manifold is not necessarily connected.

affine connection. According to Chern's formulation,[5] the affine connection is defined by a set of $n+n^2$ linear differential forms θ^i and θ^i_j $(i, j = 1, \ldots, n)$ on the bundle of frames B^* of M which satisfy the equations of structure of affine connection. If we take a local coordinate system (u^i) in M, then there corresponds a local coordinate system (u^i, X^k_j) in B^* such that the n vectors of a frame are given by $\sum_{k=1}^n X^k_j \frac{\partial}{\partial u^k}$ $(j = 1, \ldots, n)$, where the determinant of $(X^k_j) \neq 0$. Let (Y^k_j) be the inverse matrix of (X^k_j). Then the $n+n^2$ linear differential forms θ^i, θ^i_j are given as follows:

$\theta^i = \sum_{j=1}^n Y^i_j du^j$, $\theta^i_j = \sum_{k=1}^n Y^i_k (dX^k_j + \sum_{l,m=1}^n \Gamma^k_{ml} X^l_j du^m)$, where Γ^k_{ml} are so-called coefficients of the affine connection with respect to the local coordinates (u^i). Then θ^i and θ^i_j are linearly independent linear differential forms defined on the whole space B^* and satisfy the following equations of structure:

$$d\theta^i - \sum_{k=1}^n \theta^k \wedge \theta^i_k = \frac{1}{2} \sum_{l,m=1}^n P^i_{lm} \theta^l \wedge \theta^m,$$

$$d\theta^i_j - \sum_{k=1}^n \theta^k_j \wedge \theta^i_k = \frac{1}{2} \sum_{l,m=1}^n S^i_{jlm} \theta^l \wedge \theta^m$$

where

$$\frac{1}{2} P^i_{lm} = \sum_{j,p,q=1}^n Y^i_j X^p_l X^q_m T^j_{pq}$$

$$S^i_{jlm} = \sum_{k,p,q,r=1}^n Y^i_k X^p_j X^q_l X^q_m R^k_{pqr},$$

$T^j_{p,q}$, R^k_{pqr} being the components of the torsion and curvature tensors respectively.

Let B be the tangent bundle of M. For each element L_p of B, where $p \in M$ and L_p is a tangent vector at p, there exists a geodesic curve $f(t)$ defined by the so-called canonical parameter[6] t in $-\varepsilon < t < \varepsilon$ for some positive ε with origin $f(0) = p$ and tangent vector $df\left(\left(\frac{d}{dt}\right)_{t=0}\right) = L_p$. With respect to a local coordinate system (u^i) in a coordinate neighbourhood of p, $f(t)$ is represented by a set of solutions of the system of differential equations

$$\frac{d^2 u^i(f(t))}{dt^2} + \sum_{j,k=1}^n \Gamma^i_{jk}(f(t)) \frac{du^j(f(t))}{dt} \cdot \frac{duf^k((t))}{dt} = 0$$

$$(i = 1, \ldots, n)$$

with the initial conditions

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5) cf. 1) or S. Chern; Lecture note on differential geometry, Princeton.

6) Hereafter we consider geodesic curve always with the canonical parameter.

$$u^i(f(0)) = u^i(p) \quad \text{and} \quad L_p = \sum_{i=1}^{n}\left(\frac{du^i(f(t))}{dt}\right)_{t=0}\left(\frac{\partial}{\partial u^i}\right)_p .$$

If a geodesic curve $f(t)$ can be extended over the valus of the canonical parameter $-\varepsilon < t < a+\varepsilon$, there exists a certain neighbourhood $\mathfrak{U}$ of L_p in B such that for each $L_q \in \mathfrak{U}$ the geodesic curve $f_{L_q}(t)$ is defined in $-\delta < t < a+\delta$ with origin $f_{L_q}(0)=q$ and tangent vector L_q at q and the mapping η^a from $\mathfrak{U}$ into a certain neighbourhood of $f(a)$ in M defined by $\eta^a(L_q)=f_{L_q}(a)$ is differentiable.

Let $f^i(t, u, \alpha)$ be the solutions of the system of differential equations which define a geodesic curve with initial conditions $f^i(0, u, \alpha) = u^i$ and $\left(\frac{df^i}{dt}\right)_{t=0} = \alpha^i$. We put $v^i = f^i(1, u, \alpha)$ and consider (u^i, v^j) as function of (u^i, α^j), then we see that at $\alpha^i = 0$ (for all i) their functional determinant is equal to 1. Hence it follows that for any neighbourhood U of any point p in M there exist an open neighbourhood N of p contained in U and an open neighbourhood $\mathfrak{U}$ of zero tangent vector at p in B such that the mapping F from $\mathfrak{U}$ onto $N\times N$ which is defined by $F(L_q) = (q, \eta^1(L_q))$ for L_q in $\mathfrak{U}$ is a differentiable isomorphism. This open neighbourhood N is called a regular neighbourhood of p contained in U.[7] It is easily seen that the geodesic curve $f(t)$, $0 \leqq t \leqq 1$, with $f(0)\in N$, $f(1) \in N$ and with the tangent vector $F^{-1}(f(0), f(1))$ at $f(0)$ is contained in U.

For the sake of necessity we shall prove the following

LEMMA 1. *Let N_1 be a regular neighbourhood of p in M whose closure is contained in a regular neighbourhood N of p. If a geodesic curve $f_{L_q}(t)$, $-\varepsilon < t < 1+\varepsilon$, with origin q and tangent vector L_q at q is contained in N_1, then L_q is an element of $\mathfrak{U}_1 = F^{-1}(N_1\times N_1)$.*

Proof. Let T_q be the tangent space at q. The restriction of the mapping F on $T_q \cap \mathfrak{U}_1$ is also a differentiable homeomorphism from $T_q\cap\mathfrak{U}_1$ onto $q\times N_1$. As $\mathfrak{U}_1$ is open. the set of non-negative numbers $\{\lambda\,;\,\lambda \geqq 0,\ \lambda L_q \in \mathfrak{U}_1\}$ is open in the set of all non-negative real numbers. And as zero-tangent vector at q is contained in $\mathfrak{U}_1$ its connected compontent of zero $\{\lambda\,;\, 0\leqq\lambda<\lambda_1\}$ where λ_1 may be ∞, is not empty. From the definition of the mapping F, $F(\lambda L_q) = (q, f_{L_q}(\lambda))$ for $0\leqq\lambda<\lambda_1$. If λ_1 is not ∞, $\lambda_1 L_p$ is contained in $\overline{\mathfrak{U}}_1$, so $\lambda_1 L_q$ is contained in

[7] When we say simply that N is a regular neighbourhood of p it means N is a regular neighbourhood of p contained in some neighbourhood U of p. cf. [1].

$\mathfrak{U}$. Therefore F is defined at $\lambda_1 L_q$ and $F(\lambda_1 L_q) = (q, f_{L_q}(\lambda_1))$. As $\lambda_1 L_q$ is not contained in $\mathfrak{U}_1$, $f_{L_q}(\lambda_1)$ is not contained in N_1. If λ_1 were not larger than 1, from our assumption $f_{L_q}(\lambda_1)$ would be contained in N_1 and this is a contradiction. So λ_1 must be larger than 1 and surly L_q is contained in $\mathfrak{U}_1$.

Moreover the fact that for any compact sets K and K' contained in N the set $F^{-1}(K \times K')$ in B is a compact set is effectively used by K. Nomizu, and we shall follow him.

2. Let T and T' be Hausdorff spaces. We denote the set of all continuous mappings from T into T' with the compact-open topology by $C(T, T')$. And we denote the set of all homeomorphisms of T onto itself by $H(T)$. Let U and U' be open sets in T, K and K' be compact sets in T, and let $W(U, U'\ ;\ K, K')$ be the set of ϕ in $H(T)$ such that $\phi(U) \subset K$ and $\phi^{-1}(U') \subset K'$. We take the totality of sets $W(U, U'\ ;\ K, K')$ as a subbase for open sets of $H(T)$, then $H(T)$ becomes a Hausdorff space.

Let M and M' be differentiable manifolds of dimension n with affine connections. Let θ^i and θ^i_j $(i, j = 1, \ldots, n)$ be $n + n^2$ linear differential forms on the bundle B^* of frames of M, θ'^i and θ'^i_j $(i, j = 1, \ldots, n)$ be linear differential forms on the bundle B'^* of frames of M', which define the affine connections on M and M' respectively. If ϕ is a differentiable mapping from M into M' of rank n at any point, it induces a differentiable mapping $D\phi$ from B^* into B'^* and $\delta D\phi(\theta'^i)$ is equal to θ^i. We shall call ϕ an affine transformation if $\delta D\phi(\theta'^i_j) = \theta^i_j$. We denote the group of all affine transformations from M onto itself by $A(M)$. $A(M)$ is a subset of $H(M)$. We define the topology of $A(M)$ by the relative topology induced by the topology of $H(M)$, then $A(M)$ is a topological group.

Let R and R' be two n-dimensional Riemannian manifold defined by the fundamental quadratic tensor fields G and G' respectively.

A differentiable mapping ϕ from R into R' of rank n such that $\delta\phi$ maps the fundamental quadratic tensor fields G' of R' to the fundamental quadratic tensor field G of R is called an isometry. A Riemannian manifold is a metric space and an isometry is of course an metric preserving mapping.[8] Moreover, on a Riemannian manifold there is one and only one affine connection with the

[8] cf. S. Myers and N. Steenrod; The group of isometres of a Riemannian manifold, Ann. of Math. vol. 40 (1939).

following properties: 1) Its torsion tensor is zero; 2) The scalar product of two vectors remains unchanged by a parallel displacement along a curve.[9] An isometry from R into R' is an affine transformation from R into R' with respect to these connections. We denote the set of all isometries from R onto itself by $I(R)$. $I(R)$ is a topological group with the relative topology induced by $A(R)$. The point-wise convergence of a sequence of isometries and the convergence with respect to the topology of $I(R)$ are equivalent.

When an affine connection is defined on a n-dimensional differentiable manifold M by linear differential forms θ^i and θ^i_j $(i, j = 1, \ldots, n)$ on the bundle of frames B^*, we can define a Riemannian metric on B^* by the positive definite symmetric quadratic tensor field $\sum_{i=1}^{n}\theta^i \otimes \theta^i + \sum_{i,j=1}^{n}\theta^i_j \otimes \theta^i_j$, where $\otimes$ means tensor product of covariant vector fields. The induced isomorphism $D\phi$ of B^* onto itself by an affine transformation from M onto itself obviously preserves this tensor field.

LEMMA 2. *Let M and M' be two differentiable manifolds of dimension n with affine connections. Let $\{\phi_\nu\}$ be a sequence of affine transformations from M into M' which converges to a continuous mapping ϕ from M into M' with respect to the compact-open topology of $C(M, M')$. Then the image of a geodesic curve in M by ϕ is a geodesic curve in M' and ϕ has the first partial derivatives at each point in M. And the sequence $\{d\phi_\nu\}$ of mappings from the tangent bundle B of M into the tangent bundle B' of M' converges point-wise to the mapping $d\phi$ from B into B' induced by ϕ.*

Proof. We take an arbitrary point p in M, and regular neighbourhoods N and N_1 of p which have compact closure such as in lemma 1. For an arbitrary tangent vector L_p at p, there is a positive number λ such that the image of the geodesic curve $f'(t)$, $-\varepsilon < t < 1+\varepsilon$, with origin $f'(0) = p$ and tangent vector $L'_p = \lambda L_p$ at p is contained in N_1. This follows from the continuity of ϕ. By the convergence of the sequence $\{\phi_\nu\}$ with respect to the compact-open topology in $C(M, M')$, we can find a sufficiently large number ν_0 such that $\phi_\nu \circ f'(t) \in N_1$ for $\nu \geqq \nu_0$, $0 \leqq t \leqq 1$. By lemma 1 the set of tangent vectors $d\phi_\nu(L'_p)$ for $\nu \geqq \nu_0$ is contained in a compact set K in the tangent bundle B' of M'. Then we can choose a subsequence $d\phi_{\nu'}(L'_p)$ which converges to some tangent vector $L'_{\phi(p)}$ in

9) cf. S. Chern's, lecture note in 5).

K. And the geodesic curve $g'(t)$ with origin $\phi(p)$ and tangent vector $L'_{\phi(p)}$ at $\phi(p)$ can be extended over the values of the canonical parameter $-\varepsilon_1 < t < 1+\varepsilon_1$ for $\varepsilon_1 > 0$. From the continuity of η^t for each t, $0 \leqq t \leqq 1$, it is clear that $\phi_{\nu'} \circ f'(t)$ converges $g'(t)$. On the other hand $\phi_\nu \circ f'(t)$ converges to $\phi \circ f'(t)$, so we see that $g'(t) = \phi \circ f'(t)$. It is proved that the image of a geodesic curve $f(t) = f'(t/\lambda)$, $-\varepsilon/\lambda < t < 1/\lambda + \varepsilon/\lambda$, with origin p and tangent vector L_p at p is also a geodesic curve. If we take a normal coordinate system around p,[10] it is easily shown that ϕ has the first partial derivatives with respect to this coordinate system. From above considerations the sequence $\{d\phi_\nu(L'_p)\}$ is contained in a compact set K, and any convergent subsequence must converge to $L'_{\phi(p)}$, hence the original sequence $\{d\phi_\nu(L'_p)\}$ itself converges to $L'_{\phi(p)}$. As $d\phi_\nu(L_p) = d\phi_\nu(\lambda L'_p) = \lambda d\phi_\nu(L'_p)$, the sequence $\{d\phi_\nu(L_p)\}$ converges to $L_{\phi(p)} = \lambda L'_{\phi(p)}$. With respect to a coordinate system (y^i) around $\phi(p)$, $L_{\phi(p)} = \sum_{i=1}^n \left(\frac{dy^i(g(t))}{dt}\right)_{t=0} \cdot \left(\frac{\partial}{\partial y^i}\right)_{\phi(p)} = \sum_{i=1}^n \left(\frac{dy^i(\phi \circ f(t))}{dt}\right)_{t=0} \cdot \left(\frac{\partial}{\partial y^i}\right)_{\phi(p)} = d\phi(L_p)$, hence a mapping $d\phi$ is defined from the tangent bundle B into the tangent bundle B', and $\{d\phi_\nu(L_p)\}$ converges to $d\phi(L_p)$ at each element L_p in B q.e.d.

LEMMA 3. *Let M be a differentiable manifold with an affine connection. If a sequence $\{\phi_\nu\}$ of elements in $A(M)$ converges to a element ϕ in $H(M)$ with respect to the topology in $H(M)$, then ϕ is contained in $A(M)$.*

Proof. From our assumption, the sequence $\{\phi_\nu\}$ converges to ϕ with respect to the compact-open topology of $C(M, M')$ and also the sequence $\{\phi_\nu^{-1}\}$ converges to ϕ^{-1}. Then lemma 2 can be applied to these sequences. Moreover, as ϕ is a homeomorphism, the image of a geodesic curve which is not a point is also not a point and therefore $d\phi(L_p)$ is zero vector if and only if L_p is zero vector. Then ϕ induces a mapping $D\phi$ from the bundle of frames B^* of M onto itself and the sequence $\{D\phi_\nu\}$ converges point-wise to $D\phi$. As ϕ_ν is an affine transformation, $D\phi_\nu$ is an isometry of the Riemannian manifold B^* defined by the fundamental tensor field $\sum_{i=1}^n \theta^i \otimes \theta^i + \sum_{j,i=1}^n \theta^i_j \otimes \theta^i_j$. Therefore $D\phi$ is a metric preserving mapping. The situation is just same for $D\phi^{-1}$. Thus we see that $D\phi$ is a homeomorphism of B^* onto itself and consequently ϕ and ϕ^{-1} are of class C^1.

We can consider the sequence $\{D\phi_\nu\}$ is contained in $A(B^*)$, where the affine

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[10] cf. S. Chern's lecture note in [5].

connection is given by the Riemannian metric defined above. That the sequence $\{D\phi_\nu\}$ converges to $D\phi$ with respect to the topology of $H(B^*)$ follows from the fact that for the sequence of isometries of B^* the point-wise convergence and the convergence with respect to the topology of $H(B^*)$ are equivalent. Then above discussion can be applied to the sequence $\{D\phi_\nu\}$ and we can see $D\phi$ and $D\phi^{-1}$ are of class C^2. This means ϕ and ϕ^{-1} are of class C^2. Repeating the same argument for the sequence $\{D(D\phi_\nu)\}$ on the bundle of frames B^{**} of B^*, $\{D(D(D\phi_\nu))\}$ on the bundle of frames B^{***} of B^{**} and so on, we can conclude ϕ is an isomorphism of M and is an affine transformation.

3. Van Dantzig and van der Waerden proved the following theorem[11]: if a sequence of metric preserving homeomorphisms $\{\phi_\nu\}$ of a connected locally Euclidean metric space M satisfying the 2nd axiom of countability converges at a certain point in M, there can be found a subsequence $\{\phi_{\nu'}\}$ which converges to a certain metric preserving homeomorphism point-wise in M. We remark here that this theorem is true without any separability assumption in the case where M is a connected Riemannian manifold, and ϕ_ν are isometries of M. Namely we shall prove the following.

LEMMA 4. *If a sequence $\{\phi_\nu\}$ of isometries of a Riemannian manifold R converges at a certain point p_0 in R then there exists a suitable subsequence $\phi_{\nu'}$ which converges at each point of the connected component of R which contains p_0 such that $\phi_{\nu'}^{-1}$ converge at each point of the connected component of R which contains the limit point of $\phi_\nu(p_0)$. In particular if R is connected we can find the subsequence $\phi_{\nu'}$ which converges to an isometry ϕ such that the sequence $\phi_{\nu'}^{-1}$ also converges to ϕ^{-1}.*

Proof. We take a compact neighbourhood V of $\bar{p}_0$, where $\bar{p}_0$ is the limit point of $\phi_\nu(p_0)$. Then we can find an open neighbourhood U of p_0 in some coordinate neighbourhood of p_0 whose closure is compact such that $\phi_\nu(U) \subset V$ for $\nu \geqq \nu_0$, where ν_0 is some sufficiently large number. We also take a countable dense subset $\{q_\nu\}$ of U, then we can choose a subsequence $\{\phi_{\nu_{1i}}\}$ of $\{\phi_\nu\}$ such that $\phi_{\nu_{1i}}(q_1)$ converge to some point in V as i tends to infinity. Next we choose a subsequence $\{\phi_{\nu_{2i}}\}$ of $\{\phi_{\nu_{1i}}\}$ such that $\phi_{2i}(q_2)$ converge to some point of V, and so on. Then we see easily that the sequence $\{\phi_{\nu_{ii}}\}$ converges at each

[11] van Dantzig und van der Waerden, Über metrischen homogene Räume, Abh. Math. Sem. Hamburg, vol. 6 (1928).

point q_i and also at each point of U. We denote the sequence $\{\phi_{\nu_{ii}}\}$ by $\{\phi_{\nu'}\}$ and consider the maximal open set O containing U in which $\phi_{\nu'}$ converge at each point. If O did not contain the connected component K of R containing p_0, there would be a boundary point p_1 of $O_1 = O \cap K$. If the closure of the ε-neighbourhood $U(p_1, \varepsilon)$ of p_1 is compact we take a regular neighbourhood N of p_1 contained in $U(p_1, \varepsilon/4)$. As p_1 is a boundary point of O_1, $O_1 \cap N$ contains a point p_2 of O_1. By the convergence of $\phi_{\nu'}$ at p_2 we can find a sufficiently large number m such that $\rho(\phi_{\nu'}(p_2), \phi_m(p_2)) < \varepsilon/4$ for $\nu' \geqq m$, where ρ denotes the Riemann metric in R. Now it is shown that $\phi_{\nu'}(N) \subset U(\phi_m(p_1), \varepsilon)$ for $\nu' \geqq m$. For $\phi_{\nu'}(N) \subset U(\phi_{\nu'}(p_1), \varepsilon/4)$ and if $x \in U(\phi_{\nu'}(p_1), \varepsilon/4)$ then $\rho(x, \phi_{\nu'}(p_1)) < \varepsilon/4$ and

$$\begin{aligned}\rho(x, \phi_m(p_1)) &\leqq \rho(x, \phi_{\nu'}(p_1)) + \rho(\phi_{\nu'}(p_1), \phi_{\nu'}(p_2)) \\ &\qquad + \rho(\phi_{\nu'}(p_2), \phi_m(p_2)) + \rho(\phi_m(p_2), \phi_m(p_1)) \\ &= \rho(x, \phi_{\nu'}(p_1)) + \rho(p_1, p_2) + \rho(\phi_{\nu'}(p_2), \phi_m(p_2)) \\ &\qquad + \rho(p_2, p_1) < \varepsilon.\end{aligned}$$

As $\phi_m(\overline{U(p_1, \varepsilon)}) = \overline{U(\phi_m(p_1), \varepsilon)}$ is compact, $\phi_{\nu'}(N)$ is contained in a fixed compact set for all $\nu' \geqq m$. Hence we can find a subsequence $\{\phi_{\nu''}\}$ of $\{\phi_{\nu'}\}$ which converges at each point of N as the same way as we have chosen the sequence $\{\phi_{\nu'}\}$ for U. Next we take a regular neighbourhood N_1 of p contained in N then for any two points p and q of N_1 there exists a geodesic curve contained in N joining p and q. If we take arbitrary points p in N_1 and q in $N_1 \cap O_1$, we can join p and q with a geodesic curve $f(t)$, $-\varepsilon < t < 1 + \varepsilon$ contained in N, whose tangent vector at origin $f(0) = q$ is equal to L_q and the tangent vector at $f(1) = p$ is equal to L_p. From the lemma 2 $g(t) = \lim_{\nu'' \to \infty} \phi_{\nu''} \circ f(t)$ is a geodesic curve and $\{d\phi_{\nu'}(L_q)\}$ converges to the tangent vector $L_{g(0)}$ of the geodesic curve $g(t)$ at $g(0)$. Then from the continuity of the solution of a system of differential equations with respect to initial conditions $\{\phi_{\nu'} \circ f(1)\}$ converges to $g(1)$. Thus $\{\phi_{\nu'}\}$ converges at each point in the open set $N_1 \cup O$ $(\neq \phi)$ and this is a contradiction. Hence the convergence of $\{\phi_{\nu'}\}$ in K is proved. It is clear that $\{\phi_{\nu'}^{-1}\}$ converges at $\bar{p}_0$, so we can find as above a suitable subsequence $\{\phi_{\nu'''}\}$ of $\{\phi_{\nu'}\}$ which converges in K to a continuous mapping ϕ from K into R such that $\{\phi_{\nu'''}^{-1}\}$ converges in L to a continuous mapping ψ from L into R, where L is the connected component of R containing $\bar{p}_0$.

Now we assume that R is connected, then we can easily see that ϕ and ψ are homeomorphisms of R and each one is the inverse mapping of the other. The differentiability of ϕ and ψ follows from lemma 3.

4. LEMMA 5. *Let $\{\phi_\nu\}$ be a sequence of elements in $A(M)$. If the sequence $\{D\phi_\nu\}$ converges at a certain frame $(L^1_{p_0}, \ldots, L^n_{p_0})$ at p_0, then there exists a subsequence $\phi_{\nu'}$ which converges to an affine transformation ϕ with respect to the topology of $A(M)$.*

Proof. In B^* there are at most two connected components of B^* each of which contains $(L^1_{p_0}, \ldots, L^n_{p_0})$ or $(-L^1_{p_0}, L^2_{p_0}, \ldots, L^n_{p_0})$. As it is clear that $\{D\phi_\nu\}$ converges also at $(-L^1_{p_0}, L^2_{p_0}, \ldots, L^n_{p_0})$, from the proof of lemma 4 we see that there exists a subsequence of isometries $\{D\phi_{\nu'}\}$ which converges to an isometry ϕ^* at each point in B^* such that $\{D\phi_{\nu'}^{-1}\}$ also converges to ϕ^{*-1}. As ϕ^* is a fibre preserving isomorphic mapping, ϕ^* induces an isomorphism ϕ of M. It is easily verified that $\{\phi_{\nu'}\}$ converges to ϕ with respect to the topology of $H(M)$ from the fact that point-wise convergence of isometries is equivalent to the convergence with respect to the topology of $H(B^*)$. Next we shall prove that ϕ is an affine transformation, i.e. $D\phi$ leaves the linear differential forms θ^i_j invariant. For this purpose it is sufficient to show that $D\phi = \phi^*$, because according to lemma 2 $\{\delta D\phi_{\nu'}\}$ converges to $\delta\phi^*$ at each point in B^* and so ϕ^* leaves θ^i_j invariant.

At any point in B^* there exist a neighbourhood V with compact closure which is contained in some coordinate neighbourhood and an open set U in some coordinate neighbourhood such that if ν' is larger than a sufficiently large number ν_0, $D\phi_{\nu'}(\overline{V}) \subset U$. Let (u^i, X^k_j) and $(\overline{u}^i, \overline{X}^k_j)$ be local coordinate systems in V and U respectively. Then with respect to these coordinate systems the mapping $D\phi_{\nu'}$ are expressed as follows

$$\overline{u}^i(D\phi_{\nu'}(u, X)) = \overline{u}^i(\phi_{\nu'}(u))$$
$$\overline{X}^k_j(D\phi_{\nu'}(u, X)) = \sum_{l=1}^n X^l_j \frac{\partial \overline{u}^k(\phi_{\nu'}(u))}{\partial u^l}.$$

$\{D\phi_{\nu'}\}$ converges uniformly to ϕ^* on the compact $\overline{V}$ with respect to the metric. And the metric topology and the manifold topology is equivalent. Therefore the functions $\overline{u}^i(D\phi_{\nu'}(u, X))$ and $\overline{X}^k_j(D\phi_{\nu'}(u, X))$ converge uniformly to $\overline{u}^i(\phi(u)) = \overline{u}^i(\phi^*(u, X))$ and $\overline{X}^k_j(\phi^*(u, X))$ respectively. According to a theorem of differential calculus from the facts that

1) $\lim_{\nu'\to\infty}\bar{u}^i(\phi_{\nu'}(u))=\bar{u}^i(\phi(u))$,

2) $\dfrac{\partial\bar{u}^k(\phi_{\nu'}(u))}{\partial u^l}$ converge uniformly, and

3) $\dfrac{\partial\bar{u}^k}{\partial u^l}(\phi_{\nu'}(u))$ are continuous,

we can conclude that $\lim\dfrac{\partial\bar{u}^k}{\partial u^l}(\phi_{\nu'}(u))=\dfrac{\partial\bar{u}^k}{\partial u^l}(\phi(u))$, hence $D\phi=\lim_{\nu'\to\infty}D\phi_{\nu'}=\phi^*$.

5. Before proceeding to prove the theorem, it is convenient to show that in a certain neighbourhood of the identity of the group $A(M)$ the 2nd axiom of countability holds. If we take the neighbourhood W of the identity which is given by $W=\{\phi\,;\phi\in A(M),\ \phi(K)\subset U,\ \phi^{-1}(K)\subset U\}$, where a compact set K containing an open set V and an open set U whose closure is compact are contained in a certain coordinate neighbourhood and $V\subset K\subset U$, then in this neighbourhood W the axiom always holds with respect to the compact-open topology in $C(K, U)$, because K and U are separable. Therefore we have only to prove that the new topology in W coincides with the relative topology induced by the topology of $A(M)$. We shall now show that if a sequence $\{\phi_\nu\}$ in W converges to ϕ in W with respect to the new topology, then also with respect to the topology of $A(M)$. If $\{\phi_\nu\}$ did not converge to ϕ with respect to the topology of $A(M)$ there would exist a neighbourhood W_1 of ϕ in $A(M)$ and a subsequence $\{\phi_{\nu'}\}$ such that $\phi_{\nu'}\notin W_1$. But from lemma 2 $\{d\phi_{\nu'}\}$ converge to $d\phi$ because $\phi_{\nu'}$ satisfies the assumption of lemma 2. Moreover, as ϕ is a homeomorphism we can see $\{D\phi_\nu\}$ converge to $D\phi$ at each point in the same fashion as in the first part of the proof of lemma 3. Hence by lemma 5 we can find a subsequence $\{\phi_{\nu''}\}$ of $\{\phi_{\nu'}\}$ which converges to an affine transformation ψ with respect to the topology of $A(M)$. From the fact that an affine transformation which leaves a non-empty open set in M point-wise fixed must be the identily transformation,[12] it is clear that $\phi=\psi$, because $\phi(p)=\psi'(p)$ for any point p in K. Namely $\{\phi_{\nu''}\}$ converges to ϕ in $A(M)$, which is a contradiction. From this the equivalence of the two topologies is easily seen.

6. *Proof of the theorem*

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12) cf. 1).

In virtue of a theorem of S. Bochner, D. Montgomery[13] and M. Kuranishi[14] we have only to prove that $A(M)$ is locally compact and that any element of $A(M)$ which leaves a non-emply open set in M point-wise fixed must be the identity. The latter part is proved in K. Nomizu's paper as already indicated in the preceding section.

We take a neighbourhood W of the identity which we deal in the section 5 and a point p_0 in V. The induced mapping $D\phi$ is given at p_0 by

$$D\phi_{p_0}\left(\left(\frac{\partial}{\partial u^1}\right)_{p_0}, \dots, \left(\frac{\partial}{\partial u^n}\right)_{p_0}\right) = \left(\sum_{j=1}^{n}\left(\frac{\partial u^j(\phi)}{\partial u^i}\right)_{p_0}\left(\frac{\partial}{\partial u^j}\right)_{\phi(p_0)}, \dots\right)$$

where (u^i) is a local coordinate system in U. If we consider the following mapping

$$D_{p_0} : \phi \to \left(\left(\frac{\partial u^j(\phi)}{\partial u^i}\right)_{p_0}\right)$$

from W into a general linear group, it is a continuous mapping from the proof of lemma 3. Then we take a neighbourhood W' of the identity in $A(M)$ such that W' and its closure are contained in W and the image $D_{p_0}(\overline{W}')$ is contained in a compact set. We have only to prove that $\overline{W}'$ is sequentially compact. Let ϕ_ν be an arbitrary sequence in $\overline{W}'$. As $\overline{U}$ is compact there exists a subsequence $\{\phi_{\nu'}\}$ which converges at p. If we take again a suitable subsequence $\{\phi_{\nu''}\}$ of $\{\phi_{\nu'}\}$, $D_{p_0}(\phi_{\nu''})$ converge to a non-singular matrix. Then $\{D\phi_{\nu''}\}$ converges at $\left(\left(\frac{\partial}{\partial u^1}\right)_{p_0}, \dots, \left(\frac{\partial}{\partial u^n}\right)_{p_0}\right)$ in B.* From lemma 5 we get a subsquence $\{\phi_{\nu'''}\}$ which converges to an element ϕ in $A(M)$, then ϕ is contained in $\overline{W}'$. Thus the sequential compactness of $\overline{W}'$ is proved, q.e.d.

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13) S. Bochner and D. Montgomery; Locally compact groups of differentiable transformations, Ann. of Math. vol. 47 (1946).

14) M. Kuranishi; On conditions of differentiability of locally compact groups, Nagoya Math. J. vol. 1 (1950).