ON SOME ASYMPTOTIC PROPERTIES OF POISSON PROCESS

TAKEYUKI HIDA

The Poisson process $X(t, \omega)$,[1] ($\omega \in \Omega$, $0 \leqq t < \infty$), as is well-known, is a temporally and spatially homogeneous Markoff process satisfying

(1) $$X(0, \omega) = 0 \text{ and } X(t, \omega) = \text{integer} \geqq 0 \text{ for every } \omega \in \Omega,$$

(2) $$Pr\{X(t, \omega) - X(t', \omega) \geqq k\} = \sum_{i=k}^{\infty} \frac{\{\lambda(t-t')\}^i}{i\,!} e^{-\lambda(t-t')} \quad \text{for} \quad t > t',$$

where k is a non-negative integer and λ is a positive constant. In this note we consider the random variable $L_m(\omega)$ which denotes the length of t-interval such that $X(t, \omega) = m$ ($m = 0, 1, 2, \ldots$) and some of other properties concerning them.

§ 1. The known results on L_m.

Definition. We define $L_m(\omega)$, the function of m and ω, as follows,

$$L_m(\omega) = t_{m+1}(\omega) - t_m(\omega),$$

where

$$t_m(\omega) = \text{Min}\,\{\tau\,;\ X(\tau, \omega) = m\}.$$

This $t_m(\omega)$ exists almost certainly by the right continuity property of Poisson process, and furthermore it is clear that $t_m(\omega)$ is measurable. Thus $L_m(\omega)$ becomes a non-negative random variable.

THEOREM 1. *$L_0, L_1, \ldots, L_m, \ldots$ are mutually independent random variables with a common distribution function $F(l)$,*
where

(3) $$F(l) = \begin{cases} 1 - e^{-\lambda l} & \text{if} \quad l \geqq 0, \\ 0 & \textit{otherwise}. \end{cases}$$

Furthermore

(4) $$E(L_m)^{2)} = \frac{1}{\lambda}$$

(5) $$V(L_m)^{2)} = \frac{1}{\lambda^2} \qquad m = 0, 1, 2, \ldots$$

This theorem was already suggested by P. Levy [2][3] and a rigorous proof was

Received March 19, 1953.

[1] ω denotes the probability parameter.

[2] $E(\ldots)$ and $V(\ldots)$ denote the mean and the variance respectively.

[3] Numbers in brackets refer to the bibliography at the end of this note.

given by T. Nishida [1]. From this theorem we can easily conclude the following corollaries.

COROLLARY 1. *The characteristic function* $\varphi_L(z)$ *of* L_m, *and therefore that of* F, *is* $\frac{\lambda e^{iz}}{\lambda - iz}$.

COROLLARY 2. *The probability* $L_m \geqq l (\geqq l_0)$ *under the assumption* $L_m \geqq l_0$ *is* $e^{-\lambda(l-l_0)}$ *and its conditional expectation is* $\frac{1}{\lambda} + l_0$.

§ 2. The definitions and the behaviours of M_n and m_n

Definition. Let M_n *be defined by*

$$M_n(\omega) = \max\{L_0(\omega), L_1(\omega), \ldots, L_{n-1}(\omega)\}.$$

$M_n(\omega)$ is monotone non-decreasing with respect to n for every ω. The probability law of $M_n(\omega)$ is easily obtained as follows:

$$
\begin{aligned}
(6)\qquad & Pr\{M_n < x\} \qquad (= Pr\{M_n \leqq x\}) \\
& = Pr\{L_0 < x,\ L_1 < x, \ldots, L_{n-1} < x\} \\
& = Pr\{L_0 < x\} Pr\{L_1 < x\} \ldots Pr\{L_{n-1} < x\} \\
& \text{(as } L_m \text{ is mutually independent)} \\
& = (1 - e^{-\lambda x})^n.
\end{aligned}
$$

THEOREM 2. $E(M_n) = O(\log n)$.

Proof. We have

$$
\begin{aligned}
E(M_n) &= n\lambda \int_0^\infty x e^{-\lambda x}(1 - e^{-\lambda x})^{n-1} dx \\
&= -\frac{1}{\lambda} n \int_0^\infty \log(1 - e^{-y}) e^{-ny} dy \\
&= -\frac{1}{\lambda} n \int_0^\varepsilon \quad - \frac{1}{\lambda} n \int_\varepsilon^\infty
\end{aligned}
$$

where ε is arbitrary small such that $1 - e^{-y} \sim y$ when $0 \leqq y \leqq \varepsilon$. The second term is $o(\log n)$ when $n \to \infty$, and

$$
\begin{aligned}
-n \int_0^\varepsilon \log y e^{-ny} dy &= \log n \int_0^{n\varepsilon} \left(1 - \frac{\log z}{\log n}\right) e^{-z} dz \\
&= O(\log n).
\end{aligned}
$$

Hence we can conclude $E(M_n) = O(\log n)$.

THEOREM 3. $\lambda M_n / \log n$ *converges in law to the random variable* Y *which takes the value* 1 *with probability* 1.

Proof. We have

$$
(7)\qquad Pr\{\lambda M_n / \log n < x\} = (1 - 1/n^x)^n \to \begin{cases} 1 & \text{if} \quad x > 1 \\ 0 & \text{if} \quad 1 > x \geqq 0, \end{cases}
$$

as n tends to ∞.

More precisely we may prove

THEOREM 4. *If* $0<\alpha<1$, *then*

$$Pr\{\liminf_{n\to\infty} \lambda M_n/\alpha \log n \geqq 1\} = 1. \tag{8}$$

In order to prove the theorem above we need the following lemma.

LEMMA. *The series*

$$\sum_{n=1}^{\infty}(1-1/n^{\alpha})^n$$

is convergent when $0<\alpha<1$.

Proof of the Lemma. It is sufficient to prove $u_n/v_n \to 0$ $(n\to\infty)$, where

$$u_n = (1-1/n^{\alpha})^n, \quad v_n = 1/n^2.$$

Let

$$f(x) \equiv x^2(1-1/x^{\alpha})^x.$$

Then

$$\begin{aligned}\log f(x) &= 2\log x + x\log(1-1/x^{\alpha})\\ &= \frac{(2\log x)/x + \log(1-1/x^{\alpha})}{1/x}\\ &\to \frac{2((1-\log x)/x^2) + \alpha x^{\alpha-1}/(1-x^{-\alpha})}{-1/x^2} \qquad (x\to\infty)\\ &= 2(\log x - 1) - \alpha x^{-\alpha+1} - 1\\ &= \left(\frac{2(\log x-1)}{x/(x^{\alpha}-1)} - \alpha\right)\cdot\frac{x}{x^{\alpha}-1}.\end{aligned}$$

Here

$$2\,\frac{\log x - 1}{x/(x^{\alpha}-1)} \to 0, \quad \frac{x}{x^{\alpha}-1} \to \infty. \quad (x\to\infty)$$

Hence $\log f(x) \to -\infty$ and therefore $f(x)\to 0$ when $x\to\infty$. Thus $u_n/v_n \to 0$ when $n\to\infty$.

Proof of Theorem 4. We have, by (6) and by the Lemma above,

$$\sum_{n=1}^{\infty} Pr\left\{M_n < \frac{\alpha}{\lambda}\log n\right\} = \sum_{n=1}^{\infty}\left(1-\frac{1}{n^{\alpha}}\right)^n < \infty.$$

Therefore, by the Borel-Cantelli's Lemma,

$$Pr\{\liminf_{n\to\infty} E_n^c\} = 1,$$

where

$$E_n = \left\{\omega;\ M_n(\omega) < \frac{\alpha}{\lambda}\log n\right\}.$$

On the other hand

$$\liminf_{n\to\infty} E_n^c \subseteqq \left\{\omega;\ \liminf_{n\to\infty} M_n(\omega)\Big/\frac{\alpha}{\lambda}\log n \geqq 1\right\}. \tag{9}$$

This shows that (8) is valid.

Definition. Let $m_n(\omega)$ be defined by

$$m_n(\omega) = \mathrm{Min}\,\{L_0(\omega),\ L_1(\omega),\ \ldots,\ L_{n-1}(\omega)\}.$$

$m_n(\omega)$ is monotone non-increasing with respect to n for every ω. The law of m_n is calculated in the same way as M_n:

$$\begin{aligned} Pr\{m_n > x\} &\quad (= Pr\{m_n \geqq x\}) \\ &= Pr\{L_0 > x,\ L_1 > x,\ \ldots,\ L_{n-1} > x\} \\ &= (e^{-\lambda x})^n = e^{-\lambda n x}, \end{aligned} \tag{10}$$

and hence

$$Pr\{m_n < x\} = 1 - e^{-\lambda n x}.$$

THEOREM 5. *If $\beta > 1$, then*

$$Pr\{\limsup_{n\to\infty} \lambda m_n/\beta n^{-1}\log n \geqq 1\} = 0. \tag{11}$$

Proof. We have

$$Pr\{m_n \geqq \beta \log n/\lambda\, n\} = 1/n^{\beta} \quad \text{and} \quad \sum_{n=1}^{\infty} n^{-\beta} < \infty.$$

Thus, by the Borel-Cantelli's Lemma,

$$Pr\{\limsup_{n\to\infty} F_n\} = 0,$$

where

$$F_n = \{\omega;\ m_n(\omega) \geqq \beta \log n/\lambda\, n\}.$$

On the other hand

$$\limsup_{n\to\infty} F_n \supseteqq \{\omega;\ \limsup_{n\to\infty} \lambda\, m_n(\omega)/\beta\, n^{-1}\log n \geqq 1\}.$$

Thus we obtain (11).

§ 3. Asymptoric properties of Z_n

Let $Z_n(\omega)$ be defined by

$$Z_n(\omega) = (L_0(\omega) + L_1(\omega) + \ldots + L_{n-1}(\omega))/M_n(\omega).$$

Remembering

$$Pr\{L_0 = M_n\} = Pr\{L_1 = M_n\} = \ldots = Pr\{L_{n-1} = M_n\} = 1/n,$$[4]

we see that Z_n has the first and the second (absolute) moments:

$$E(Z_n) = \int_0^{\infty} dx_1 \cdot n \Big(\overbrace{\int_0^{x_1}\ldots\int_0^{x_1}}^{n-1\ \text{ple}} \frac{x_1 + x_2 + \ldots + x_n}{x_1} \times \\ \times \lambda^n e^{-\lambda x_1} e^{-\lambda x_2} \ldots e^{-\lambda x_n} dx_2 \ldots dx_n\Big). \tag{12}$$

[4] See e.g. D. A. Darling [3].

$$(13) \qquad E(Z_n^2)=\int_0^\infty dx_1\cdot n\Big(\overbrace{\int_0^{x_1}\cdots\int_0^{x_1}}^{n-1\text{ ple}}\Big(\frac{x_1+x_2+\cdots+x_n}{x_1}\Big)^2\times \times\lambda^n e^{-\lambda x_1}e^{-\lambda x_2}\cdots e^{-\lambda x_n}dx_2\cdots dx_n\Big).$$

The characteristic function $\varphi_{Z_n}(z)$ of Z_n satisfies

$$(14) \qquad \varphi_{Z_n}(z)=E(e^{iz}(L_0+L_1+\cdots+L_{n-1})/M_n)$$

$$=n\lambda^n\int_0^\infty dx_1\Big(\overbrace{\int_0^{x_1}\cdots\int_0^{x_1}}^{n-1\text{ ple}}e^{iz\frac{x_1+x_2+\cdots+x_n}{x_1}}\times \times e^{-\lambda(x_1+x_2+\cdots+x_n)}dx_2\cdots dx_n\Big)$$

$$=n\lambda^n\int_0^\infty e^{iz-\lambda x_1}dx_1\Big(\int_0^{x_1}e^{iz\frac{x}{x_1}-\lambda x}dx\Big)^{n-1}$$

(as $L_1, L_2, \ldots, L_{n-1}$ are mutually independent)

$$=n\lambda^n\int_0^\infty e^{iz-\lambda x_1}x_1^{n-1}\Big(\frac{e^{iz-\lambda x_1}-1}{iz-\lambda x_1}\Big)^{n-1}dx_1$$

$$=n\lambda^n e^{iz}\int_0^\infty e^{-\lambda x}x^{n-1}\Big(\frac{e^{iz-\lambda x}-1}{iz-\lambda x}\Big)^{n-1}dx.$$

$$(15) \qquad \frac{1}{i}\frac{d\varphi_{Z_n}(z)}{dz}=n\lambda^n e^{iz}\int_0^\infty e^{-\lambda x}x^{n-1}\Big(\frac{e^{iz-\lambda x}-1}{iz-\lambda x}\Big)^{n-1}dx$$

$$+n(n-1)\lambda^n e^{iz}\int_0^\infty e^{-\lambda x}x^{n-1}\Big(\frac{e^{iz-\lambda x}-1}{iz-\lambda x}\Big)^{n-2}\times \times\frac{(iz-\lambda x)e^{iz-\lambda x}-e^{iz-\lambda x}+1}{(iz-\lambda x)^2}dx.$$

$$(16) \qquad \Big(\frac{1}{i}\Big)^2\frac{d^2\varphi_{Z_n}(z)}{dz^2}=n\lambda^n e^{iz}\int_0^\infty e^{-\lambda x}x^{n-1}\Big(\frac{e^{iz-\lambda x}-1}{iz-\lambda x}\Big)^{n-1}dx$$

$$+2n(n-1)\lambda^n e^{iz}\int_0^\infty e^{-\lambda x}x^{n-1}\Big(\frac{e^{iz-\lambda x}-1}{iz-\lambda x}\Big)^{n-2} \times\frac{(iz-\lambda x)e^{iz-\lambda x}-e^{iz-\lambda x}+1}{(iz-\lambda x)^2}dx$$

$$+n(n-1)(n-2)\lambda^n e^{iz}\int_0^\infty e^{-\lambda x}x^{n-1}\Big(\frac{e^{iz-\lambda x}-1}{iz-\lambda x}\Big)^{n-3} \times\frac{\{(iz-\lambda x)e^{iz-\lambda x}-e^{iz-\lambda x}+1\}^2}{(iz-\lambda x)^4}dx$$

$$+n(n-1)\lambda^n e^{iz}\int_0^\infty e^{-\lambda x}x^{n-1}\Big(\frac{e^{iz-\lambda x}-1}{iz-\lambda x}\Big)^{n-2} \times\frac{(iz-\lambda x)^2e^{iz-\lambda x}-2\{(iz-\lambda x)e^{iz-\lambda x}-e^{iz-\lambda x}+1\}}{(iz-\lambda x)^3}dx.$$

The differentiations in (15) and (16) are possible since Z_n has the first and the second moments.

THEOREM 6. *Z_n has the first and second absolute moments. And if n is* *sufficiently large, the mean and the standard deviation of Z_n are both of order* $n/\log n$.

Proof. The first half of the theorem is proved above. Thus

$$E(Z_n)=\frac{1}{i}\left(\frac{d\varphi_{Z_n}(z)}{dz}\right)_{z=0}=n\lambda^n\int_0^\infty e^{-\lambda x}x^{n-1}\left(\frac{e^{-\lambda x}-1}{-\lambda x}\right)^{n-1}dx \tag{17}$$

$$+n(n-1)\lambda^n\int_0^\infty e^{-\lambda x}x^{n-1}\left(\frac{1-e^{-\lambda x}}{\lambda x}\right)^{n-2}\frac{-\lambda xe^{-\lambda x}-e^{-\lambda x}+1}{\lambda^2x^2}dx$$

$$=\varphi_{Z_n}(0)-n(n-1)\lambda\int_0^\infty e^{-2\lambda x}(1-e^{-\lambda x})^{n-2}dx$$

$$+n(n-1)\int_0^\infty x^{-1}e^{-\lambda x}(1-e^{-\lambda x})^{n-1}dx.$$

Here

$$\varphi_{Z_n}(0)=1,$$

$$n(n-1)\lambda\int_0^\infty e^{-2\lambda x}(1-e^{-\lambda x})^{n-2}dx$$

$$=n(n-1)\lambda\int_0^\infty e^{-\lambda x}(1-e^{-\lambda x})^{n-2}dx-n(n-1)\lambda\int_0^\infty e^{-\lambda x}(1-e^{-\lambda x})^{n-1}dx$$

$$=n(n-1)\lambda\left(\frac{1}{\lambda(n-1)}-\frac{1}{\lambda n}\right)=1.$$

The last term of (19), $I_n=\int_0^\infty n(n-1)x^{-1}e^{-\lambda x}(1-e^{-\lambda x})^{n-1}dx$, is of order $n/\log n$. It is proved as follows. We have

$$I_n=\int_0^\infty\frac{n(n-1)e^{-\lambda x}}{x}(1-e^{-\lambda x})^{n-1}dx=n(n-1)\int_0^\infty\frac{e^{-ny}}{-\log(1-e^{-y})}dy.$$

Let a be sufficiently small such that $e^{-t}\sim 1-t$ when $0<t<a$, and let K be sufficiently large such that $\log(1-e^{-t})\sim e^{-t}$ when $t>K$. Then

$$\frac{I_n}{n(n-1)}=\int_0^a+\int_a^K+\int_K^\infty \tag{18}$$

$$\int_0^a\frac{e^{-nt}}{-\log(1-e^{-t})}dt\sim\int_0^a\frac{e^{-nt}}{-\log t}dt$$

$$=\frac{1}{n\log n}\int_0^{na}\frac{e^{-y}}{1-\log y/\log n}dy\qquad(nt=y)$$

$$<\frac{1}{n\log n}\int_0^\infty\frac{e^{-y}}{1-\log y/\log n}dy=O\left(\frac{1}{n\log n}\right),$$

since

$$\lim_{n\to\infty}\int_0^\infty\frac{e^{-y}}{1-\log y/\log n}dy=\int_0^\infty\lim_{n\to\infty}\frac{e^{-y}}{1-\log y/\log n}dy=\int_0^\infty e^{-y}dy=1.$$

On the other hand, we have

$$\frac{1}{n\log n}\int_0^{na}\frac{e^{-y}}{1-\log y/\log n}dy\geqq\frac{1}{n\log n}\int_1^{na}\frac{e^{-y}}{1-\log y/\log n}dy$$

$$\geqq \frac{1}{n\log n}\int_1^{na} e^{-y}dy = \frac{1}{n\log n}(e^{-1}-e^{-na}).$$

Hence

$$\int_0^a \frac{e^{-nt}}{-\log(1-e^{-t})}dt = O(1/n\log n),$$

$$\int_a^K \frac{e^{-nt}}{-\log(1-e^{-t})}dt = C(e^{-na}-e^{-nK}) = o(e^{-n}),$$

where $0<1/-\log(1-e^{-a}) \leqq C \leqq 1/-\log(1-e^{-K})<\infty$.

Therefore

$$\int_K^\infty \frac{e^{-nt}}{-\log(1-e^{-t})}dt \sim \int_K^\infty e^{-(n-1)t}dt = o(e^{-n}),$$

and thus

$$I_n = O(n/\log n).$$

This proves

$$E(Z_n) = O(n/\log n).$$

Similarly we have

$$E(Z_n^2) = \left(\frac{1}{i}\right)^2\left(\frac{d^2\varphi_{Z_n}(z)}{dz^2}\right)_{z=0}$$

$$= \varphi_{Z_n}(0) + 2n(n-1)\lambda^2\int_0^\infty e^{-\lambda x}x^{n-1}\left(\frac{1-e^{-\lambda x}}{\lambda x}\right)^{n-2}\frac{1-e^{-\lambda x}-\lambda x e^{-\lambda x}}{\lambda^2 x^2}dx$$

$$+ n(n-1)(n-2)\lambda^n\int_0^\infty e^{-\lambda x}x^{n-1}\left(\frac{1-e^{-\lambda x}}{\lambda x}\right)^{n-3}\frac{(-\lambda x e^{-\lambda x}-e^{-\lambda x}+1)^2}{\lambda^4 x^4}dx$$

$$= O(n/\log n) + \frac{n(n-1)(n-2)}{\lambda}\int_0^\infty \frac{e^{-\lambda x}}{x^2}(1-e^{-\lambda x})^{n-3}(1-e^{-\lambda x}-\lambda x e^{-\lambda x})^2 dx$$

$$- \frac{n(n-1)}{\lambda}\int_0^\infty x^{-2}e^{-\lambda x}(1-e^{-\lambda x})^{n-2}\{\lambda^2 x^2 e^{-\lambda x} - 2(1-e^{-\lambda x}-\lambda x e^{-\lambda x})\}dx$$

$$= O(n/\log n) + n(n-1)(n-2)\lambda\int_0^\infty e^{-3\lambda x}(1-e^{-\lambda x})^{n-3}dx$$

$$- 2n(n-1)(n-2)\int_0^\infty x^{-1}e^{-2\lambda x}(1-e^{-\lambda x})^{n-2}dx$$

$$+ \frac{n(n-1)(n-2)}{\lambda}\int_0^\infty x^{-2}e^{-\lambda x}(1-e^{-\lambda x})^{n-1}dx$$

$$- n(n-1)\lambda\int_0^\infty e^{-2\lambda x}(1-e^{-\lambda x})^{n-2}dx - 2n(n-1)\int_0^\infty x^{-1}e^{-2\lambda x}(1-e^{-\lambda x})^{n-2}dx$$

$$+ \frac{2n(n-1)}{\lambda}\int_0^\infty x^{-2}e^{-\lambda x}(1-e^{-\lambda x})^{n-1}dx.$$

Since

$$n(n-1)(n-2)\lambda\int_0^\infty e^{-3\lambda x}(1-e^{-\lambda x})^{n-2}dx = 2,$$

$$n(n-1)\lambda\int_0^\infty e^{-2\lambda x}(1-e^{-\lambda x})^{n-2}dx = 1,$$

we obtain

$$E(Z_n^2) = O(n/\log n) + 2\,n(n-1)^2\frac{1}{\lambda}\int_0^\infty x^{-2}e^{-\lambda x}(1-e^{-\lambda x})^{n-1}dx$$
$$-2\,n(n-1)^2\int_0^\infty x^{-1}e^{-2\lambda x}(1-e^{-\lambda x})^{n-2}dx.$$

Thus we have

$$(19)\qquad \begin{aligned} V(Z_n) &= E(Z_n^2) - (E(Z_n))^2 \\ &= O(n/\log n) + \frac{2\,n(n-1)^2}{\lambda}\int_0^\infty x^{-2}e^{-\lambda x}(1-e^{-\lambda x})^{n-1}dx \\ &\quad -2\,n(n-1)^2\int_0^\infty x^{-1}e^{-2\lambda x}(1-e^{-\lambda x})^{n-2}dx \\ &\quad -\{n(n-1)\int_0^\infty x^{-1}e^{-\lambda x}(1-e^{-\lambda x})^{n-1}dx + o(n/\log n)\}^2 \\ &\geqq O(n/\log n) + \frac{n(n-1)^2}{\lambda}\int_0^\infty x^{-2}e^{-\lambda x}(1-e^{-\lambda x})^{n-1}dx \\ &\quad -2\,n(n-1)^2\int_0^\infty x^{-1}e^{-2\lambda x}(1-e^{-\lambda x})^{n-2}dx, \end{aligned}$$

since, by the Schwarz's inequality,

$$\left\{\int_0^\infty x^{-1}e^{-\lambda x}(1-e^{-\lambda x})^{n-1}dx\right\}^2 \leqq \int_0^\infty x^{-2}e^{-\lambda x}(1-e^{-\lambda x})^{n-1}dx\int_0^\infty e^{-\lambda x}(1-e^{-\lambda x})^{n-1}dx$$
$$= \frac{1}{n\lambda}\int_0^\infty x^{-2}e^{-\lambda x}(1-e^{-\lambda x})^{n-1}dx.$$

Similarly as in the proof of (18), we obtain

$$(20)\qquad \frac{n(n-1)^2}{\lambda}\int_0^\infty \frac{e^{-\lambda x}}{x^2}(1-e^{-\lambda x})^{n-1}dx = O(n^2/(\log n)^2).$$

There exists a large number M such that

$$(21)\qquad \frac{e^{-2\lambda x}}{x}(1-e^{-\lambda x})^{n-2} < \frac{e^{-\lambda x}}{x^2}(1-e^{-\lambda x})^{n-1}$$

whenever $x > M$. This fact implies that $J_n = 2\,n(n-1)^2\int_0^\infty x^{-1}e^{-2\lambda x}(1-e^{-\lambda x})^{n-2}dx$ is, when $n \to \infty$, negligible in the formula (19).
Therefore $E(Z_n^2)$ and $V(Z_n)$ are of order $n^2/(\log n)^2$.

Bibliography

[1] T. Nishida, On the inverse function of Poisson process. Mathematica Japonicae. Vol. II No. 3. (1952) pp. 135-142.

[2] P. Lévy, Processus stochastiques et mouvement brownien. (1948) p. 182.

[3] D. A. Darling, The influence of the maximum term in the addition of independent random variables. Transactions of the American Mathematical Society. Vol. **73**. No. 1 (1952) pp. 95-107.

[4] T. Ōta, On some properties of Poisson process I. Bulletin of the Aichi Gakugei University. Vol. **2**. (1953). pp. 1-6.

Mathematical Institute,
Aichi Gakugei University