NOTE ON THE HARMONIC MEASURE OF THE ACCESSIBLE BOUNDARY OF A COVERING RIEMANN SURFACE

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Introduction. The following relation was set up in [5] for an open covering Riemann surface $\mathfrak{R}$ with positive boundary over an abstract Riemann surface $\underline{\mathfrak{R}}$:[1]

$$(1) \qquad \mu(P,\ \mathfrak{A}(\mathfrak{R})) = \mu(P,\ \mathfrak{A}(\tilde{\mathfrak{R}})) \geqq \mu(P,\ \mathfrak{A}(\mathfrak{R}^\infty)) \geqq \mu(P,\ \mathfrak{A}(\tilde{\mathfrak{R}}^\infty)) \equiv \omega(P),$$

when the universal covering surface $\underline{\mathfrak{R}}'^\infty$ of the projection is not of hyperbolic type; when $\underline{\mathfrak{R}}'^\infty$ is of hyperbolic type this relation is reduced to

$$(2) \qquad \mu(P,\ \mathfrak{A}(\mathfrak{R})) \geqq \mu(P,\ \mathfrak{A}(\mathfrak{R}^\infty)) \equiv \omega(P).$$

In the present note we shall give some contributions to the clarification of these relations in two special cases.

1. *We suppose first that* $\mathfrak{R}$ *has a positive boundary, that* $\underline{\mathfrak{R}}'^\infty$ *is not of hyperbolic type, but that* $\mathfrak{R}$ *covers a finite number of points* $\{\underline{P}_n\}$ *of* $\underline{\mathfrak{R}}$ *only in finite times, where the universal covering surface* $(\underline{\mathfrak{R}} - \{\underline{P}_n\})^\infty$ *is of hyperbolic type. Under these hypotheses we shall show*

$$(3) \qquad \mu(P,\ \mathfrak{A}(\mathfrak{R}^\infty)) = \mu(P,\ \mathfrak{A}(\tilde{\mathfrak{R}}^\infty)).$$

For that purpose it is sufficient to prove $\mu(P,\ \mathfrak{A}(\mathfrak{R}^\infty)) \leqq \mu(P,\ \mathfrak{A}(\tilde{\mathfrak{R}}^\infty))$ on account of (1).

Map $\mathfrak{R}^\infty$ conformally onto $U: |z| < 1$ and denote by $f(z)$ the function which corresponds to $U \rightarrow \mathfrak{R}^\infty \rightarrow \mathfrak{R} \rightarrow \underline{\mathfrak{R}}$. Let l be an image in U of any determining curve of an accessible boundary point of $\mathfrak{R}$ relative to $\underline{\mathfrak{R}}$. If it is shown that

i) l terminates at a point on $\Gamma: |z| = 1$;[2]

ii) $f(z)$ has an angular limit at every point of $E - E_1$, where E is the image on Γ of $\mathfrak{A}(\mathfrak{R})$ and E_1 is a set of linear measure zero;

iii) E is linearly measurable;

then Lemma in [5] will give $\mu(z,\ E) \leqq \mu(P,\ \mathfrak{A}(\tilde{\mathfrak{R}}^\infty))$. On the other hand, the

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[1] We shall follow the definitions and notations in [5] and make use of results in it without proofs.

[2] This point is called an image of a point of $\mathfrak{A}(\mathfrak{R})$.

same reasoning as in Theorem 1 of [5] yields $\mu(z, E)=\mu(P, \mathfrak{A}(\mathfrak{R}^{\infty}))$. Thus there will follow the required inequality $\mu(P, \mathfrak{A}(\mathfrak{R}^{\infty}))\leqq\mu(P, \mathfrak{A}(\tilde{\mathfrak{R}}^{\infty}))$. In the following we shall prove i), ii), iii) stepwise.

i) Suppose that l oscillates in U, and let γ be an open arc to which l clusters. According to Theorem 3.5 of [4], the function mapping U onto $\mathfrak{R}$ is univalent in a sufficiently small vicinity of every regular point on Γ.[3] Hence $f(z)$ does not take $\{\underline{P}_n\}$ near it, because $\mathfrak{R}$ covers $\{\underline{P}_n\}$ only in finite times. On mapping $(\underline{\mathfrak{R}}-\{\underline{P}_n\})^{\infty}$ onto a circular disk and applying Koebe's theorem, we see that l does not oscillate near any regular point. Therefore γ consists of singular points only. Since the case in which $\mathfrak{R}$ is conformally equivalent to a sphere minus three points is excluded at present, hyperbolic fixed points exist and are dense in γ. Let z_0 be any hyperbolic fixed point of γ. An image of a closed curve on $\mathfrak{R}$ terminates at z_0 and l intersects it in any neighborhood of z_0. This contradicts the fact that every determining curve of an accessible boundary point of $\mathfrak{R}$ tends to the ideal boundary of $\mathfrak{R}$. Thus it has been shown that l terminates at a point on Γ.

ii) If $\mathfrak{R}$ is simply-connected, it is mapped conformally onto U. Since the function $f(z)$ does not take $\{\underline{P}_n\}$ near Γ, it has always an angular limit at every point of E.

Hence we suppose that $\mathfrak{R}$ is not simply-connected. A Green's function $G(P)$ exists on it, because it has a positive boundary. The function $G(P(z))$ considered in U has angular limit zero everywhere on Γ minus E_1 with linear measure $m(E_1)=0$. Let z_0 be any point of $E-E_1$, and l be the image, terminating at z_0, of a curve determining a point of $\mathfrak{A}(\mathfrak{R})$. This curve converges to an ideal boundary component $P_{\mathfrak{C}}$ of $\mathfrak{R}$.[4] We take a domain $\mathfrak{R}_1$ of the determining sequence of $P_{\mathfrak{C}}$ such that $\mathfrak{R}_1$ does not cover $\{\underline{P}_n\}$ and $\mathfrak{R}-\mathfrak{R}_1^a$ ($\mathfrak{R}_1^a$ = closure of $\mathfrak{R}_1$) is not simply-connected, and we denote its relative boundary by C, which is a simple closed curve. Every image in U of $\mathfrak{R}_1$ is a simply-connected domain bounded by some points on Γ and by cross-cuts of U which are images of C.[5] Let A be any angular domain at z_0. Since $G(P(z))\to 0$ as $A\ni z\to z_0$, the images in U of C do not intersect A near z_0. Further they have no common point with l near z_0. Therefore there exists a simply-connected domain A_1, whose closure contains parts near z_0 of both A and l and is contained in an image in U of $\mathfrak{R}_1$. Since $\mathfrak{R}_1$ does not cover $\{\underline{P}_n\}$ and $f(z)$ tends to a value of $\underline{\mathfrak{R}}$ along l, $f(z)$ tends to this value when z approaches z_0 from the insideof any angular subdomain of A, with its boundary contained in A. By the arbitrariness of A it is concluded that $f(z)$ has an angular limit

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[3] For regular and singular points on Γ, see [4], Chap. III, § 4.

[4] For an ideal boundary component, see [4], Chap. III, § 5.

[5] Details of the boundary correspondence of ideal boundary components in the conformal mapping will be found in a paper, which is now in preparation.

at z_0.

iii) Map the universal covering surface $\underline{\mathfrak{R}}^\infty$ of $\underline{\mathfrak{R}}$ onto $D: |w|<1$ or $|w|<\infty$ or $|w| \leqq \infty$, and denote any branch of $w(f(z))$ by $w(z)$. $f(z)$ has a radial limit at a point on Γ if and only if $w(z)$ has there a radial limit lying inside D. By the aid of the theory of functions of real variables (cf. [2], pp. 270-175), the set E_2 where $w(z)$ has radial limits in D is linearly measurable. Since $E-E_2 \subset E_1$ and $m(E_1)=0$, E is measurable too. Thus the proof of (3) is completed.

2. Next *consider the case in which $\mathfrak{R}$ is a subdomain of $\underline{\mathfrak{R}}$ and has a positive boundary*. Then $\underline{\mathfrak{R}}'^\infty = \mathfrak{R}^\infty$ and is clearly of hyperbolic type. *We now want to show*

$$\mu(P, \mathfrak{A}(\mathfrak{R})) = \mu(P, \mathfrak{A}(\mathfrak{R}^\infty)). \tag{4}$$

When $\mathfrak{R}$ is compact in $\underline{\mathfrak{R}}$, $\underline{\mathfrak{R}}-\mathfrak{R}$ is of positive capacity on $\underline{\mathfrak{R}}$. Hence $\mathfrak{R}$ is of F-type by a theorem due to R. Nevanlinna [3] (cf. Theorem 3.3 of [4]). Therefore $\omega(P) \equiv \mu(P, \mathfrak{A}(\mathfrak{R}^\infty)) = \mu(P, \mathfrak{A}(\mathfrak{R})) \equiv 1$ by (2).

In the following we assume that $\mathfrak{R}$ is non-compact in $\underline{\mathfrak{R}}$. If $\underline{\mathfrak{R}}-\mathfrak{R}$ is of capacity zero on $\underline{\mathfrak{R}}$, it is shown that $\mu(P, \mathfrak{A}(\mathfrak{R})) \equiv 0$ as follows. Cover $\underline{\mathfrak{R}}-\mathfrak{R}$ by a sequence of neighborhoods $\{\underline{N}_k\}$, in each of which a local parameter is defined. By Evans' theorem [1] there is a harmonic function $h_k(P)>0$ in every $\mathfrak{R} \cap \underline{N}_k$ such that $h_k(P) \to +\infty$ as $P \to (\underline{\mathfrak{R}}-\mathfrak{R}) \cap \underline{N}_k$. We can extend this to a positive function $H_k(P)$ on $\mathfrak{R}$ by Theorem 2.1 of [4], because $\mathfrak{R}$ has a positive boundary by Lemma 1.3 of [4]. For an arbitrary point $P_0 \in \mathfrak{R}$ set $H(P) = \sum_k \frac{1}{k^2} \cdot \frac{H_k(P)}{H_k(P_0)}$. This function is positive harmonic in $\mathfrak{R}$ and tends to $+\infty$ as $P \to \underline{\mathfrak{R}}-\mathfrak{R}$. Therefore $\mu(P, \mathfrak{A}(\mathfrak{R})) \leqq \varepsilon H(P)$ for any $\varepsilon>0$. By $\varepsilon \to 0$ there follows $\mu(P, \mathfrak{A}(\mathfrak{R})) \equiv 0$. Thus $\mu(P, \mathfrak{A}(\mathfrak{R}^\infty)) = \mu(P, \mathfrak{A}(\mathfrak{R})) \equiv 0$ by (2).

We pass to the case when $\underline{\mathfrak{R}}-\mathfrak{R}$ is of positive capacity on $\underline{\mathfrak{R}}$. Let $\mathfrak{U}(\mathfrak{R})$ be the class of all the non-negative continuous subharmonic functions $\{u(P)\}$ on $\mathfrak{R}$ such that $u(P) \leqq 1$ and $\lim u(P)=0$ as $\mathfrak{R} \ni P$ tends to the ideal boundary of $\underline{\mathfrak{R}}$, and denote the upper cover of $\mathfrak{U}(\mathfrak{R})$ by $\underline{\mu}(P, \mathfrak{A}(\mathfrak{R}))$. This is harmonic on $\mathfrak{R}$ by Perron-Brelot's principle. Similary as above, cover the boundary $\mathfrak{R}^b$ of $\mathfrak{R}$ in $\underline{\mathfrak{R}}$ by $\{\underline{N}_k\}$. Replace any $u(P) \in \mathfrak{U}(\mathfrak{R})$ in $\underline{N}_k \cap \mathfrak{R}$ by the solution of the ordinary Dirichlet problem with boundary value $u(P)$ on $\underline{N}_k^b \cap \mathfrak{R}$ and 1 on $\mathfrak{R}^b \cap \underline{N}_k$, where $\underline{N}_k^b$ denotes the boundary of $\underline{N}_k$. The replacing function still belongs to $\mathfrak{U}(\mathfrak{R})$ and tends to 1 as the variable approaches every regular point of $\mathfrak{R}^b \cap \underline{N}_k$. Therefore also $\underline{\mu}(P, \mathfrak{A}(\mathfrak{R}))$ has this property. Similarly as in Lemma 4.1 of [4] we can find a positive harmonic function in $\mathfrak{R}$ which tends to $+\infty$ as P approaches every irregular point of $\mathfrak{R}^b \cap \underline{N}_k$. Then we obtain as above a positive harmonic function $H'(P)$ in $\mathfrak{R}$ which tends to $+\infty$ as P approaches every irregular point of $\mathfrak{R}^b$. Therefore $\min(1, \underline{\mu}(P, \mathfrak{A}(\mathfrak{R}))+\varepsilon H'(P))$ for any $\varepsilon>0$ belongs to the upper class $\mathfrak{B}(\mathfrak{R})$. ε being arbitrarily small, we have

$$\underline{\mu}(P, \mathfrak{A}(\mathfrak{R})) \geqq \mu(P, \mathfrak{A}(\mathfrak{R})). \tag{5}$$

Let us take any $u(P) \in \mathfrak{U}(\mathfrak{R})$ and $v(P) \in \mathfrak{V}(\mathfrak{R}^\infty)$ and put $u(P) - v(P) = u_1(P)$, where $u(P)$ is considered on $\mathfrak{R}^\infty$. $u_1(P)$ is continuous subharmonic on $\mathfrak{R}^\infty$ and $\overline{\lim}\, u_1(P) \leqq 0$ as $P \to \mathfrak{A}(\mathfrak{R}^\infty)$ or as the projection into $\underline{\mathfrak{R}}$ of P tends to the ideal boundary of $\underline{\mathfrak{R}}$. Suppose $u_1(P_0) > 0$ at a certain point $P_0 \in \mathfrak{R}^\infty$, and let D be any component of the open set $\{P\,;\, u_1(P) > u_1(P_0)\}$ on $\mathfrak{R}^\infty$. The projection of D into $\underline{\mathfrak{R}}$ is compact in $\underline{\mathfrak{R}}$, and does not contain any points of $\underline{\mathfrak{R}} - \mathfrak{R}$, which is of positive capacity. Therefore by Theorem 3.3 in [4] D is of F-type relatively to $\underline{\mathfrak{R}}$ and hence is of D-type (cf. Theorem 4.2 of [4], or §6 in [5]). Consequently $u_1(P) - u_1(P_0) \leqq 0$ in D, because every accessible boundary point Q of D relative to $\underline{\mathfrak{R}}$ lies above $\mathfrak{R}$ and so $\lim u_1(P) = u_1(P_0)$ as P approaches Q. But it contradicts the definition of D. Thus there holds $u_1(P) \leqq 0$ everywhere on $\mathfrak{R}^\infty$, that is, $u(P) \leqq v(P)$. Accordingly $\underline{\mu}(P, \mathfrak{A}(\mathfrak{R})) \leqq \mu(P, \mathfrak{A}(\mathfrak{R}^\infty))$. This inequality together with (2) and (5) yields (4).

BIBLIOGRAPHY

[1] G. Evans: Potentials and positively infinite singularities of harmonic functions, Monat. Math. Phys., 43 (1936), pp. 419-424.

[2] F. Hausdorff: Mengenlehre, 3 Aufl., Berlin and Leipzig (1935).

[3] R. Nevanlinna: Über die Lösbarkeit des Dirichletschen Problems für eine Riemannsche Fläche, Nachr. Gött., 1 (1938), pp. 181-193.

[4] M. Ohtsuka: Dirichlet problems on Riemann surfaces and conformal mappings, Nagoya Math. Journ., 3 (1951), pp. 91-137.

[5] M. Ohtsuka: On a covering surface over an abstract Riemann surface, Nagoya Math. Journ., 4 (1952), pp. 109-118.

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