ON KRULL'S CONJECTURE CONCERNING VALUATION RINGS

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Introduction. Previously W. Krull conjectured[1] that every completely integrally closed primary[2] domain of integrity is a valuation ring. The main purpose of the present paper is to construct in §1 a counter example against this conjecture. In §2 we show a necessary and sufficient condition that a field is a quotient field of a suitable completely integrally closed primary domain of integrity which is not a valuation ring.

By a ring we mean a commutative ring with identity. We refer to the notations like $\mathfrak{o}_{\mathfrak{p}}$ as the ring of quotients of $\mathfrak{p}$ with respect to $\mathfrak{o}$ when $\mathfrak{o}$ is a ring and $\mathfrak{p}$ is a prime ideal of $\mathfrak{o}$.

1. A counter example.

Let K be an algebraically closed field with a non-trivial special valuation w whose value group G does not fill up all real numbers. Take a positive number α which is not in G. Consider a rational function field $K(x)$ of one variable x with constant field K. Let us define the following two types of valuations of $K(x)$ which are extensions of w: (1) For every element e of K such that $\alpha < w(e) < 2\alpha$,[3] we define a valuation w_e (of $K(x)$) such that

$$w_e\Big(\sum_{i=0}^{n} a_i(x+e)^i\Big) = \min\,(w(a_i) + 2\alpha i) \quad (a_i \in K).^{4)}$$

(2) For every real number λ such that $\alpha \leqq \lambda \leqq 2\alpha$, we define a valuation w_λ such that

$$w_\lambda\Big(\sum_{i=0}^{n} a_i x^i\Big) = \min\,(w(a_i) + \lambda i) \quad (a_i \in K).$$

THEOREM 1. *Let $\mathfrak{o}_e$ and $\mathfrak{o}_\lambda$ be the valuation rings determined by w_e snd w_λ respectively $(\alpha < w(e) < 2\alpha,\ \alpha \leqq \lambda \leqq 2\alpha)$ and let $\mathfrak{o}$ be the intersection of all such $\mathfrak{o}_e$ and $\mathfrak{o}_\lambda$. Then $\mathfrak{o}$ is completely integrally closed and primary, but $\mathfrak{o}$ is not a*

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1) W. Krull, Beiträge zur Arithmetik kommutativer Integritätsbereiche II, Math. Zeit. 41 (1936). p, 670.

2) A ring is called primary if it has at most one proper prime ideal.

3) Observe the fact that $2\alpha \notin G$, because K is algebraically closed.

4) Since $2\alpha \notin G$, w_e is uniquely determined by the relation $w_e(x+e) = 2\alpha$.

valuation ring.

Proof. Let $c(\neq 0)$ be an element of $\mathfrak{o}$. First we prove that (1) if $w_{\lambda_0}(c)=0$ for some λ_0 $(\alpha \leqq \lambda_0 \leqq 2\alpha)$, then $w_\lambda(c)=0$ and $w_e(c)=0$ for every w_λ and w_e, and that (2) if $w_\alpha(c)>0$, there exist the least and the largest values $\varepsilon>0$ and δ among values of c taken by w_λ and w_e $(\alpha \leqq \lambda \leqq 2\alpha,\ \alpha<w(e)<2\alpha)$.

Since K is algebraically closed, c is of the form

$$c_0\prod_{i=1}^{n}(x+a_i)/\prod_{j=1}^{m}(x+b_j) \quad (c_0,\ a_i,\ b_j \in K).$$

Every factor $x+d$ $(d=a_i$ or $b_j)$ such that $w(d)>2\alpha$ may be replaced by x, since we only consider the values of c taken by w_λ and w_e. Similarly we may replace by d every factor $x+d$ $(d=a_i$ or $b_j)$ such that $w(d)<\alpha$. Therefore we may assume without loss of generality that (i) $\alpha<w(a_i)<2\alpha$ or $a_i=0$, $\alpha<w(b_j)<2\alpha$ or $b_j=0$ for each i and j $(1\leqq i\leqq n,\ 1\leqq j\leqq m)$, (ii) $a_i\neq b_j$ for every pair (i, j) and (iii) $w(a_i)\leqq w(a_{i+1})$, $w(b_j)\leqq w(b_{j+1})$ $(1\leqq i<n,\ 1\leqq j<m)$.

First we assume that $w_{\lambda_0}(c)=0$ for some λ_0 $(\alpha\leqq\lambda_0\leqq 2\alpha)$. If there exists one j_1 such that $w(b_{j_1})=\lambda_0$, then we have $w_{b_{j_1}}(c)<0$, which is a contradiction. Therefore no $w(b_j)$ is equal to λ_0. Assume that $w(a_i)<\lambda_0$ if $i\leqq i_0$, $w(a_i)=\lambda_0$ if $i_0<i\leqq i_0+s$, $w(a_i)>\lambda_0$ if $i>i_0+s$; $w(b_j)<\lambda_0$ if $j\leqq j_0$, $w(b_j)>\lambda_0$ if $j>j_0$. Set $\lambda_1=\max(\alpha,\ w(a_{i_0}),\ w(b_{j_0}))$, $\lambda_2=\min(2\alpha,\ w(a_{i_0+s+1}),\ w(b_{j_0+1}))$.
Then

$$w_{\lambda_1}(c)=w(c_0)+\sum_{i\leqq i_0}w(a_i)-\sum_{j\leqq j_0}w(b_j)+(n-i_0)\lambda_1-(m-j_0)\lambda_1\geqq 0,$$
$$w_{\lambda_0}(c)=w(c_0)+\sum_{i\leqq i_0}w(a_i)-\sum_{j\leqq j_0}w(b_j)+(n-i_0)\lambda_0-(m-j_0)\lambda_0=0,$$
$$w_{\lambda_2}(c)=w(c_0)+\sum_{i\leqq i_0}w(a_i)-\sum_{j\leqq j_0}w(b_j)+s\lambda_0+(n-i_0-s)\lambda_2-(m-j_0)\lambda_2\geqq 0.$$

Hence we have

$$w_{\lambda_1}(c)=w_{\lambda_1}(c)-w_{\lambda_0}(c)=(n-i_0)(\lambda_1-\lambda_0)-(m-j_0)(\lambda_1-\lambda_0)\geqq 0,$$
whence $n-i_0\leqq m-j_0$.[5]

Similarly we have

$$w_{\lambda_2}(c)=w_{\lambda_2}(c)-w_{\lambda_0}(c)=(n-i_0-s)(\lambda_2-\lambda_0)-(m-j_0)(\lambda_2-\lambda_0)\geqq 0,$$
whence $n-i_0-s\geqq m-j_0$.[5]

Thus we have $s=0$ and $n-i_0=m-j_0$. $s=0$ shows that no $w(a_i)$ is equal to λ_0. Further, $n-i_0=m-j_0$, $s=0$ show $w_{\lambda_1}(c)=w_{\lambda_2}(c)=0$. Therefore neither $w(a_i)$ nor $w(b_j)$ are equal to λ_1 or λ_2, by the above observation. This means that $\lambda_1=\alpha$ and $\lambda_2=2\alpha$. From $\lambda_1=\alpha$ we have that $i_0=j_0=0$, whence $m=n$; From $\lambda_2=2\alpha$ we have that $a_i=0$, $b_j=0$ $(1\leqq i\leqq n,\ 1\leqq j\leqq m)$. By our assumption

5) If $\alpha=\lambda_0$ or $2\alpha=\lambda_0$, we see easily that $n-i_0=m-j_0$ because $\alpha\notin G$. In this case, $s=0$ is also clear.

that $a_i \neq b_j$, it follows that $m=n=0$, i.e., $c=c_0 \in K$. Since $w_{\lambda_0}(c)=0$, we have $w(c)=0$. This proves (1). Next assume that $w_\alpha(c)>0$. Let us consider $w_\lambda(c)$ as a function of variable λ $(\alpha \leqq \lambda \leqq 2\alpha)$. Then it is evidently continuous, and it takes the least and the largest values ε_1 and δ_1 in $\alpha \leqq \lambda \leqq 2\alpha$. By virtue of (1), we see that ε_1 is positive. Then (2) follows easily from the fact that $w_e(c) \neq w_{w(e)}(c)$ holds only if e is one of a_i or b_j and in this case $w_e(c) \notin G$, whence $w_e(c) \neq 0$.

These being proved, we see that $\mathfrak{o}$ is primary. Let $a(\neq 0)$ and $b(\neq 0)$ be two non-units in $\mathfrak{o}$. Then there exist positive numbers ε and δ such that $w_\lambda(a) \geqq \varepsilon$, $w_e(a) \geqq \varepsilon$, $w_\lambda(b) \leqq \delta$, $w_e(b) \leqq \delta$ $(\alpha \leqq \lambda \leqq 2\alpha,\ \alpha < w(e) < 2\alpha)$. Let k be an integer such that $k\varepsilon > \delta$. Then we have $w_\lambda(a^k/b) \geqq 0$, $w_e(a^k/b) \geqq 0$ $(\alpha \leqq \lambda \leqq 2\alpha,\ \alpha < w(e) < 2\alpha)$, whence $a^k/b \in \mathfrak{o}$, i.e., $a^k \in b\mathfrak{o}$.

It is evident that $\mathfrak{o}$ is completely integrary closed, because $\mathfrak{o}$ is an intersection of special valuation rings. That $\mathfrak{o}$ is not a valuation ring follows from that $e/x \notin \mathfrak{o}$, $x/e \notin \mathfrak{o}$ if $\alpha < w(e) < 2\alpha$.

2. An existence theorem.

LEMMA 1. Let $\mathfrak{r}$ be an integrally closed integral domain which has only one maximal ideal $\mathfrak{p}_0$. Let K be the quotient field of $\mathfrak{r}$. If Z is a field containing K, $\mathfrak{o}_\mathfrak{p} \cap K = \mathfrak{r}$, where $\mathfrak{o}$ is the totality of $\mathfrak{r}$-integers in Z and $\mathfrak{p}$ a maximal ideal of $\mathfrak{o}$.

Proof. We may assume without loss of generality that Z is algebraic over K because the quotient field of $\mathfrak{o}$ is algebraic over K.

First we assume that Z is finite normal over K. Let $\{\sigma_1, \ldots, \sigma_n\}$ be the totality of automorphisms of Z over K. We show that every maximal ideal of $\mathfrak{o}$ is one of $\mathfrak{p}^{\sigma_i}$: Assume that a maximal ideal $\mathfrak{q}$ of $\mathfrak{o}$ is none of $\mathfrak{p}^{\sigma_i}$. Then there exists an element c of $\mathfrak{q}$ such that $c \notin \mathfrak{p}^{\sigma_i}$ for every $i=1, \ldots, h$. A power e of $\prod_{i=1}^{h} c^{\sigma_i}$ is in K, whence in $\mathfrak{r}$. Since $c \in \mathfrak{q}$, we have $e \in \mathfrak{p}_0$, whence $e \in \mathfrak{p}$.[6] Therefore one of c^{σ_i} must be in $\mathfrak{p}$, i.e., c is in some $\mathfrak{p}^{\sigma_i}$, which is a contradiction. This being shown, we have $\mathfrak{o} = \bigcap_{i=1}^{h} (\mathfrak{o}_\mathfrak{p})^{\sigma_i}$. Therefore $\mathfrak{o}_\mathfrak{p} \cap K = (\mathfrak{o}_\mathfrak{p})^{\sigma_i} \cap K = (\bigcap_{i=1}^{h} (\mathfrak{o}_\mathfrak{p})^{\sigma_i}) \cap K = \mathfrak{o} \cap K = \mathfrak{r}$.

Next we assume that Z is finite algebraic over K. Let Z^* be a field containing Z which is finite normal over K. Let $\mathfrak{o}^*$ be the totality of $\mathfrak{r}$-integers in Z^* and let $\mathfrak{p}^*$ be a maximal ideal of $\mathfrak{o}^*$ which contains $\mathfrak{p}\mathfrak{o}^*$. Then evidently $\mathfrak{o}^*_{\mathfrak{p}^*} \geqq \mathfrak{o}_\mathfrak{p}$. Since $\mathfrak{o}^*_{\mathfrak{p}^*} \cap K = \mathfrak{r}$, we have $\mathfrak{o}_\mathfrak{p} \cap K = \mathfrak{r}$.

Making use of this, we prove the general case. Let c be an element of $\mathfrak{o}_\mathfrak{p} \cap K$. c may be written in a form a/b $(a, b \in \mathfrak{o}, b \notin \mathfrak{p})$. We consider $Z^* = K(a, b)$. We set $\mathfrak{o}^* = \mathfrak{o} \cap Z^*$, and $\mathfrak{p}^* = \mathfrak{p} \cap \mathfrak{o}^*$. Then $\mathfrak{p}^*$ is a maximal ideal because $\mathfrak{o}$

6) Because $\mathfrak{o}$ is integral over $\mathfrak{r}$, $\mathfrak{p}_0 = \mathfrak{r} \cap \mathfrak{p} = \mathfrak{r} \cap \mathfrak{q}$.

is integral over $\mathfrak{o}^*$. It is clear that $a, b \in \mathfrak{o}^*$, $b \notin \mathfrak{p}^*$ whence $\mathfrak{o}^*_{\mathfrak{p}^*} \ni c$. Since Z^* is finite over K, we have $\mathfrak{o}^*_{\mathfrak{p}^*} \cap K = \mathfrak{r} \ni c$, which proves our assertion.

LEMMA 2. Let K be a field with a valuation ring $\mathfrak{v}$ and let Z be a field containing K which is algebraic over K. Let $\mathfrak{o}$ be the totality of $\mathfrak{v}$-integers in Z and let $\{\mathfrak{p}_\lambda ; \lambda \in \Lambda\}$ be the totality of maximal ideals of $\mathfrak{o}$. Then every valuation ring $\mathfrak{w}$ of Z, such that the valuation given by $\mathfrak{w}$ is an extension of that given by $\mathfrak{v}$, is one of $\mathfrak{o}_{\mathfrak{p}_\lambda}$ $(\lambda \in \Lambda)$. Conversely, every $\mathfrak{o}_{\mathfrak{p}_\lambda} (\lambda \in \Lambda)$ is a valuation ring.

Proof. It is clear that any such valuation ring $\mathfrak{w}$ contains one of $\mathfrak{o}_{\mathfrak{p}_\lambda}$. Hence we have only to prove the converse part. But this follows immediately from the following facts:

(1)[7] An integrally closed domain $\mathfrak{m}$ of integrity is a multiplication ring if and only if $\mathfrak{m}_\mathfrak{p}$ is a valuation ring for every maximal ideal $\mathfrak{p}$ of $\mathfrak{m}$.

(2)[8] Let $\mathfrak{m}$ be a multiplication ring with quotient field K. If a field Z containing K is algebraic over K, then the totality $\mathfrak{o}$ of $\mathfrak{m}$-integers in Z is also a multiplication ring and Z is the quotient field of $\mathfrak{o}$.

LEMMA 3. Let $\mathfrak{r}$ be a completely integrally closed integral domain with quotient field K. If Z is a field containing K, the totality $\mathfrak{o}$ of $\mathfrak{r}$-integers in Z is also cmpletely integrally closed.

Proof. Assume that Z is finite normal (algebraic) over K. Let $\{\sigma_1, \ldots, \sigma_h\}$ be the totality of automorphisms of Z over K. Set $r = [Z:K]/h$. Assume that $(a/b)^n c \in \mathfrak{o}$ for every natural number n, where a, b and c are non-zero elements of $\mathfrak{o}$. Let f be an arbitrary elementary symmetric formula of $[(a/b)^{\sigma_1}]^r, \ldots, [(a/b)^{\sigma_h}]^r$, and set $c' = (\prod_{i=1}^{h} c^{\sigma_i})^r$. Then $f^n c' \in \mathfrak{o}$, whence $f^n c' \in \mathfrak{r}$ for every natural number n. This shows that $f \in \mathfrak{r}$, whence a/b satisfies a monic equation with coefficient in $\mathfrak{r}$, i.e., $a/b \in \mathfrak{o}$, which proves our assertion when Z is finite normal over K. This being proved, we can reduce our problem to the ganeral case by the same way as in the proof of Lemma 1.

THEOREM 2. *Let K be a field. Then there exists a completely integrally closed primary domain of integrity which is not a valuation ring such that its quotient field is K if and only if K satisfies one of the following two conditions:*

(1) K is of characteristic 0 and K is not algebraic over its prime field.

(2) K is of characteristic p $(\neq 0)$ and K contains at least two algebraically independent elements over its prime field.

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7) W. Krull, Beiträge zur Arithmetik kommutativer Integritätsbereiche, Math. Zeit. 41 (1936), Theorem 7 (p. 554).

8) Prüfer, Untersuchungen über die Teilbarkeitseigenschaften in Körpern, Crelle 168, p. 31 or 1. c. note 6) Theorem 8 (p. 555).

Proof. (I) The case where K satisfies neither of these conditions. Let $\mathfrak{o}$ be any integrally closed[9] primary domain of integrity with quotient field K. When K is algebraic over its prime field, let K_0 be its prime field. When K is not algebraic over its prime field, let K_0 be its subfield which is isomorphic to the rational function field of one variable with its prime field as the constant field. Then evidently $\mathfrak{o} \cap K_0$ is a valuation ring. Then by Lemma 2 it follows that $\mathfrak{o}$ is also a valuation ring.

(II) Assume that K satisfies one of the above two conditions. Then it is easy to see that there exists a subfield K_0 of K such that K_0 has a non-trivial discrete special valuation and such that K has transcendental degree 1 over K_0, that is, there exists an element x of K such that x is not algebraic over K_0 and K is algebraic over $K_0(x)$. Let $\overline{K}_0$ and $\overline{K}$ be the algebraic closures of K_0 and K respectively. Then by Theorem 1 we can construct a completely integrally closed primary domain $\mathfrak{r}$ of integrity which is not a valuation ring and whose quotient field is $\overline{K}_0(x)$. Let $\overline{\mathfrak{o}}$ be the totality of $\mathfrak{r}$-integers in $\overline{K}$ and let $\overline{\mathfrak{p}}$ be a maximal ideal of $\overline{\mathfrak{o}}$. Set $\mathfrak{o} = \overline{\mathfrak{o}}_{\overline{\mathfrak{p}}} \cap K$. Then since $\mathfrak{r}$ is completely integrally closed, $\overline{\mathfrak{o}}$ is so too by Lemma 3. Therefore $\mathfrak{o}$ is also completely integrally closed. Since $\mathfrak{r}$ is primary, so is $\overline{\mathfrak{o}}_{\overline{\mathfrak{p}}}$ too, whence $\mathfrak{o}$ is primary. On the other hand, since $\overline{\mathfrak{o}}_{\overline{\mathfrak{p}}} \cap \overline{K}_0(x) = \mathfrak{r}$ by Lemma 1, $\overline{\mathfrak{o}}_{\overline{\mathfrak{p}}}$ is not a valuation ring and therefore $\mathfrak{o}$ is not a valuation ring again by virtue of Lemma 2. Thus our proof is complete.

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[9] We need not assume here that $\mathfrak{o}$ is "completely" integrally closed.