A REMARK ON FINITELY GENERATED MODULES

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Theorem 5 of Azumaya's recent article[1] can be formulated in the following generalized form:

I. Let R be a ring. Let $\mathfrak{m}$ be a finitely generated right-module of R such that $\mathfrak{m}R=\mathfrak{m}$. Assume that $\mathfrak{m}=u_1\mathfrak{r}+u_2\mathfrak{r}+\ldots+u_m\mathfrak{r}$ for every generating system $u_1, u_2, \ldots, u_m$ of $\mathfrak{m}$ and for every maximal right-ideal $\mathfrak{r}$ of R. Then $\mathfrak{m}=0$.

For the proof, we first consider the case where R possesses a unit element 1. Then the assertion can be proved quite similarly as in Azumaya, l. c. Let namely $u_1, u_2, \ldots, u_m$ be any finite generating system of $\mathfrak{m}$; $\mathfrak{m}=u_1R+u_2R+\ldots+u_mR$. Let $\mathfrak{r}_0$ be the right-ideal of R consisting of all elements x of R such that

$$u_1x\in u_2R+\ldots+u_mR.$$

Suppose $\mathfrak{r}_0\neq R$. There exists a maximal right-ideal $\mathfrak{r}$ which contains $\mathfrak{r}_0$, and we have $\mathfrak{m}=u_1\mathfrak{r}+u_2\mathfrak{r}+\ldots+u_m\mathfrak{r}$, whence $=u_1\mathfrak{r}+u_2R+\ldots+u_mR$ much the more, by our assumption. There is an element a in $\mathfrak{r}$ such that $1-a\in\mathfrak{r}_0\subseteq\mathfrak{r}$, which is a contradiction. Hence necessarily $\mathfrak{r}_0=R$ and $\mathfrak{m}=u_2R+\ldots+u_mR$. Now the assertion can be proved by an induction with respect to the minimal number of generating elements.

Let next R be general. Let R^* be the ring which is as module a direct sum of R and the ring of rational integers and in which $1x=x1=x$ $(x\in R)$. If $\mathfrak{r}^*$ is a maximal right-ideal of R^*, then $\mathfrak{r}^*\cap R$ is either R or a maximal right-ideal of R. Thus $\mathfrak{m}=u_1\mathfrak{r}^*+u_2\mathfrak{r}^*+\ldots+u_m\mathfrak{r}^*$, much the more, and the assertion can be reduced to the above case.

It is perhaps of interest to observe that from this generalization Jacobson's theorem[2] may be derived:

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[1] G. Azumaya, On maximally central algebras, Nagoya Math. J. **2** (1951).

[2] N. Jacobson, The radical and semi-simplicity for arbitrary rings, Amer. J. Math. **67** (1945), Theorem 10; the present formulation is in 1), l. c.

II. Let R be a ring and N be its radical. If $\mathfrak{m}$ is a finitely generated right-module of R and if $\mathfrak{m}=\mathfrak{m}N$, then $\mathfrak{m}=0$.

For, the radical N^* of R^*, as above, contains (coincides with, as a matter of fact) N. Hence $\mathfrak{m}N^*=\mathfrak{m}$. Then $\mathfrak{m}=u_1N^*+u_2N^*+\ldots+u_mN^*$ for any generating system $u_1, u_2, \ldots, u_m$. Since every maximal right-ideal of R^* contains N^*, the assertion is an immediate consequence of I.

Next we want to observe:

III. In I we may restrict ourselves to those maximal right-ideals r which contain the radical N.

We consider namely the residue-module $\mathfrak{m}/\mathfrak{m}N$ and find, by virtue of I, $\mathfrak{m}=\mathfrak{m}N$. Then $\mathfrak{m}=0$ because of II. (It would also be possible to prove III directly.)

Now, it is clear that the radical is the largest two-sided ideal possessing the property of II. Namely:

IV. If M is a two-sided ideal and if $\mathfrak{m}=0$ is the only finitely generated module with $\mathfrak{m}=\mathfrak{m}M$, then $M\subseteqq N$.

For, if $M\not\subseteqq N$, there exists a maximal right-ideal r with left modulo-unit a. such that $r\not\supseteqq M$. The right-module $\mathfrak{m}=R/r$ of R, generated by a (mod r), satisfies $\mathfrak{m}=\mathfrak{m}M$. since $aM\equiv M\equiv R$ mod r.

Similary, the family of right-ideals of III gives the "natural boundary," i.e.:

V. *A family of right-ideals $\{r'\}$ of R possesses the property of* I *(with $\{r'\}$ in place of $\{r\}$) if and only if for every maximal right-ideal r with left modulo-unit there exists an r' in the family such that $r'\subseteqq r$.*

It is evident that if this is the case then $\{r'\}$ possesses the required property. On the other hand, if there exists a maximal right-ideal r with left modulo-unit a such that $r\not\supseteqq r'$ for every $r'\in\{r'\}$, then $\mathfrak{m}=R/r$ satisfies $\mathfrak{m}=\mathfrak{m}r'$ for every r', as in IV.

Needless to say that I, II, III and IV are special cases of V. Further, we may replace "finitely generated module" by ("cyclic module," or) "minimal module" in V (and IV) (and then the statement includes also a well known characterization of the radical).

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