ON A HOPF HOMOTOPY CLASSIFICATION THEOREM

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There are various generalizations of Hopf's brilliant theorem, which may be stated, as newly formulated by Alexandroff; all the homotopy classes of the mappings of a compact Hausdorff space X with $\dim X \leqq n$ into an n-sphere S^n are in a $(1-1)$-correspondence with the elements of the n-dimensional Čech cohomology group $H^n(X)$ with integer coefficients.

The object of the present work is to build up a generalization of Hopf's theorem. Let X be a compact Hausdorff space with $\dim X \leqq n$ and let Y be a connected absolute neighbourhood retract satisfying $\pi_r(Y)=0$ for each $r<n$. Making use of Hu's bridge operation introduced recently, addition can be defined in the homotopy classes of mappings of X into Y, so that the set of all the homotopy classes forms a group $\tilde{\mathfrak{H}}_n(X)$. It is also shown that this group is isomorphic to the n-th Čech cohomology group $H^n(X, \pi_n(Y))$ of X with coefficient group $\pi_n(Y)$.

1. Let A be an n-dimensional finite geometric complex, whose r-skelton, for $r \leqq n$, is usually designated by A^r, and let Y be an arcwise connected topological space with $\pi_r(Y)=0$ for each $r<n$. The set Ω of all the mappings of X into Y are seperated by the homotopy concept into the mutually disjoint homotopy classes, each of which contains at least one normal mapping f such that $f(X^{n-1})=y_0$, a fixed point of Y. Throughout the present paper mappings are assumed to be normal.

2. The simplest case where the n-th homotopy group $\pi_n(Y)$ $(n>1)$ of Y has a finite base, each element of which is free.

Let us denote a base of $\pi_n(Y)$ by $\{\alpha_1, \ldots, \alpha_\lambda\}$ and denote a normal mapping by $f:(A, A^{n-1}) \to (Y, y_0)$. Then we have a characteristic cocycle $c^n(f) = \sum_i (f, \sigma_i^n)\sigma_i^n$ such that $(f, \sigma_i^n) = \sum_{j=1}^{\lambda} r_{ij}\alpha_j$, where r_{ij} is an integer. Considering a complex $P^n = S_1^n \vee S_2^n \vee \ldots \vee S_\lambda^n$ constructed by joining n-dimensional spheres S_i^n $(i=1 \ldots \lambda)$ at a point $*$, we define a mapping $h:(P^n, *) \to (Y, y_0)$ such

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that $h \mid S_i^n$ (for each $i=1, \ldots, \lambda$) represents $\alpha_i \in \pi_n(Y, y_0)$. A mapping $\psi_f: (A, A^{n-1}) \to (P^n, *)$ can be also defined in such a way that ψ_f maps an n-simplex σ_i^n onto S_j^n $(j=1, \ldots, \lambda)$ with degree r_{ij} (in notation: $\psi_f(\sigma_i^n) = \sum_{j=1}^{\lambda} r_{ij} S_j^n$). It is easily seen that $h \cdot \psi_f : (A, A^{n-1}) \to (Y, y_0)$ is homotopic to f.

LEMMA 1. *Let us consider two normal mappings $f, g: (A, A^{n-1}) \to (Y, y_0)$. Then f is homotopic to g if and only if ψ_f is homotopic to ψ_g.*

Proof. It is evident that $f \sim g$ if $\psi_f \sim \psi_g$. Thus it is sufficient to prove that when $f \sim g$, we have $\psi_f \sim \psi_g$. It is well known that if $f \sim g$, $c^n(f)$ is cohomologous to $c^n(g)$, where $c^n(f) = \sum_i (f, \sigma_i^n)\sigma_i^n = \sum_i (\sum_{j=1}^{\lambda} r_{ij}\alpha_j)\sigma_i^n$, and $c^n(g) = \sum_i (g, \sigma_i^n)\sigma_i^n = \sum_i (\sum_{j=1}^{\lambda} r'_{ij}\alpha_j)\sigma_i^n$, so that there exists an $(n-1)$-cochain $d^{n-1} = \sum_i (\sum_{j=1}^{\lambda} s_{ij}\alpha_j)\sigma_i^{n-1}$, whose coboundary is equal to $c^n(f) - c^n(g)$. From the definition of the mappings ψ_f, ψ_g we have, $c^n(\psi_f) = \sum_i (\psi_f, \sigma_i^n)\, \sigma_i^n = \sum_i (\sum_{j=1}^{\lambda} r_{ij} S_j^n)\sigma_i^n$, and $c^n(\psi_g) = \sum_i (\sum_{j=1}^{\lambda} r'_{ij} S_j^n)\sigma_i^n$. Putting $\bar{d}^{n-1} = \sum_i (\sum_j s_{ij} S_j^n)\sigma_i^{n-1}$, we have $\delta \bar{d}^{n-1} = c^n(\psi_f) - c^n(\psi_g)$, so that $\psi_f \sim \psi_g$. Thus the proof of Lemma has been established.

3. The case where $\pi_n(Y)$ is a cyclic group $\{\overset{m}{\alpha}\}$ of order m. Again, let f be a normal mapping, then we have a characteristic cocycle $c^n(f) = \sum_i (r_i \alpha)\sigma_i^n$, where r_i $(m > r_i \geqq 0)$ is an integer. Defining a mapping $h: (S^n, *) \to (Y, y_0)$ such that h represents the generator α, and constructing a mapping $\psi_f: (A, A^{n-1}) \to (S^n, *)$ in such a way that ψ_f maps σ_i^n, onto S^n with degree r_i (in notation: $\psi_f(\sigma_i^n) = r_i S^n$), we have $f \sim h \cdot \psi_f$. Let us denote by $Q^{n+1} = E^{n+1} \cup S^n$, where E^{n-1} is attached to S^n by a mapping: $\partial E^{n+1} \to S^n$ of degree m. Then a mapping h can be extended to a mapping $\bar{h}: (Q^{n+1}, *) \to (Y, y_0)$.

LEMMA 2. *For two mappings $f, g: (A, A^{n-1}) \to (Y, y_0)$, f is homotopic to g if and only if ψ_f is homotopic to ψ_g in Q^{n+1}.*

Proof. In virtue of an extended mapping $\bar{h}$, it is clear that if $\psi_f \sim \psi_g$ in Q^{n+1}, f is homotopic to g. Next, we shall prove the converse statement. Since f is homotopic to g, we have $\delta d^{n-1} = c^n(f) - c^n(g)$, where $d^{n-1} = \sum_i (s_i \alpha)\sigma_i^{n-1}$ and $m > s_i \geqq 0$ is an integer. Putting $\bar{d}^{n-1} = \sum_i (s_i S^n)\sigma_i^{n-1}$, we have $\bar{\delta} d^{n-1}(\sigma_i^n) = c^n(\psi_f)(\sigma_i^n) - c^n(\psi_g)(\sigma_i^n) + pmS^n$, where p is an integer. As pmS^n represents zero element of $\pi_n(Q^{n+1})$, we have $\delta \bar{d}^{n-1} = c^n(\psi_f) - c^n(\psi_g)$. Therefore Lemma 2 has been proved.

4. The most general case where $\pi_n(Y)$ has a countable infinite base. As the consequence of the direct combination of two lemmas referred to above, we have the following Theorem.

THEOREM 1. *For two mappings* $f, g : (A, A^{n-1}) \to (Y, y_0)$, f *is homotopic to* g *if and only if* $\psi_f \sim \psi_g$ *in* R^{n+1}, *where* $R^{n+1} = (S_1^n \vee S_2^n \vee \ldots \vee S_\lambda^n \ldots) \vee (Q_1^{n+1} \vee Q_2^{n+1} \vee \ldots \vee Q_\mu^{n+1} \vee \ldots)$.

5. Definition of Addition

LEMMA 3. *Let us consider two cell complexes* $Q_1^{n+1} = E_1^{n+1} \cup S_1^n$, and $Q_2^{n+1} = E_2^{n+1} \cup S_2^n$, *where* E_i^n $(i=1, 2)$ *is attached to* S_i^n *by a mapping* $\partial E_i^{n+1} \to S_i^n$ *of degree* m_i. *Then the product complex* $Q_1^{n+1} \times Q_2^{n+1}$ *can be deformed into* $S_1^n \vee S_2^n$, *removing from* $Q_1^{n+1} \times Q_2^{n+1}$ *four points contained in cells of dimensions not less than* $2n$.

Also we can prove a more general case

LEMMA 4. *Let* R^{n+1} *be a complex referred to in 4. The product complex* $R^{n+1} \times R^{n+1}$ *can be deformed into* $\mathcal{S}^n \vee \mathcal{S}^n$ *by removing discrete points involved in cells of dimensions not less than* $2n$, *where* $\mathcal{S}^n = S_1^n \vee S_2^n \vee \ldots$.

Proof. The proof of Lemmas 3, 4 can be easily verified and so is ommited.

Now, for two mappings $\alpha, \beta : (A, A^{n-1}) \to (R^{n+1}, p)$, $\alpha \times \beta : (A, A^{n-1}) \to (R^{n+1} \times R^{n+1}, p \times p)$ is defined such that $\alpha \times \beta(x) = (\alpha(x), \beta(x))$. Then we have

LEMMA 5. (*Existence of normalizing homotopy.*) *There exists a normalization* $f : (A, A^{n-1}) \to (\mathcal{S}^n \vee \mathcal{S}^n, p \times p)$ *of* $\alpha \times \beta$, *such that* $\alpha \times \beta$ *is homotopic to* f *rel.* $(\alpha \times \beta)^{-1}(\mathcal{S}^n \vee \mathcal{S}^n)$, *where* $n > 1$ *is assumed.*

Proof. Let us denote by σ_i one of the simplexes of dimensions not less than $2n$, which contains one of the removed points mentioned in Lemma 4 as its inner point and does not intersect with $\mathcal{S}^n \vee \mathcal{S}^n$. Then it is easily verified that there exists a mapping $h : (A, A^{n-1}) \to (R^{n+1} \times R^{n+1}, p \times p)$ such that $h(A) \subset R^{n+1} \times R^{n+1} - \sum_i \sigma_i$ and $\alpha \times \beta \sim h$ rel. $(\alpha \times \beta)^{-1}(R^{n+1} \times R^{n+1} - \sum_i \sigma_i)$. As $\mathcal{S}^n \vee \mathcal{S}^n$ is a deformation retract of $R^{n+1} \times R^{n+1} - \sum_i \sigma_i$, we have $\alpha \times \beta \sim D_1 h$ rel. $(\alpha \times \beta)^{-1}(\mathcal{S}^n \vee \mathcal{S}^n)$, where D_t $(1 \geqq t \geqq 0)$ is a retracting deformation. Thus we have a desired normalization $f = D_1 h : (A, A^{n-1}) \to (\mathcal{S}^n \vee \mathcal{S}^n, p \times p)$.

Let us define a mapping $\Omega : (\mathcal{S}^n \vee \mathcal{S}^n, p \times p) \to (\mathcal{S}^n, p)$ in such a way that $\Omega(x, p) = x$ for $(x, p) \in \mathcal{S}^n \times p$ and $\Omega(p, x') = x'$ for $(p, x') \in p \times \mathcal{S}^n$. Then we have the following Lemma.

LEMMA 6. If $\alpha, \beta, \alpha', \beta' : (A, A^{n-1}) \to (R^{n+1}, p)$ *with* $\alpha \sim \alpha'$ *and with* $\beta \sim \beta'$, *and if* f *and* f' *are normalization of* $\alpha \times \beta$ *and* $\alpha' \times \beta'$ *respectively, then* $\Omega f \sim \Omega f'$, *where* $n > 2$ *is assumed.*

Proof. From the assumptions it is evident that $f \sim \alpha \times \beta \sim \alpha' \times \beta' \sim f'$. Thus there exists a mapping $F : (A \times I, A^{n-1} \times I) \to (R^{n+1} \times R^{n+1}, p \times p)$ such that $F(x, 0) = f(x)$ for $x \in A$ and $F(x, 1) = f'(x)$ for $x \in A$. Moreover we have $F^{-1}(\mathcal{S}^n \vee \mathcal{S}^n)$

$\supset(A\times 0)\cup(A\times 1)$. It follows from Lemma 5 that if $n>2$, there exists a normalization $G:(A\times I,\ A^{n-1}\times I)\to(\mathcal{S}^n\vee\mathcal{S}^n,\ p\times p)$ of F such that $F\sim G$ *rel.* $F^{-1}(\mathcal{S}^n\vee\mathcal{S}^n)$. Thus we have $G(x, 0)=f(x)$ and $G(x)=f'(x)$. That ΩG is a homotopy between Ωf and $\Omega f'$, gives the complete proof of Lemma.

THEOREM 2. *Let A be a finite geometric complex with* $\dim A\leqq n$. *For two normal mappings $f, g:(A, A^{n-1})\to(Y, y_0)$ we have, as was referred to in 2, 3, 4, mappings $\psi_f, \psi_g:(A, A^{n-1})\to(R^{n+1}, p)$. Then the homotopy classes $\{f\}$ of f form an abelian group $\mathfrak{H}^n(A)$ with the law of composition $\{f\}+\{g\}=\{h\Omega\alpha\}$, where α is an arbitrary normalization of $\psi_f\times\psi_g$ and $h:(R^{n+1}, p)\to(Y, y_0)$ is a mapping as used in 2, 3, 4.*

Proof. From Lemma 5 there exists a normalization α of $\psi_f\times\psi_g$, if $n>1$. If $f\sim f'$ and $g\sim g'$, we have $\psi_f\sim\psi_{f'}$ and $\psi_g\sim\psi_{g'}$ from Theorem 1. Thus, if α and α' are normalizations of $\psi_f\times\psi_g$ and $\psi_{f'}\times\psi_{g'}$ respectively, we have $h\Omega\alpha\sim h\Omega\alpha'$ from Lemma 6 in case $n>2$. This proves the uniqueness of the law of composition. It is easily verified that with respect to this composition the homotopy classes form an abelian group.

6. Some preliminary remarks of Hu's results. Hereafter we shall assume that X is a compact Hausdroff space with $\dim X\leqq n$ and Y is a connected A.N.R. (and hence arcwise connected). Here we make some preliminary preparations on Hu's bridge operation, which will be used in the present work, Let $f:X\to Y$ be a given mapping and α a covering of X. Let A_α be a geometric nerve of a covering α. A mapping $\xi^f_\alpha:A_\alpha\to Y$ is called a bridge mapping for f, if $\xi^f_\alpha\varphi_\alpha$ is homotopic with f for each canonical mapping $\varphi_\alpha:X\to A_\alpha$ of the covering α. If such a bridge mapping ξ^f_α exists, α is said to be a bridge for the mapping f. Hu has proved the following theorems on bridge operation.

i) Bridge Refinement Theorem. For a given mapping $f:X\to Y$, any refinement β of a bridge α is also a bridge.

ii) Bridge Existence Theorem. Every mapping $f:X\to Y$ has a bridge α.

iii) Bridge Homotopy Theorem. If α, β be two bridges for a given mapping $f:X\to Y$, *and if $\xi^f_\alpha:A_\alpha\to Y$, $\xi^f_\beta:A_\beta\to Y$ be bridge mappings; then there exists* a common refinement γ of α and β such that $\xi^f_\alpha p_{\gamma\alpha}$ and $\xi^f_\beta p_{\gamma\beta}$ are homotopic, where $p_{\gamma\alpha}:A_\gamma\to A_\alpha$, $p_{\gamma\beta}:A_\gamma\to A_\beta$ are arbitrary simplicial projections.

7. Main Results. Now we assume that Y is a connected A.N.R. with $\pi_i(Y)=0$ for each $i<n$. We shall define a law of composition in the set of the homotopy classes of the mapping $f:X\to Y$. For two mappings $f, g:X\to Y$ there exists a common bridge γ for them in virtue of i), ii). Let ξ^f_γ, ξ^g_γ be bridge mappings for f, g respectively such that $\xi^f_\gamma(A^{n-1}_\gamma)=\xi^g_\gamma(A^{n-1}_\gamma)=y_0$. From Theorem 2

we have a representative $\xi_\gamma^{f+g}: A_\gamma \to Y$ of the class $\{\xi_\gamma^f\}+\{\xi_\gamma^g\}$. Then we have

THEOREM 3. *The homotopy classes $\{f\}$ of $f: X\to Y$ form an abelian group with the law of comosition $\{f\}+\{g\}=\{\varphi_\gamma\xi_\gamma^{f+g}\}$ where $\varphi_\gamma: X\to A_\gamma$ is an arbitrary canonical mapping.*

Proof. It can be shown that this definition of composition does not depend on the choice of the bridge γ, bridge mappings ξ_γ^f, ξ_γ^g, representative mappings f, g, and the canonical mapping φ_γ. First, the independency of the choice of the bridge γ is shown as follows. Let γ' be another common bridge for f and g, and let $\xi_{\gamma'}^f$, $\xi_{\gamma'}^g$, be bridge mappings for f, g respectively. In virtue of bridge homotopy theorem iii) there exists a common refinement δ of γ and γ' such that $\xi_\gamma^f p_{\delta\gamma}\sim\xi_{\gamma'}^f p_{\delta\gamma'}$ and $\xi_\gamma^g p_{\delta\gamma}\sim\xi_{\gamma'}^g p_{\delta\gamma'}$, where $p_{\delta\gamma}: A_\delta\to A_\gamma$, $p_{\delta\gamma'}: A_\delta\to A_{\gamma'}$ are arbitrary simplical projections. It has already been proved in Theorem 2 that $\{\xi_\gamma^f p_{\delta\gamma}\}+\{\xi_\gamma^g p_{\delta\gamma}\}=\{\xi_{\gamma'}^f p_{\delta\gamma'}\}+\{\xi_{\gamma'}^g p_{\delta\gamma'}\}$.

Moreover it is easily verified that $\{\xi_\gamma^f p_{\delta\gamma}\}+\{\xi_\gamma^g p_{\delta\gamma}\}=\{\xi_\gamma^{f+g} p_{\delta\gamma}\}$ and $\{\xi_{\gamma'}^f p_{\delta\gamma'}\}+\{\xi_{\gamma'}^g p_{\delta\gamma'}\}=\{\xi_{\gamma'}^{f+g} p_{\delta\gamma'}\}$. Thus we have $\xi_\gamma^{f+g}p_{\delta\gamma}\sim\xi_{\gamma'}^{f+g}p_{\delta\gamma'}$. This proves that $\xi_\gamma^{f+g}p_{\delta\gamma}\varphi_\delta\sim\xi_{\gamma'}^{f+g}p_{\delta\gamma'}\varphi_\delta$ where $\varphi_\delta: X\to A_\delta$ is an arbitrary canonical mapping. As $p_{\delta\gamma}\varphi_\delta: X\to A_\gamma$ and $p_{\delta\gamma'}\varphi_\delta: X\to A_{\gamma'}$ are canonical mappings, the independency of the choice of γ has been established. As the bridges for homotopic mappings may be the same from their definition, the rule of this composition does not depend on the choice of representative mappings. Because all the canonical mappings $\varphi_\gamma: X\to A_\gamma$ are homotopic, the independency of the choice of φ_γ is also proved. Thus it is easily verified that this rule of composition may be considered to define a group operation in the set of all the homotopy classes $\{f\}$ of $f: X\to Y$.

As was referred in the introduction of this paper, we have the following Main Theorem.

MAIN THEOREM 4. *The group $\tilde{\mathfrak{H}}_n(X)$ is isomorphic to the n-th Čech cohomology group $H^n(X, \pi_n(Y))$ of X with coefficient group $\pi_n(Y)$.*

Proof. Let α be a bridge for $f: X\to Y$ and let $\xi_\alpha^f: A_\alpha\to Y$ be a bridge mapping for f such that $\xi_\alpha^f(X^{n-1})=y_0$, then we have a characteristic cocycle $c^n(\xi_\alpha^f)=\sum_i(\xi_\alpha^f, \sigma_i^n)\sigma_i^n$ of the n-th cohomology group $H^n(A_\alpha, \pi_n(Y))$ of a nerve A_α of α. Correspond to the homotopy class $\{f\}$ of f the element $\{c^n(\xi_\alpha^f)\}$ of the n-th Čech cohomology group $H^n(X, \pi_n(Y))$ which is represented by the cocycle $c^n(\xi_\alpha^f)$. This correspondence λ does not depend on the choice of a bridge α and of a representative f of $\{f\}$. Let us prove this. For another bridge β for another representative g of $\{f\}$, we have a common refinement γ of α and β such that $\xi_\alpha^f p_{\gamma\alpha}$ is homotopic to $\xi_\beta^g p_{\gamma\beta}$ where $p_{\gamma\alpha}: A_\gamma\to A_\alpha$ $p_{\gamma\beta}: A_\gamma\to A_\beta$

be arbitrary simplicial projections, and $\xi_\alpha^f : A_\alpha \to Y$ and $\xi_\beta^g : A_\beta \to Y$ be normal bridge mappings for f and g respectively. Then it is easily seen that $c^n(\xi_\alpha^f p_{\gamma\alpha}) = p_{\gamma\alpha}^* c^n(\xi_\alpha^f)$ and $c^n(\xi_\beta^g p_{\gamma\beta}) = p_{\gamma\beta}^* c^n(\xi_\beta^g)$, where $p_{\gamma\alpha}^*$ and $^*_{\gamma\beta}$ denote the homomorphisms of cocycles induced by simplicial projections $p_{\alpha\gamma}$ and $p_{\gamma\beta}$ and that $c^n(\xi_\alpha^f p_{\alpha\gamma})$ is cohomologous to $c^n(\xi_\beta^g p_{\gamma\beta})$ in virtue of the first homotopy theorem of Eilenberg. It follows that $c^n(\xi_\beta^f)$ and $c^n(\xi_\alpha^\gamma)$ represent the same element of $H^n(X, \pi_n(Y))$. Moreover it can be shown that this correspondence $\lambda : \tilde{\mathfrak{H}}_n(X) \to H^n(X, \pi_n(Y))$ is the desired isomorphism. This completes the proof.

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