SOME CONSEQUENCES OF MARTIN'S AXIOM AND THE NEGATION OF THE CONTINUUM HYPOTHESIS

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§0. W. Sierpisnki [3] demonstrated 82 propositions, called $C_1$--$C_{82}$, with the aid of the continuum hypothesis. D. A. Martin and R. M. Solovay remarked in [2] that 48 of these propositions followed from Martin's axiom (MA), 23 were refuted by $\text{MA} + 2^{\omega_1} > \aleph_1$ and three were independent of $\text{MA} + 2^{\omega_1} > \aleph_1$. But the relation of the remaining eight propositions to $\text{MA} + 2^{\omega_1} > \aleph_1$ has been unsettled.

In this paper, we shall show at least five of them ($C_8$, $C_{13}$, $C_{61}$, $C_{62}$, and $C_{70}$) are also refuted by $\text{MA} + 2^{\omega_1} > \aleph_1$.

The following table gives the relation of $C_1$--$C_{82}$ to $\text{MA} + 2^{\omega_1} > \aleph_1$.

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By $O$, we denote the propositions following from MA, by $\times$ the propositions refuted by $\text{MA} + 2^{\omega_1} > \aleph_1$, by $\triangle$ the propositions independent of $\text{MA} + 2^{\omega_1} > \aleph_1$ and by ? the propositions whose relation to $\text{MA} + 2^{\omega_1} > \aleph_1$ we do not know about at present.

Let $\mathcal{P} = \langle P, \leq \rangle$ be a partially ordered set. A subset $X$ of $P$ is said to be dense in $\mathcal{P}$ if, for every $p \in P$, there is $q \in X$ such that $p \leq q$. If $\mathcal{F}$ is a collection of dense subsets of $P$, a subset $G$ of $P$ is said to be an $\mathcal{F}$-generic filter on $\mathcal{P}$ if $G$ has the following properties:

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(1) if \( p, q \in P \), \( p \in G \) and \( q \leq p \), then \( q \in G \);

(2) if \( p, q \in G \), then there is \( r \in G \) such that \( p \leq r \) and \( q \leq r \);

(3) if \( X \in \mathcal{F} \), then \( X \cap G \neq 0 \).

If \( p, q \in P \), then \( p \) and \( q \) are said to be compatible if there is \( r \in P \) such that \( p \leq r \) and \( q \leq r \). \( P \) is said to have the countable chain condition if every collection of pairwise incompatible elements of \( P \) is countable.

Martin's axiom (MA) is the following statement:

If \( \mathcal{P} = \langle P, \leq \rangle \) is a partially ordered set having the countable chain condition and \( \mathcal{F} \) is a collection of dense open subsets of \( P \) of cardinality \( < 2^{\aleph_0} \), then there exists an \( \mathcal{F} \)-generic filter on \( \mathcal{P} \).

§ 1. In this section, we shall show \( C_8, C_9, C_{60} \), and \( C_{62} \) are refuted by MA + \( \neg \text{CH} \). From [2], we quote the following lemma.

**Lemma 1.** Let \( A \) and \( B \) be collections of subsets of \( \omega \), each of cardinality \( < 2^{\aleph_0} \), such that if \( x \in B \) and \( K \) is a finite subset of \( A \) then \( x - \bigcup K \) is infinite. If we assume MA, then there exists a subset \( t \) of \( \omega \) such that \( x \cap t \) is finite if \( x \in A \) and infinite if \( x \in B \).

Let \( \omega^\omega \) be the set of all functions from \( \omega \) into \( \omega \), (more generally, \( x^y \) be the set of all functions from \( x \) into \( y \)). Following Sierpinski [3], we define a partial ordering \( < \) on \( \omega^\omega \) as follows:

\[ f < g \leftrightarrow (\exists k \in \omega)(\forall n \geq k)[f(n) < g(n)] \]

The following lemma is due to K. Kunen [1].

**Lemma 2.** Let \( F \) be a subset of \( \omega^\omega \) of cardinality \( < 2^{\aleph_0} \). If we assume MA, then there exists \( g \in \omega^\omega \) such that if \( f \in F \) then \( f < g \).

From Lemma 2, we have the following proposition, which is the negation of \( C_9 \).

**Proposition 1 (Assume MA and \( 2^{\aleph_0} > \aleph_1 \)).** Let \( E \) be an uncountable subset of \( \mathbb{R} \), the set of reals, and \( \langle f_n : n \in \omega \rangle \) be a convergent sequence of functions from \( E \) to \( \mathbb{R} \). Then there exists an uncountable subset \( N \) of \( E \) such that \( \langle f_n : n \in \omega \rangle \) is uniformly convergent on \( N \).

**Proof.** We may assume \( E \) is of cardinality \( \aleph_1 \). Let \( f \) be the limit of \( \langle f_n : n \in \omega \rangle \). Then for any \( x \in E \) and \( m \in \omega \), there is \( k \in \omega \) such that if \( n \geq k \) then \( |f_n(x) - f(x)| < 1/m + 1 \). Take such \( k \in \omega \) and denote it by \( \varphi_x(m) \). Then we can define \( \aleph_1 \) functions \( \varphi \) from \( \omega \) into \( \omega \). Using
Lemma 2, we can find $\varphi \in {}^\omega \omega$ such that $\varphi_x < \varphi$ for all $x \in E$. For each $x \in E$, let $k_x$ denote the least $k \in \omega$ such that $\varphi_x(m) < \varphi(m)$ for all $m \geq k$. Since $E$ is uncountable, there is $k \in \omega$ and an uncountable subset $N$ of $E$ such that if $x \in N$ then $k_x = k$. Then for any $x \in N$ and $m \geq k$, if $n \geq \varphi(m)$ then $|f_x(x) - f(x)| < 1/m + 1$. This means $\langle f_n : n \in \omega \rangle$ converges uniformly to $f$ on $N$.

Since $C_8$ and $C_9$ are equivalent, $C_8$ is also refuted by $\text{MA} + 2^{\aleph_0} > \aleph_1$.

Recall that an $F_\sigma$-set is the union of a countable family of closed sets and a $G_\delta$-set is the intersection of a countable family of open sets.

**Lemma 3.** Let $X$ be a separable metric space of cardinality $< 2^{\aleph_0}$. If we assume $\text{MA}$, then every subset of $X$ is $F_\sigma$ and $G_\delta$ in $X$.

**Proof.** Let $D$ be any subset of $X$ and $\{B_i : i \in \omega\}$ be a basis for open sets of $X$ such that all $B_i$ are non-empty. For each $x \in X$, let $s_x = \{i \in \omega : x \in B_i\}$. If we put $A = \{s_x : x \in X - D\}$ and $B = \{s_y : y \in D\}$, then $A$ and $B$ are of cardinality $2^{\aleph_0}$. It is easily checked that if $y \in D$ and $x_1, \ldots, x_n \in X - D$ then $s_y - (s_{x_1} \cup \cdots \cup s_{x_n})$ is infinite. By Lemma 1, we can find a subset $t$ of $\omega$ such that $s_x \cap t$ is finite if $x \in X - D$ and $s_y \cap t$ is infinite if $y \in D$. For each $n \in \omega$, let

$$K_n = \bigcup_{i > n} B_i.$$  

And let $K = \bigcap_{n \in \omega} K_n$. Then $K$ is a $G_\delta$-set of $X$. In order to prove that $D$ is a $G_\delta$-set of $X$, it suffices to prove the following (1) and (2):

1. $D \subseteq K$
2. $(X - D) \cap K = 0$.

Let $y$ be an arbitrary element of $D$ and $n \in \omega$. Since $t \cap s_y$ is infinite, there is $i \in t \cap s_y$ such that $i > n$. Then $y \in B_i$ and $B_i \subseteq K_n$, so $y \in K_n$. Since $y$ and $n$ are arbitrary, we have (1). Let $x$ be any element of $X - D$. Since $t \cap s_x$ is finite, there is $n \in \omega$ such that if $i \in t$ and $i > n$ then $i \in s_x$. For such $n \in \omega$, we have $x \notin K_n$, and so $x \notin K$. Thus we have (2).

Replacing $D$ with $X - D$, we have that $X - D$ is a $G_\delta$-set of $X$. Hence $D$ is an $F_\sigma$-set of $X$. Therefore $D$ is $F_\sigma$ and $G_\delta$ in $X$.

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*This lemma is a slight generalization of that of J. Silver.*
The following proposition is the negation of C_{62}.

**PROPOSITION 2.** (Suppose MA and \(2^{\aleph_0} > \aleph_1\)). Let \(E\) be any uncountable set of reals and \(f\) be any function from \(E\) into \(\mathbb{R}\), the set of reals. Then there exists an uncountable subset \(N\) of \(E\) such that \(f \upharpoonright N\), the restriction of \(f\) to \(N\), is continuous on \(N\).

**Proof.** We may assume \(E\) is of cardinality \(\aleph_1\). Let \(F\) be an arbitrary closed set in \(\mathbb{R}\). Then, by Lemma 3, \(f^{-1}(F)\), the inverse image of \(F\), is a \(G_\delta\)-set of \(E\). Thus \(f_n\) is Baire function of class \(\leq 1\). As is well-known, every Baire function of class \(\leq 1\) whose range is a subset of \(\mathbb{R}\) is the limit of a sequence of continuous functions. Let \(\langle f_n : n \in \omega \rangle\) be a sequence of continuous functions from \(E\) to \(\mathbb{R}\) which converges to \(f\). Then, by Proposition 1, there exists an uncountable subset \(N\) of \(E\) such that \(\langle f_n : n \in \omega \rangle\) converges uniformly to \(f\) on \(N\). Since each \(f_n \upharpoonright N\) is continuous on \(N\), so is \(f \upharpoonright N\).

This proposition implies the following proposition, which is the negation of C_{61}.

**PROPOSITION 3.** (Suppose MA and \(2^{\aleph_0} > \aleph_1\)). There is a subset \(F\) of \(\mathbb{R}\) of cardinality \(2^{\aleph_0}\) such that if \(g \in \mathbb{R}\) then for some \(f \in F\) the set \(\{x \in \mathbb{R} : f(x) = g(x)\}\) is uncountable.

**Proof.** Let \(F\) be the set of Baire functions from \(\mathbb{R}\) into \(\mathbb{R}\). Then clearly, \(F\) is of cardinality \(2^{\aleph_0}\). By Proposition 2, if \(g \in \mathbb{R}\), then there exists an uncountable subset \(N\) of \(\mathbb{R}\) such that \(g \upharpoonright N\) is continuous on \(N\). The following is a well-known theorem.

Let \(X\) be an arbitrary metric space, let \(Y\) be a complete separable space and \(A\) be a subset of \(X\). Then every Baire function from \(A\) to \(Y\) can be extended to a Baire function from \(X\) into \(Y\).

Since \(f \upharpoonright N\) is a Baire function on \(N\), by this theorem, there exists \(f \in F\) such that \(f \upharpoonright N = g \upharpoonright N\). Thus the set \(\{x \in \mathbb{R} : f(x) = g(x)\}\) includes \(N\), and is uncountable.

\(§2\). Let \([\omega]^{\aleph_0}\) denote the set of all infinite subsets of \(\omega\). We define a relation \(\subseteq^*\) on \([\omega]^{\aleph_0}\) as follows:

\[ a \subseteq^* b \leftrightarrow a - b \text{ is finite, where } a, b \in [\omega]^{\aleph_0}. \]

Intuitively \(a \subseteq^* b\) iff \(a \subseteq b\) almost everywhere.
LEMMA. Suppose MA. Let $\Theta$ be an ordinal such that $\Theta < 2^{\omega_1}$, and let $\langle a_\alpha : \alpha < \Theta \rangle$ be a sequence of elements of $[\omega]^{\omega_1}$ such that if $\alpha < \beta < \Theta$ then $a_\beta \subseteq^* a_\alpha$. Then there exists $a \in [\omega]^{\omega_1}$ such that if $\alpha < \Theta$ then $a \subseteq^* a_\alpha$.

Proof. Let $A = \{ \omega - a_\alpha : \alpha < \Theta \}$ and $B = \{ a_\alpha : \alpha < \Theta \}$. Then clearly, $A$ and $B$ are of cardinality $< 2^{\omega_1}$. If $a, a_1, \cdots, a_n < \Theta$, then

$$a_\alpha - \bigcup_{i=1}^{n} (\omega - a_\alpha) = a_\alpha \cap a_{a_1} \cap \cdots \cap a_{a_n}.$$

It is easily checked the intersection of finite elements of $B$ is an element of $[\omega]^{\omega_1}$. Thus $A$ and $B$ satisfy the condition of Lemma 1 of § 1. Therefore there is a subset $a$ of $\omega$ such that $a - a_\alpha$ is finite and $a \cap a_\alpha$ is infinite for any $\alpha < \Theta$. For such $a \subseteq \omega$, we have $a \in [\omega]^{\omega_1}$ and $a \subseteq^* a_\alpha$.

From this lemma, we obtain the following proposition, which is the negation of $C_{13.}$

PROPOSITION. (Assume MA and $2^{\omega_1} > \aleph_1$). Let $\langle f_n : n \in \omega \rangle$ be a sequence of functions from $\mathbb{R}$ to $\mathbb{R}$. Then there exists a sequence $\langle m_k : k \in \omega \rangle$ of natural numbers such that $m_0 < m_1 < \cdots < m_k < \cdots$ and the set $\{ x \in \mathbb{R} : \langle f_{m_k}(x) : k \in \omega \rangle$ converges to a finite or infinite value $\}$ is uncountable.

Proof. For each $a \in [\omega]^{\omega_1}$, let $a'$ denote the sequence $\langle n_k : k \in \omega \rangle$ such that $n_0 < n_1 < \cdots < n_k < \cdots$ and $a = \{ n_k : k \in \omega \}$. By the limit of the sequence $\langle f_n(a) : n \in \omega \rangle$, we mean the limit of the sequence $\langle f_{m_k}(x) : k \in \omega \rangle$ in the usual sense, where $\langle n_k : k \in \omega \rangle = a'$. Let $E$ be a subset of $\mathbb{R}$ of cardinality $\aleph_1$. Order $E$ into a transfinite sequence of type $\omega_1$ as follows:

$$x_0, x_1, \cdots, x_\alpha, \cdots \quad (\alpha < \omega_1)$$

By transfinite induction on $\alpha$, we define a sequence $\langle a_\alpha : \alpha < \omega_1 \rangle$ of elements of $[\omega]^{\omega_1}$ such that $a_\beta \subseteq^* a_\alpha$ if $\alpha < \beta < \omega_1$ and the sequences $\langle f_n(x_\alpha) : n \in a_\alpha \rangle$ with $\alpha \in \omega_1$ are convergent. The sequence $\langle f_n(x_\alpha) : n \in \omega \rangle$ includes a convergent subsequence $\langle f_{m_k}(x_\alpha) : k \in \omega \rangle$, whose limit is finite or infinite. So, we define $a_\alpha$ to be $\{ n_k : k \in \omega \}$. Assume that $a_\beta$ with $\beta < \alpha$ are defined and $a_\gamma \subseteq^* a_\beta$ if $\beta < \gamma < \alpha$. Then, by the above lemma, we can find $a \subseteq [\omega]^{\omega_1}$ such that $a \subseteq^* a_\beta$ for all $\beta < \alpha$. The sequence $\langle f_\iota(x_\alpha) : \iota < \alpha \rangle$:

\footnote{It was pointed out by the referee that this lemma could be proved from Lemma 2 of § 1.}

\footnote{This proof was suggested to the author by Professor Kanji Namba.}
$i \in a$ includes a convergent subsequence $\langle f_k(x_k) : k \in \omega \rangle$. So, we define $a_n$ to be $\{i_k : k \in \omega\}$.

By the lemma of this section, let $b$ be an element of $[\omega]^\omega$ such that $b \subseteq^* a_n$ for all $\alpha < \omega_1$. For every $\alpha < \omega_n$, since $b \subseteq^* a_n$, the sequence $\langle f_m(x) : m \in b \rangle$ is convergent. If we put $\langle m_k : k \in \omega \rangle = b'$, then the set $\{x : \langle f_{m_k}(x) : k \in \omega \rangle \text{ is convergent} \}$ includes $E$, and is uncountable.

§3. Let $E$ be a subset of $\mathbb{R}$ and $a \in R$. By $E(a)$ we denote the set $\{x + a : x \in E\}$.

Without MA, we can prove the following proposition.

**Proposition.** (Suppose $2^{\aleph_0} > \mathfrak{c}$.) If $E$ is an uncountable subset of $\mathbb{R}$ such that its complement is of cardinality $2^{\aleph_0}$, then there exists $a \in \mathbb{R}$ such that $E(a) \triangle E$, the symmetric difference of $E(a)$ and $E$, is uncountable.

**Proof.** Suppose, on the contrary, that for any $a \in \mathbb{R}$, $E(a) \triangle E$ is countable. Let $N$ be a subset $E$ of cardinality $\aleph_0$. Then we show $\bigcap_{x \in N} [R - E(-x)] \neq 0$. If $\bigcap_{x \in N} [R - E(-x)] = 0$, then $R = \bigcup_{x \in N} E(-x)$.

On the other hand
\[
\bigcup_{x \in N} E(-x) = \bigcup_{x, y \in N} [E(-x) \triangle E(-y)] \cup \bigcap_{x \in N} E(-x).
\]

Therefore,
\[
A = \bigcup_{x \in N} E(-x) = R,
\]
where $A = \bigcup_{x, y \in N} [E(-x) \triangle E(-y)]$.

Since $A$ and $\bigcap_{x \in N} E(-x)$ are disjoint, we have $R - \bigcap_{x \in N} E(-x) = A$. Let $x$ be an arbitrary element of $N$. Then we have $R - E(-x) \subseteq A$. Note that each $E(a) \triangle E(b)$ is countable because $E(a) \triangle E(b) = J(a) \cup K(b)$, where $J = E(b - a) \triangle E$, $K = E(a - b) \triangle E$. Therefore $A$ is of cardinality $\leq \mathfrak{c}$. This contradicts the hypothesis that the complement of $E$ is of cardinality $2^{\aleph_0}$. Thus $\bigcap_{x \in N} [R - E(-x)] \neq 0$.

Let $a \in \bigcap_{x \in N} [R - E(-x)]$, then $N \subseteq R - E(-a)$ because $a \in E(-x)$ iff $x \in E(-a)$. Therefore $E(-a) \triangle E$ includes $N$, and is uncountable.

The following corollary is the negation of $C_{\aleph_0}$.

**Corollary.** (Suppose MA and $2^{\aleph_0} > \mathfrak{c}$.) Let $E$ be a non-measurable set of reals. Then for some $a \in \mathbb{R}$, $E(a) \triangle E$ is uncountable.

**Proof.** If we assume MA, then every set of reals of cardinality...
<2^{\kappa} is of Lebesgue measure 0 ([2, § 4]). Hence, if \( E \) is non-measurable, the \( E \) and its complement are of cardinality \( 2^{\kappa} \). Thus \( E \) satisfies the condition of the proposition.

§ 4. A set \( E \) of reals is said to have the property \((M)\)\(^\dagger\) if, for any collection \( \mathcal{U} \) of open sets satisfying the condition

\[ (\forall x \in E)(\forall \varepsilon > 0)(\exists U \in \mathcal{U})[\delta(U) < \varepsilon \land x \in U] \]

where \( \delta(U) \) is the diameter of \( U \), there is a sequence \( \langle U_n : n \in \omega \rangle \) of elements of \( \mathcal{U} \) such that \( E \subseteq \bigcup_{n \in \omega} U_n \) and \( \lim_{n \to \omega} \delta(U_n) = 0 \).

As a direct application of MA, we have the following proposition.

**Proposition.** (Suppose MA). Every set of reals of cardinality \(< 2^{\kappa} \) has the property \((M)\).

**Proof.** Let \( E \) be a set of reals of cardinality \(< 2^{\kappa} \), and \( \mathcal{U} \) be a collection of open sets satisfying the condition \((*)\). For each \( n \in \omega \), there is a sequence \( \langle U_{nm} : m \in \omega \rangle \) of elements of \( \mathcal{U} \) such that \( E \subseteq \bigcup_{m \in \omega} U_{nm} \) and \( \delta(U_{nm}) < 1/n + 1 \) for all \( m \in \omega \). We define a partially ordered set \( \mathcal{P} = \langle P, \leq \rangle \) as follows:

\[
P = \{ p : p \text{ is a finite function with } \text{dom}(p) \cup \text{rang}(p) \subseteq \omega \},
\]

\[
p \leq q \iff p \subseteq q.
\]

Then clearly, \( \mathcal{P} \) satisfies the countable chain condition. For each \( x \in E \), if we put \( X_x = \{ p \in P : x \in \bigcup_{n \in \text{dom}(p)} U_{np(n)} \} \), then \( X_x \) is dense in \( \mathcal{P} \). Let \( \mathcal{F} = \{ X_x : x \in E \} \). Then \( \mathcal{F} \) is of cardinality \(< 2^{\kappa} \), so there is an \( \mathcal{F} \)-generic filter \( G \) on \( \mathcal{P} \). If we put \( f = \bigcup G \), then \( f \) is a function with \( \text{dom}(f) \subseteq \omega \) and \( \text{rang}(f) \subseteq \omega \). We define \( U_n \) as follows:

\[
U_n = \begin{cases} 
U_{nf(n)} & \text{if } n \in \text{dom}(f) \\
U_{n0} & \text{otherwise}
\end{cases}
\]

Then, clearly, \( U_n \in \mathcal{U} \) and \( \lim_{n \to \omega} \delta(U_n) = 0 \). Let \( x \) be an arbitrary element of \( E \). Since \( X_x \cap G \neq 0 \), there is \( p \in G \) such that \( x \in \bigcup_{n \in \text{dom}(p)} U_{np(n)} \). Since \( P \in G \), we have \( \bigcup_{n \in \text{dom}(p)} U_{np(n)} \subseteq \bigcup_{n \in \omega} U_n \), so \( x \in \bigcup_{n \in \omega} U_n \). Therefore \( E \) has the property \((M)\).

\(^\dagger\) See [3, p. 48]
§ 5. A set $E$ of reals is said to have the property ($\lambda$) if every countable subset of $E$ is a $G_\delta$-set of $E$.

In this section, we shall show there is a non-measurable set of reals of cardinality $2^{\aleph_0}$ with the property ($\lambda$).

A set $E$ of reals is said to have the property ($S^*$) if, for every set $N$ of Lebesgue measure 0, $E \cap N$ is of cardinality $< 2^{\aleph_0}$. If a set $E$ is measurable and has positive measure, then $E$ includes a set of measure 0 and cardinality $2^{\aleph_0}$. If we assume MA, then every set of reals of cardinality $< 2^{\aleph_0}$ is of Lebesgue measure 0. Therefore every set of reals of cardinality $2^{\aleph_0}$ with the property ($S^*$) is non-measurable. The existence of a non-measurable set of reals of cardinality $2^{\aleph_0}$ with the property ($\lambda$) follows from the following proposition.

**Proposition.** (Suppose MA). There is a set $E$ of reals of cardinality $2^{\aleph_0}$ with the property ($S^*$) such that every subset of $E$ of cardinality $< 2^{\aleph_0}$ is $G_\delta$ in $E$.

**Proof.** Order the set of all $G_\delta$-sets of measure 0 into a transfinite sequence of type $2^{\aleph_0}$ as follows:

$$N_0, N_1, \ldots, N_t, \ldots, (\xi < 2^{\aleph_0}).$$

By transfinite induction on $\alpha$, we define a sequence $\langle x_\alpha: \alpha < 2^{\aleph_0} \rangle$ of reals and a sequence $\langle K_\alpha: \alpha < 2^{\aleph_0} \rangle$ of $G_\delta$-sets of measure 0. Let $K_0 = N_0$ and $x_0$ be an arbitrary element of $\mathbb{R}$. Suppose $x_\beta$ and $K_\beta$ with $\beta < \alpha$ are defined, and let

$$S_\alpha = \bigcup_{\beta < \alpha} K_\beta \cup \{x_\beta: \beta < \alpha\} \cup N_\alpha.$$

Then, by MA, $S_\alpha$ is of measure 0, so $R - S_\alpha \neq 0$. Let $x_\alpha$ be an arbitrary element of $R - S_\alpha$ and $K_\alpha$ be the first $N_\alpha$ such that $S_\alpha \cup \{x_\alpha\} \subseteq N_\alpha$.

Let $E$ be the set $\{x_\alpha: \alpha < 2^{\aleph_0}\}$. Then we have

1. $E$ is of cardinality $2^{\aleph_0}$;
2. for each $\alpha < 2^{\aleph_0}$, $E \cap N_\alpha$ is of cardinality $< 2^{\aleph_0}$;
3. $K_\alpha \subseteq K_\beta$ if $\alpha < \beta < 2^{\aleph_0}$.

From (1) and (2), $E$ is a set of cardinality $2^{\aleph_0}$ with the property ($S^*$).

Let $D$ be an arbitrary subset of $E$ of cardinality $< 2^{\aleph_0}$. Since $2^{\aleph_0}$ is a regular cardinal, there is $\alpha < 2^{\aleph_0}$ such that $D \subseteq \{x_\beta: \beta \leq \alpha\}$. Put

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1) See [3, p. 94]

2) Cf. [3, p. 81]
$X = \{ x_\beta : \beta \leq \alpha \}$. Then, by Lemma 3 of § 1, $D$ is a $G_\delta$-set in $X$. Since $X = E \cap K_\alpha$ and $K_\alpha$ is $G_\delta$ in $R$, $X$ is $G_\delta$ in $E$. Therefore $D$ is a $G_\delta$-set in $E$.

REFERENCES


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