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Φ-BOUNDED HARMONIC FUNCTIONS AND THE CLASSIFICATION OF HARMONIC SPACES

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1. By a *harmonic space* we mean a pair (X, H) where X is a locally compact, non-compact, connected, locally connected Hausdorff space; and H is a sheaf of *harmonic functions* defined as follows: Suppose to each open set $\Omega \subset X$ there corresponds a linear space $H(\Omega)$ of finitely-continuous real-valued functions defined on Ω. Then $H = \{H(\Omega)\}_\Omega$ must satisfy the three axioms of Brelot (1) and in addition Axiom 4 of Loeb (4): 1 is H-superharmonic in X.

Denote by $\Phi(t)$ a nonnegative real-valued function defined on $[0, \infty)$. We stress that except for the condition $\Phi(t) \geqslant 0$ nothing is required of $\Phi(t)$ such as continuity and measurability. A harmonic function u on X (when H is well-understood we simply refer to X itself as the harmonic space) is called *Φ-bounded* if the composite function $\Phi(|u|)$ possesses a harmonic majorant on X. The notion of Φ-boundedness is due to Parreau (9) who considered the special case of an increasing, convex Φ. Later Nakai (6), using general Φ, completely determined the class $O_{H\Phi}$ of Riemann surfaces for which every Φ-bounded harmonic function reduces to a constant. Recently Ow (8) considered the classification of harmonic spaces with respect to Φ-bounded harmonic functions using a stronger assumption that Loeb's Axiom 4; namely it was assumed that $1 \in H$.

Since the case $1 \in H$ has already been considered, as mentioned above, throughout this paper we will make the following assumption:

$$1 \notin H .$$

This condition occurs, for example, in the study of the harmonic space of solutions of the elliptic partial differential equation $\Delta u = Pu$, where $P \not\equiv 0$ is a nonnegative function on a manifold X.

The main object of this paper is to show that in view of the con-

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dition $1 \notin H$, the assumed existence of a harmonic function u on X with positive infimum is essential in the classification of harmonic spaces with respect to Φ-boundedness. Furthermore, it is shown that is sometimes necessary to further assume that the function u above is bounded in order to obtain inclusion relations similar to those in (8). Before proceeding further it is necessary to give some preliminary results.

2. If K is a compact subset of X and E the family of all regular regions Ω (cf. (4)) containing K then by a theorem of Loeb (4), E is an exhaustion of X. We will always assume that X is countable at the ideal boundary and therefore there exists a countable exhaustion of X by regular regions $\{\Omega_n\}_1^\infty$ such that $\bar{\Omega}_n \subset \Omega_{n+1}$ and $X = \bigcup_{n=1}^\infty \Omega_n$.

We now state some results of Loeb-Walsh (5) using their terminology. Let e be the greatest H-harmonic minorant of 1 and assume that $e \neq 0$. Denote by $HB = HB(X)$ the Banach lattice of bounded functions in H. Note that $HB \neq \{0\}$. Let X^* be the HB-compactification of $X, \Gamma = X^* - X$, and

$$\Delta = \{t \in \Gamma \mid e(t) = 1 \text{ and } f \wedge_H g(t) = f \wedge g(t) \text{ for all } f, g \in HB\} .$$

It is shown in (5) that Δ is regular for the Dirichlet problem and is also equivalent to the harmonic boundary of Constantinescu-Cornea (3). Also it is shown in (5) that the restriction mapping of HB onto $C(\Delta)$ is an isometric isomorphism which preserves positivity and lattice operations.

3. If Ω is a subregion of X then we will say that $\Omega \notin SO_{HB}$ provided Ω contains a neighborhood of some point $p \in \Delta$. We then have the following generalization of the well-known two-domain criterion for Riemann surfaces (cf. e.g. (10)):

LEMMA 1. *There exists at least $k \geqslant 1$ disjoint regions $\Omega_i \subset X$ with $\Omega_i \notin SO_{HB}$ if and only if* $\dim HB \geqslant k$.

Proof. It follows from the definition that if there exist at least $k \geqslant 1$ disjoint regions $\Omega_i \notin SO_{HB}$ then Δ contains at least k points and hence $\dim HB = \dim C(\Delta) \geqslant k$. Conversely suppose $\dim HB \geqslant k$. Then there exists at least k points $p_j \in \Delta$. Let f be a Wiener function, i.e. a bounded, continuous, harmonizable function on X (cf. (10)) such that $f(p_j) = j$. Set $G_j^* = \{p \in X^* \mid j - \frac{1}{2} < f(p) < j + \frac{1}{2}\}$ and $G_j = G_j^* \cap X$. Then $G \notin SO_{HB}$ and the G_j are disjoint. This completes the proof.

4. As an immediate consequence of a result of Constantinescu-Cornea (cf. (3), p. 32) the following maximum principle of Nakai (10) is also valid for harmonic spaces:

LEMMA 2. *Let Ω be a subregion of X and s a superharmonic function on Ω bounded from below. If*

$$\liminf_{z \in \Omega,\ z \to p} s(z) \geqslant 0$$

for every point $p \in (\Delta \cap \bar{\Omega}) \cup \partial\Omega$ then $s \geqslant 0$ on Ω. Here $\bar{\Omega}$ means the closure of Ω in X^* while $\partial\Omega$ denotes the boundary of Ω relative to X.

5. Denote by $H\Phi = H\Phi(X)$ the family of all Φ-bounded harmonic functions on X and by $O_{H\Phi}$ the totality of harmonic spaces on which every Φ-bounded harmonic function reduces to a constant. Similarly denote by $HP = HP(X)$, $HB = HB(X)$ the class of functions on X which are nonnegative harmonic and bounded harmonic, respectively; and by O_{HP} (resp. O_{HB}) the class of harmonic spaces X for which the class HP (resp. HB) consists only of constants. We define

$$\bar{d}\Phi = \limsup_{t \to \infty} \Phi(t)/t \quad \text{and} \quad \underline{d}\Phi = \liminf_{t \to \infty} \Phi(t)/t\ .$$

Suppose that there exists a positive harmonic function on X with positive infimum. We then note first that if Φ is bounded on $[0, \infty)$ then any nonconstant harmonic function on X is a nonconstant $H\Phi$-function, and consequently, $O_{H\Phi}$ consists only of trivial harmonic spaces. On the other hand if $\Phi(t)$ is completely unbounded on $[0, \infty)$, i.e. if $\Phi(t)$ is not bounded in any neighborhood of any point of $[0, \infty)$ then $O_{H\Phi}$ must consist of all harmonic spaces. Having dispensed with these cases we now prove a result similar to one obtained for Riemann surfaces by Nakai (6).

THEOREM 1. *Assume there exists a bounded harmonic function u_0 on X with $\inf_X u_0 > 0$. Then if Φ is not bounded nor completely unbounded on $[0, \infty)$, $O_{H\Phi} = O_{HP}$ (resp. $O_{H\Phi} = O_{HB}$) provided that $\bar{d}(\Phi)$ is finite (resp. infinite).*

A proof of Theorem 1 will be given in section 7. Using stronger assumptions on Φ, Chow-Glasner (2) have obtained results similar to Theorem 1 in their investigation on Φ-bounded solutions of $\Delta u = Pu$, $P \geqslant 0$, on Riemannian manifolds. Namely they assume that Φ is convex, positive, and increasing.

6. The next theorem shows the effect of omitting either the boundedness condition or the condition $\inf_X u_0 > 0$ as was required of the function u_0 in Theorem 1.

THEOREM 2. *Assume Φ is not bounded nor completely unbounded on* $[0, \infty)$.

a) If $\bar{d}(\Phi) < \infty$ then $O_{HP} \subset O_{H\Phi}$.

b) If $\bar{d}(\Phi) < \infty$ and if there exists an HP-function u_1 with $\inf_X u_1 > 0$, *then $O_{H\Phi} \subset O_{HP}$. But if $\bar{d}(\Phi) < \infty$, if there exists a nonconstant HP-function, and if $u \in HP$ implies* $\inf_X u = 0$, *then $O_{H\Phi} \subset O_{HP}$ is not necessarily true.*

c) If $\bar{d}(\Phi) = \infty$ then $O_{HB} \subset O_{H\Phi}$.

d) If $\bar{d}(\Phi) = \infty$ and there exists an HP-function u_0 such that u_0 is bounded and $\inf_X u_0 > 0$, *then $O_{H\Phi} \subset O_{HB}$. However, if $\bar{d}(\Phi) = \infty$ and every HP-function u is either unbounded or* $\inf_X u = 0$ *then $O_{H\Phi} \subset O_{HB}$ is not necessarily true.*

A proof of Theorem 2 appears in section 8. The existence of u_1 is also considered by Schiff (12) in the special case concerning solutions of $\Delta u = Pu$ on a Riemann surface.

7. *Proof of Theorem* 1. First assume $\bar{d}(\Phi) < \infty$. Then there exists a $c > 0$ such that $\Phi(t) \leqslant ct$ for $t \geqslant t_0$. If u is a nonconstant HP-function on X then for a suitable constant $k > 0$ the function $v = u + ku_0$ is a nonconstant $H\Phi$-function, and so $O_{H\Phi} \subset O_{HP}$.

Conversely if u is a nonconstant $H\Phi$-function on X then there exists an HP-function v on X with $\Phi(|u|) \leqslant v$ on X. Since $1 \notin H, v$ is nonconstant. Hence $O_{HP} \subset O_{H\Phi}$, completing the first part of the proof.

Now consider the case where $\bar{d}(\Phi) = \infty$. Suppose u is a nonconstant HB-function on X. By hypothesis Φ is bounded in some interval $(a, b) \subset [0, \infty)$ within which $\Phi(t) \leqslant c = \text{const}$. Then for suitable constants c_1 and c_2 the range of $v = c_1u + c_2u_0$ is contained in (a, b), and consequently $O_{H\Phi} \subset O_{HB}$.

Conversely, if we assume u is a nonconstant $H\Phi$-function on X then there exists an HP-function v on X such that $\Phi(|u|) \leqslant v$ on X. If v is bounded we are done. If u is not bounded then following the approach of Nakai (6) we show that $X \notin O_{HB}$. Suppose to the contrary that

$X \in O_{HB}$. Then $\bar{d}(\Phi) = \infty$ implies that there is a strictly increasing sequence $\{t_n\}_1^\infty$ of positive numbers for which $\lim_n t_n = \infty$, $\lim_n t_n/\Phi(t_n) = 0$ and

$$G_n = \{p \in X \mid |u(p)| < t_n\} \neq \phi .$$

Then $G_1 \subset G_2 \subset \cdots$ and $X = \bigcup_1^\infty G_n$. Now $G_n \notin SO_{HB}$ for all sufficiently large n. For if not, consider the function $a_n v - |u|$ where $a_n = t_n/\Phi(t_n)$. Then $a_n v - |u|$ is superharmonic, bounded from below on G_n, and nonnegative on ∂G_n. Hence $G_n \in SO_{HB}$ implies $a_n v - |u| \geqslant 0$ on G_n by Lemma 2. Since $a_n \to 0$ and $G_n \uparrow X$ we have $u \equiv 0$ on X, a contradiction. Hence $G_n \notin SO_{HB}$ for $n \geqslant n_1$, say, and so we may as well assume

$$G_n \notin SO_{HB} , \qquad n = 1, 2, \cdots$$

If $G_n - \bar{G}_1 \notin SO_{HB}$ for some $n > 1$ then by Lemma 1, $X \notin O_{HB}$, contradicting our original assumption. Hence

$$G_n - \bar{G}_1 \in SO_{HB} , \qquad n = 2, 3, \cdots$$

The function $w_n = a_n v + r_1 - |u|$ is superharmonic, bounded from below on G_n as well as $G_n - \bar{G}_1$. Also $w_n \geqslant 0$ on ∂G_n. Since $G_n - \bar{G}_1 \in SO_{HB}$ this implies $w_n \geqslant 0$ on G_n, i.e.

$$|u| \leqslant a_n v + r_1$$

on G_n. Hence $|u| \leqslant r_1$ on X, contradicting our assumption $X \in O_{HB}$. Hence $X \notin O_{HB}$, completing the proof.

8. *Proof of Theorem* 2. Parts a) and c) are proved exactly as in the proof of Theorem 1 since the function u_0 is not involved. The first part of b) follows exactly as in Theorem 1 since only the condition $\inf_X u_0 > 0$ is used there. For the second half of b) consider the following example:

EXAMPLE 1. Define $\Phi(t) = 1/t^2$, $t > 0$; $\Phi(0) = 0$. Then $\bar{d}(\Phi) = 0 < \infty$. Also for any harmonic function u, either $u \in HP$ or $-u \in HP$ or u assumes the value 0 on X. In either case $\inf_X |u| = 0$. It follows that $\Phi(|u|)$ has no HP-majorant on X, i.e. $O_{H\Phi} \not\subset O_{HP}$.

The first assertion in d) constitutes part of Theorem 1. For the second part of d) consider the following example in the complex plane $\boldsymbol{C}$:

EXAMPLE 2. Let $X = \{z \in \boldsymbol{C} \mid 0 < |z| < 1\}$ and H consist of all solutions of the elliptic partial differential equation $\Delta u = Pu$ on X, where $P = 4/|z|^2$

and $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, $z = x + iy$. Note that $u_1 = |z|^2 \in H$, $u_2 = 1/|z|^2 \in H$, but $1 \notin H$. Since $u_1 \in H$ the equation $\Delta u = Pu$ has no bounded solution u with $\inf_X u > 0$. However, u_2 is an unbounded positive solution with $\inf_X u_2 > 0$. Let Φ be a nonnegative real-valued function on $[0, \infty)$ which is unbounded at the points $1/n$, $n = 1, 2, \cdots$, and also at the points n, $n = 2, 3, \cdots$. Then since any member of H must either be unbounded on X or have zero infimum in its absolute value on X, it follows that there are no nonconstant Φ-bounded solutions on X, i.e. $O_{H\Phi} \not\subset O_{HB}$. This completes the proof.

9. A harmonic function u on X is called *essentially positive* if u can be represented as a difference of two HP-functions on X, or equivalently, if $|u|$ has a harmonic majorant on X. Let $HP'(X)$ be the vector lattice of essentially positive harmonic functions on X with lattice operations $\vee$ and $\wedge$, where for two functions u and v in $HP'(X)$ we denote by $u \vee v$ (resp. $u \wedge v$) the least harmonic majorant (resp. the greatest harmonic minorant) of u and v. Clearly $HP(X) \subset HP'(X)$.

For any $u \in HP(X)$ we define the function Bu by

$$Bu(p) = \sup \{v(p) \mid v \in HB(X), v \leqslant u \text{ on } X\} .$$

If $u \in HP'(X)$ we define $Bu = Bu_1 - Bu_2$ where $u = u_1 - u_2$ and $u_1, u_2 \in HP(X)$. An HP' function u is called *quasi-bounded* (resp. *singular*) if $Bu = u$ (resp. $Bu = 0$). We denote the class of quasi-bounded (resp. singular) functions on X by $HB'(X)$ (resp. $HP''(X)$). We then have the direct decomposition

$$HP'(X) = HB'(X) + HP''(X) .$$

Quasi-bounded and singular harmonic functions as well as the decomposition were introduced by Parreau (9). We now give relations between the classes $H\Phi, HB'$, and HP'. The following theorem is similar to that obtained by Nakai (7) for Riemann surfaces:

THEOREM 3. *Assume there exists an HP-function u_1 on X with* $\inf_X u_1 > 0$.

a) *If $\underline{d}(\Phi) > 0$ then $H\Phi(X) \subset HP'(X)$.*

b) *If, however, $\underline{d}(\Phi) = 0$ then $H\Phi(X) \subset HP'(X)$ is not necessarily true.*

Proof. To prove a) we set $d(\Phi) = 2c > 0$ and choose $t_0 \in (0, \infty)$ so that $\Phi(t) > ct$ for $t > t_0$. If $u \in H\Phi(X)$ then $\Phi(|u|)$ has a harmonic majorant v on X. It follows that for a suitable constant $k > 0$ we have

$$v + cku_1 \geqslant \Phi(|u|) + ct_0 \geqslant c|u|$$

on X and $|u|$ possesses a harmonic majorant on X; so $u \in HP'(X)$, thereby proving a).

To prove b) we consider the following example in the plane:

EXAMPLE 3. As in Example 2 let $X = \{z \in \boldsymbol{C} \mid 0 < |z| < 1\}$ and H consist of solutions of $\Delta u = Pu$, with $P = 4/|z|^2$. Recall that $u_1 = 1/|z|^2$ is an HP-function on X with $\inf_X u_1 > 0$. Consider the function u in H given by

$$u(z) = \cos(\sqrt{5}\,\theta)/r^3\,, \qquad z = re^{i\theta}\,,$$

and also the function $\Phi(t) = \max(\log t, 0)$ on $[0, \infty)$. Then $\underline{d}(\Phi) = 0$, $u \in H\Phi(X)$ but $u \notin HP'(X)$. This completes the proof.

The following theorem of Nakai (7) is also valid for harmonic spaces:

THEOREM 4. *If* $\bar{d}(\Phi) = \infty$ *then* $H\Phi(X) \cap HP'(X) \subset HB'(X)$.

Proof. For $u \in H\Phi(X) \cap HP'(X)$ there exists an HP-function v on X with $\Phi(|u|) \leqslant v$. Define $Mu = u \vee 0 + (-u) \vee 0$. Since B commutes with the operations $M, \vee$, and $\wedge$ we need only show

$$BMu = Mu\,.$$

Since $\bar{d}(\Phi) = \infty$ there is an increasing sequence $\{t_n\}_1^\infty$ of positive numbers with $\Phi(t_n) > 0$ and $a_n = t_n/\Phi(t_n) \to 0$. Setting $G_n = \{p \in X \mid |u(p)| < t_n\}$ we have $G_n \uparrow X$. Let $\{\Omega_m\}$ be an exhaustion of X. Let w_m be harmonic on $\Omega_m \cap G_n$ with $w_m \mid (\partial\Omega_m) \cap G_n = \min(Mu - BMu, t_n)$ and $w_m \mid (\partial G_n) \cap \bar{\Omega}_m = 0$. Here the values of w_m on $\partial(\Omega_m \cap G_n)$ need only be prescribed at the points regular for the Dirichlet problem. If we further define $w_m \mid (\Omega_m - G_n) = 0$ then w_m is subharmonic on Ω_m, and hence

$$w_m \geqslant w_{m+1}$$

on Ω_m (cf. Loeb-Walsh (5)). Also let w'_m be harmonic on Ω_m with boundary values $w'_m \mid (\partial\Omega_m) \cap G_n = \min(Mu - BMu, t_n)$ and $w'_m \mid (\partial\Omega_m - G_n) = 0$. Then $\{w'_m\}$ is a bounded sequence and $0 \leqslant w'_m \leqslant Mu - BMu$, $m = 1, 2, \cdots$. It follows from a theorem of Loeb-Walsh (5) that if $\Omega \subset X$ is a region and the family $T = \{h \in H(\Omega) \mid 0 \leqslant h\}$ is bounded then T is equicontinuous on Ω. Consequently by the Arzelà-Ascoli theorem T is a normal family. Hence $\{w'_m\}$ has a convergent subsequence with limit function w'. We obtain $0 \leqslant Bw' \leqslant B(Mu - BMu) = 0$. Since w' is bounded and nonnegative,

$$w' \equiv Bw' \equiv 0$$

on X. In addition $w'_m \geqslant w_m \geqslant 0$ implies

$$\lim_m w_m = 0$$

on X. Now on $(\partial\Omega_m) \cap G_n$ we have $|u| \leqslant t_n$ and $|u| \leqslant Mu = BMu + (Mu - BMu)$. Hence on $(\partial\Omega_m) \cap G_n$, $|u| - BMu \leqslant \min(Mu - BMu, t_n) = w_m$. On ∂G_n, $|u| = t_n = a_n\Phi(|u|) \leqslant a_n v$, and so

$$|u| \leqslant a_n v + BMu + w_m$$

on $\partial(\Omega_m \cap G_n)$ and hence on $\Omega_m \cap G_n$. Upon letting $m \to \infty$ and then $n \to \infty$ we obtain

$$|u| \leqslant BMu$$

on X. Since Mu is the least harmonic majorant of $|u|$ on X we must have $Mu \leqslant BMu$ and hence $BMu = Mu$ as was to be shown. This completes the proof.

Remark. Note that the existence of a function u_1 as in Theorem 3 is not required here.

Upon combining Theorem 3 and Theorem 4 we have the following

COROLLARY. Assume there exists an HP-function u_1 on X with $\inf_X u_1 > 0$. Then if $\bar{d}(\Phi) = \infty$ and $\underline{d}(\Phi) > 0$, we have $H\Phi(X) \subset HB'(X)$.

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