H. Morikawa.
Nagoya Math. J.
Vol. 44 (1971), 17-37

ON PROJECTIVE DIFFERENTIAL EQUATIONS ON COMPLEX ANALYTIC MANIFOLDS

HISASI MORIKAWA

Introduction.

Linear differential equations have been studied more throughly than any other class. They posses a group of characteristic properties: the invariance of linearity by linear transformations, the linearly dependence of solutions on their initial values, e.t.c. The next simple type of differential equations is quadratic type

$$\frac{\partial y_i}{\partial u_\lambda} = \sum_{l,h=1}^{n} a_{i;lh}(u) y_l y_h + \sum_{l=1}^{n} a_{i,l}(u) y_l + a_i(u)$$
$$(l \leq i \leq n\,;\ l \leq \lambda \leq r).$$

The totality of solutions of a quadratic type of differential equations is too big for the standard of our knowledge, so we should choose a nice properly defined family of solutions on which a reasonable theory can be expected. The projective point of view, on which we shall be concerned with in this paper, is a standard way to pick up compact family of solutions.

Before to interpret the main idea we introduce some terminologies briefly. M denote a connected complex analytic manifold of dimension r. A holomorphic linear differential equation of rank n on M means a system of differential equations for $y = (y_1, \cdots, y_n)$

$$dy - y\Omega = 0$$

where Ω is an $n \times n$-matrix whose entries are holomorphic differential 1-forms on M. A holomorphic projective differential equation of rank n[1] on M is a system of differential equations for $y = (y_0, y_1, \cdots, y_n)$

$$y \wedge dy - \frac{1}{2}\omega(y) = 0,$$

Received November 30, 1970

[1] We change the terminology in [3].

where

$$(y\wedge dy)_{ij}=\frac{1}{2}(y_i dy_j-y_j dy_i),$$

$$\omega(y)_{ij}=\sum_{l,h=0}^{n}\omega_{ij;lh}y_l y_h \qquad (0\leq i\leq j\leq n)$$

with holomorphic differential 1-forms $\omega_{ij;lh}$ on M. The projective differential equation is equivalent to the next overdetermined system of partial differential equations

$$d\left(\frac{y_j}{y_i}\right)=\sum_{l,h=0}^{n}\omega_{ij;lh}\frac{y_l}{y_i}\frac{y_h}{y_i} \qquad (0\leq i,\ j\leq n).$$

A projective solution is the ratio $[\varphi_0(u):\cdots:\varphi_n(u)]$ of a non-vanishing solution $(\varphi_0(u),\cdots,\varphi_n(u))$ of the equation. The initial variety W_{u_0} at a point u_0 is defined as the set of all the point $w=[w_0:\cdots:w_n]$ in the projective n-space such that one can choose a formal power series solution $(\varphi_0(u),\cdots,\varphi_n(u))$ satisfying the initial condition $[\varphi_0(u):\cdots:\varphi_n(u)]=[w_0:\cdots:w_n]$ at u_0.

It must be, first of all, notice that for a holomorphic linear equation of rank n $dy-y\Omega=0$ we can associate a projective equation of rank n for $(y_0,y)=(y_0,y_1,\cdots,y_n)$

$$(y_0,y)\wedge(dy_0,dy)-(y_0,y)\wedge(0,y\Omega)=0$$

This equation is equivalent to the pair of equations

$$\left(d\left(\frac{y_1}{y_0}\right),\cdots,d\left(\frac{y_n}{y_0}\right)\right)-\left(\frac{y_1}{y_0},\cdots,\frac{y_n}{y_0}\right)\Omega=0$$

and

$$y\wedge dy-y\wedge y\Omega=0.$$

The solutions of the linear equation $dy-y\Omega=0$ correspond bijectively to the projective solutions $[1:\varphi_1(u):\cdots:\varphi_n(u)]$ of the projective equation with non-vanishing first component $\varphi_0(u)$ and the projective solutions of $y\wedge dy-y\wedge y\Omega=0$ are the rations $[\psi_1(u):\cdots:\psi_n(u)]$ of non-trivial solution $(\psi_1(u),\cdots,\psi_n(u))$ of the linear equation. This means that linear differential equations may be regarded as special type of projective differential equations in some sense.

We are now able to explain the characteristic properties of a holomorphic projective differential equation on M

$$y\wedge dy-\frac{1}{2}\omega(y)=0$$

which correspond to the fundamental properties of linear equations :

I INVARIANCE OF PROJECTIVITY: If φ is a solution, then $\varphi\alpha$ is a solution of

$$y\wedge dy-\frac{1}{2}\{2y\wedge(y\alpha)^{-1}d\alpha)+\omega(y\alpha^{-1})\alpha\wedge\alpha\}=0$$

for any holomorphic everywhere non-singular $(n+1)\times(n\times1)$-matirix α.

II UNIQUENESS AND ANALYTICITY OF PROJECTIVE SOLUTIONS: For each point $w=[w_0:\cdots:w_n]$ in an initial variety W_{u_0} there exists a unique formal projective solution $[\varphi_0(u|u_0,w):\cdots:\varphi_n(u|u_0,w)]$ satisfying the initial condition $[\varphi_0(u_0|u_0,w):\cdots:\varphi_n(u_0|u_0,w)]=[w_0:\cdots:w_n]$ and moreover the projective solution $[\varphi_0(u|u_0,w):\cdots:\varphi_n(u|u_0,w)]$ is analytic everywhere on M, i.e. it can be analytically continuated freely on M.

III RATIONAL DEPENDENCE OF PROJECTIVE SOLUTIONS ON THEIR INITIAL CONDITIONS:

If an initial variety is not empty, then all the initial varieties W_u are projective varieties in the projective n-space which are biregularly and birationally equivalent each other such that the equivalence of W_{u_0} to W_u is given by means of the projective solution

$$w=[w_0:\cdots:w_n]\rightarrow[\varphi_0(u|u_0,w):\cdots:\varphi_n(u|u_0,w)],$$

where the equivalence, of course, depends on the path of analytic continuation.

IV INVARIANT CASE: Assume that i) M is simply conneted, ii) a connected complex Lie group G acts transitively on M, iii) the differential forms $\omega_{ij:lh}$ $(0\leq i,\ j,\ l,\ h\leq n)$ are invariant by the action of G. Then for a given point u_0 on M there exists a holomorphic group homomorphism ρ of G into the group of automorphisms of the initial variety W_{u_0} such that

$$[\varphi_0(g^{-1}u_0|u_0, w) : \cdots : \varphi_n(g^{-1}u_0|u_0, w)] = \rho(g)(w)$$
$$(w \in W_{u_0})$$

Notations.

M : a connected complex analytic manifold of dimension r,

$\boldsymbol{E}_{n+1}$: the vector space of complex $(n+1)$-row vector,

$\boldsymbol{P}_n$: the complex projective n-space whose points are ratios $[a_0 : \cdots : a_n]$ of non-zero vector $(a_0, \cdots, a_n)$ in $\boldsymbol{E}_{n+1}$,

$\boldsymbol{E}_{n+1} \wedge \boldsymbol{E}_{n+1}$: the exterior product of $\boldsymbol{E}_{n+1}$ with $\boldsymbol{E}_{n+1}$,

$\alpha \wedge \alpha$: the exterior product of α with α with α where α is an endomorphism of $\boldsymbol{E}_{n+1}$.

§ 1. Projective differential equations and projective solutions.

Though we have already touched on several concepts in Introduction, we repeat here the precise definition of fundamental terminologies.

DEFINITION 1. *A holomorphic projective differential equation of rank n on M* is a system of differential equations for $y = (y_0, \cdots, y_n)$

(1) $$y \wedge dy - \frac{1}{2}\omega(y) = 0,$$

where

$$(y \wedge dy)_{ij} = \frac{1}{2}(y_i dy_j - y_j dy_i),$$

$$\omega(y)_{ij} = \sum_{l,h=0}^{n} \omega_{ij;lh} y_l y_h \qquad (0 \leq i, \ j \leq n)$$

with holomorphic differential 1-forms $\omega_{ij;lh}$, $0 \leq i, j, l, h \leq n$ on M.

For each holomorphic functions vector $\xi(u) = (\xi_0(u), \cdots, \xi_n(u))$ the notations $\xi(u) \wedge d\xi(u)$ and $\omega(\xi(u))$ are differential 1-forms with values in the vector space $\boldsymbol{E}_{n+1} \wedge \boldsymbol{E}_{n+1}$.

DEFINITION 2. *The inhomogeneous expression of the projective differential equation* (1) means the system of holomorphic differential equations for the quotient y_j/y_i $(0 \leq i, \ j \leq n)$

(2) $$d\left(\frac{y_j}{y_i}\right) = \sum_{l,h=0}^{n} \omega_{ij;lh} \frac{y_l}{y_i} \frac{y_h}{y_j} \qquad (0 \leq i, \ j \leq n)$$

which are obtained from (1) by dividing by the square of y_i $(0 \leq i \leq n)$.

We have another expression of (1) as a usual system of differential equations plus a system of algebraic relation as follows

$$dy_{ij} = \sum_{l,h=0}^{n} \omega_{ij:lh} y_{il} y_{jh} \tag{3}$$

$$y_{ii} = 1, \qquad y_{ij} y_{jk} = y_{ik} \qquad (0 \leq i,\ j,\ k \leq n). \tag{4}$$

DEFINITION 3. *A projective solution of* (1) is the ratio $[\varphi_0 : \cdots : \varphi_n]$ of a system of a non-zero solution $[\varphi_0, \cdots, \varphi_n)$ of (1).

This definition makes sense by virtue of the next proposition:

PROPOSITION 1. *If* $(\varphi = \varphi_0, \cdots, \varphi_n)$ *is a solution of* (1) *and* f *is a holomorphic scalar function, then the function vector* $f\varphi = (f\varphi_0, \cdots, f\varphi_n)$ *is also a solution of* (1).

Proof. Since $\varphi \wedge \omega = 0$ and $\omega(f\varphi) = f^2 \omega(\varphi)$, we have

$$(f\varphi) \wedge d(f\varphi) - \frac{1}{2}\omega(f\varphi) = f\varphi \wedge (df\varphi + f\varphi \wedge fd\varphi - \frac{1}{2} f^2 \omega(\varphi)$$

$$= (fdf)\varphi \wedge \varphi + f^2 \left\{\varphi \wedge d\varphi - \frac{1}{2}\omega(\varphi)\right\} = 0.$$

DEFINITION 4. *The initial variety*, denoted by W_{u_0}, *of* (1) *at a point* u_0 *of* M is the subset in $\boldsymbol{P}_n$ consisting of all the point $w = [w_0 : \cdots : w_n]$ such that one can choose a formal power series solution $(\varphi_0(u), \cdots, \varphi_n(u))$ of (1) satisfying $[\varphi_0(u_0) : \cdots : \varphi_n(u_0)] = [w_0 : \cdots : w_n]$, where formal power series mean those with respect to local coordinates of M with the origin at u_0 and $\varphi_i(u_0)$ mean the constant terms of $\varphi_i(u)$ respectively.

PROPOSITION 2. (Invariance of projectivity). *Let* α *be a holomorphic* $(n+1) \times (n+1)$*-matrix such that* $\det \alpha$ *does not vanish on* M. *If* φ *is a solution of* (1), *then* $\varphi\alpha$ *is a solution of the next projective differential equation*

$$y \wedge dy - \frac{1}{2}\{2y \wedge (y\alpha^{-1} d\alpha) + \omega(y\alpha^{-1})\alpha \wedge \alpha\} = 0.$$

Proof. Replacing φ by $(\varphi\alpha)\alpha^{-1}$, we see that

$$\varphi \wedge d\varphi - \frac{1}{2}\omega(\varphi) = (\varphi\alpha)\alpha^{-1} \wedge d(\varphi\alpha)\alpha^{-1}) - \frac{1}{2}\omega((\varphi\alpha)\alpha^{-1})$$

$$= (\varphi\alpha)\alpha^{-1}\wedge d(\varphi\alpha)\alpha^{-1} - (\varphi\alpha)\alpha^{-1}\wedge(\varphi\alpha)\alpha^{-1}\cdot d\alpha\cdot\alpha^{-1} - \frac{1}{2}\omega((\varphi\alpha)\alpha^{-1}$$

$$\left\{(\varphi\alpha)\wedge d(\varphi\alpha) - \frac{1}{2}(2(\varphi\alpha)\wedge(\varphi\alpha)\alpha^{-1}d\alpha + \frac{1}{2}\omega((\varphi\alpha)\alpha^{-1})\alpha\wedge\alpha\right\}(\alpha\wedge\alpha)^{-1}.$$

This proves Proposition 2.

§ 2. Analyticity of projective solutions.

This paragraph is the main part of this paper and contains a rather long process of the estimations of coefficients of power series solutions of projective differential equations.

THEOREM 1. (*Uniqueness of projective solution*). *Let* $w = [w_0 : \cdots : w_n]$ *be a point in initial variety* W_{u_0} *of a holomorphic projective differential equation. Then the ratio* $[\varphi_0(u) : \cdots : \varphi_n(u)]$ *of a formal power series solution* $(\varphi_0(u), \cdots, \varphi_n(u))$ *satisfying* $[\varphi_0(u_0) : \cdots : \varphi_n(u_0)] = [w_0 : \cdots : w_n]$ *is uniquely determined by* (u_0, w).

Proof. By virtue of Proposition 2 we may assume that $w_0 = 1$ without loss of generality. Choosing a system of local coordinates $t_1, \cdots, t_r$ on M with the origin at u_0, we express $\omega(y)$ explicitely

$$\omega(y)_{ij} = \sum_{\lambda=1}^{n}\sum_{l,h=0}^{n} g_{\lambda;ij;lh}(t)y_j y_h dt_\lambda \qquad (0\leq i,\ j\leq n)$$

with local holomorphic functions $g_{\lambda;ij;lh}(t)$. Let $(\varphi_0(t), \cdots, \varphi_n(t))$ be a formal power series solution of $y\wedge dy - \frac{1}{2}\omega(y)=0$ satisfying the initial condition $[\varphi_0(0) : \cdots : \varphi_n(0) = [w_0 : \cdots : w_n]$. Since $\varphi_0(0)\neq 0$, the quotients $\varphi_1(t)/\varphi_0(t), \cdots, \varphi_n(t)/\varphi_0(t)$ are regarded as formal power series in $t_1, \cdots, t_r$ and $(\varphi_1(t)/\varphi_0(t), \cdots, \varphi_n(t)/\varphi_0(t))$ is a formal power series solution of the system of partial differential equations

$$\frac{\partial z_i}{\partial t_\lambda} = \sum_{l,h=0}^{n} g_{\lambda;0i;lh}(t)z_l z_h + \sum_{l=1}^{n}(g_{\lambda;0i;l0}(t) + g_{\lambda;0i;0l}(t))z_l + g_{\lambda;0i;00}(t)$$
$$(1\leq i\leq n).$$

Successive application of these partial differential equations makes possible for us to determine all the higher derivatives of $\varphi_i(t)/\varphi_0(t)$ $(1\leq i\leq n)$ at the origin from the given initial value $(w_1, \cdots, w_n)$. This means the uniqueness of the formal power series solution $(\varphi_1(t)/\varphi_0(t), \cdots, \varphi_n(t)/\varphi_0(t))$ and thus the ratio $[\varphi_0(t) : \cdots : \varphi_n(t)]$ is uniquely determined by (u_0, w).

DEFINITION 5. For each point w in the initial variety W_{u_0} we denote by

$$[\varphi(u|u_0, w)] = [\varphi_0(u|u_0, w) : \cdots : \varphi_n(u|u_0, w)]$$

the unique projective solution in Theorem 1 satisfying $[\varphi(u_0|u_0, w)] = w$ and call it *the projective solution with the initial point w at u_0.*

Let us recollect the definition of an associated convergence radious of a power series

$$\varphi(t_1, \cdots, t_r) = \sum_{l_1, \cdots, l_r=0}^{\infty} a_{l_1,\dots,l_r} t_1^{l_1} \cdots t_r^{l_r}$$

which is a system $(\rho_1, \cdots, \rho_r)$ of positive real numbers such that the polydisk $|t_\lambda| < \rho_\lambda (1 \leq \lambda \leq r)$ is a maximal polydisk where $\varphi(t_1, \cdots, t_r)$ converges absolutely.

CAUCHY-HADAMARD FORMULA[2]. *An associated convergence radious $(\rho_1, \cdots, \rho_r)$ of a power series*

$$\sum_{l_1, \cdots, l_r=0}^{\infty} a_{l_1,\dots,l_r} t_1^{l_1} \cdots t_r^{l_r}$$

is characterized by the relation

$$\overline{\lim} (|a_{l_1,\dots,l_r}| \rho_1^{l_1} \cdots \rho_r^{l_r})^{\frac{1}{l_1+\cdots+l_r}} = 1. \tag{5}$$

The next elementary result is very powerful for the estimations of coefficients of power series solutions of differential equations of quadratic type.

LEMMA 1. *Let ρ be a positive real number less than one and $\gamma_{l_1,\dots,l_r}$ $(l_1, \cdots, l_r = 0, 1, 2, \cdots)$ be non-negative real numbers such that*

$$\gamma_{0,\dots,0} \leqq 1$$

and

$$(l_\lambda + 1)\gamma_{l_1,\dots l_{\lambda-1}, l_\lambda+1, l_{\lambda+1},\dots,l_r}$$
$$\leqq \sum_{0 \leq p_\mu \leq l_\mu} \sum_{0 \leq q_\mu \leq p_\mu} \rho^{l_1+\cdots+l_r - p_1 - \cdots - p_r} \gamma_{p_1-q_1,\dots, p_r-q_r} \gamma_{q_1,\dots,q_r}$$
$$(l_1, \cdots, l_r = 0, 1, 2, \cdots).$$

---

[2] See standard text books on several complex variables, for an example [2].

Then

$$\gamma_{l,\ldots,l_r} \leq \left(\frac{r}{(1-\rho)^r}\right)^{l_1+\cdots l_r} \qquad (l_1, \cdots, l_r = 0, 1, 2, \cdots).$$

Proof. Let us introduce an auxiliary system of positive real numbers $\alpha_{l_1,\ldots,l_r}(l_1, \cdots, l_r = 0, 1, 2, \cdots)$ which are defined by

$$\alpha_{l_1,\ldots,l_r} = \frac{(l_1 + \cdots + l_r)!}{l_1! \cdots l_r!} \left(\frac{1}{1-\rho}\right)^{r(l_1+\cdots+l_r)}$$

$$(l_1, \cdots, l_r = 0, 1, 2, \cdots).$$

They are also defined by the power series expansion

$$g(t) = \frac{1}{1-(1-\rho)^{-r}(t_1 + \cdots + t_r)}$$

$$= \sum_{l_1,\cdots,l_r=0} \alpha_{l_1,\ldots,l_r} t_1^{l_1} \cdots t_r^{l_r}$$

This function $g(t)$ satisfies the partial differential equation

$$\frac{\partial g(t)}{\partial t_\lambda} = \frac{g(t)^2}{(1-\rho)^r} \qquad (1 \leq \lambda \leq r).$$

Comparing the coefficients of $t_1^{l_1} \cdots t_r^{l_r}$ of the both sides, we obtain the relation

(*)
$$(l_\lambda + 1)\alpha_{l_1,\ldots,l_{\lambda-1},l_\lambda+1,l_{\lambda+1},\ldots,l_r}$$

$$= \frac{1}{(1-\rho)^r} \sum_{0 \leq q_\mu \leq l_\mu} \alpha_{l_1-q,\ldots,l_r-q_r} \alpha_{q_1,\ldots,q_r}$$

$$(l_1, \cdots, l_r = 0, 1, 2, \cdots).$$

Since $(1-\rho)^{-1} > 1$
and

$$\frac{(l_1 + \cdots + l_r)!}{l_1! \cdots l_r!} \geq \frac{(h_1 + \cdots + h_r)!}{h_1! \cdots h_r!} \quad \text{for} \quad h_\lambda \leq l_\lambda \ (1 \leq \lambda \leq r),$$

we get inequalities

(**)
$$\alpha_{l_1,\ldots,l_r} \geq \alpha_{h_1,\ldots,h_r} \quad \text{for} \quad h_\lambda \leq l_\lambda \quad (1 \leq \lambda \leq r).$$

Let us now prove inductively

(***)
$$\gamma_{l_1,\ldots,l_r} \leq \alpha_{l,\ldots,l_r} \qquad (l_1, \cdots, l_r = 0, 1, 2, \cdots)$$

This holds evidently for $(0, \cdots, 0)$. Assume this inequality for $(h_1, \cdots, h_r)$

satisfying $h_\lambda \leq l_\lambda$ $(1 \leq \lambda \leq r)$. Then by virtue of (**) and the assumption of the induction it follows that

$$
\begin{aligned}
&(l_\lambda + 1)\gamma_{l_1,\dots,l_{\lambda-1},\, l_\lambda+1,\, l_{\lambda+1},\dots l_r} \\
&\leq \sum_{0\leq p_\mu \leq l_\mu} \sum_{0\leq q_\mu \leq p_\mu} \rho^{l_1+\cdots l_r - p_1 - \cdots - p_r} \gamma_{p_1-q_1,\dots,p_r-q_r} \gamma_{q_1,\dots,q_r} \\
&\leq \sum_{0\leq p_\mu \leq l_\mu} \sum_{0\leq q_\mu \leq p_\mu} \rho^{l_1+\cdots+l_r - p_1 - \cdots - p_r} \alpha_{p_1-q_1,\dots,p_r-q_r} \alpha_{q_1,\dots,q_r} \\
&\leq \sum_{0\leq q_\mu \leq l_\mu} \sum_{q_\mu\leq p_\mu \leq l_\mu} \rho^{l_1+\cdots+l_r - p_1 - \cdots - p_r} \alpha_{l_1-q_1,\dots,l_r-q_r} \alpha_{q_1,\dots,q_r} \\
&\leq \Big(\sum_{0\leq p_\mu \leq l_\mu} \rho^{(l_1-p_1)+\cdots+(l_r-p_r)} \Big) \sum_{0\leq q_\mu \leq l_\mu} \alpha_{l-q_1,\dots,l_r-q_r} \alpha_{q_1,\dots,q_r} \\
&\leq \sum_{0\leq p_\mu} \rho^{p_1+\cdots+p_r} \sum_{0\leq q_\mu \leq l_\mu} \alpha_{l_1-q,\dots,l_r-q_r} \alpha_{q_1,\dots,q_r} \\
&= \frac{1}{(1-\rho)^r} \sum_{0\leq q_\mu \leq l_\mu} \alpha_{l_1-q_1,\dots,\, l_r-q_r} \alpha_{q_1,\dots,q_r}
\end{aligned}
$$

Hence by virtue of the equality (*) we have

$$(l_\lambda + 1)\gamma_{l_1,\dots,l_{\lambda-1},\, l_{\lambda+1},\dots,l_r} \leq (l_\lambda + 1)\alpha_{l_1,\dots,l_{\lambda-1},\, l_\lambda+1,\, l_{\lambda+1},\dots,l_r}$$

and thus

$$\gamma_{l_1,\dots,l_{\lambda-1},\, l_\lambda+1,\, l_{\lambda+1},\dots,l_r} \leq \alpha_{l_1,\dots,l_{\lambda-1},\, l_\lambda+1,\, l_{\lambda+1},\dots,\, l_r}.$$

This proves

$$\gamma_{k_1,\dots,k_r} \leq \alpha_{k_1,\dots,k_r} \qquad (k_1, \cdots, k_r = 0, 1, 2, \cdots).$$

On the other hand

$$\alpha_{l_1,\dots,l_r} = \Big(\frac{1}{1-\rho}\Big)^{r(l_1+\cdots+l_r)} \frac{(l_1 + \cdots + l_r)!}{l_1! \cdots l_r!}$$

and

$$r^{l_1+\cdots+l_r} = \sum_{\Sigma h_\mu = \Sigma l_\mu} \frac{(h_1 + \cdots + h_r)!}{h_1! \cdots h_r!} \geq \frac{(l_1 + \cdots + l_r)!}{l_1! \cdots l_r!}$$

Hence we can conclude that

$$\gamma_{l,\dots,l_r} \leq \alpha_{l_1,\dots,l_r} \leq \Big(\frac{r}{(1-\rho)^r}\Big)^{l_1+\cdots+l_r} \qquad (l_1, \cdots, l_r = 0, 1, 2, \cdots)$$

The next is the key stone result in this paper with which we can prove the analyticity of projective solutions.

Proposition 3. *Let* $g_{\lambda:i;lh}(t)$ $(1 \leq \lambda \leq r\,;\, 1 \leq i \leq n\,;\, 0 \leq l, h \leq n)$ *be holomorphic functions in a neighbourhood of the origin* $t = (0, \cdots, 0)$ *and let* $\varphi(t) =$

$(\varphi_1(t), \cdots, \varphi_n(t))$ *be a formal power series solution in* $t=(t_1, \cdots, t_r)$ *of the system of partial differential equations*

$$\frac{\partial z_i}{\partial t_\lambda} = \sum_{l,h=1}^{n} g_{\lambda;i;lh}(t) z_l z_h + \sum_{l=1}^{n} (g_{\lambda;i;l0}(t) + g_{\lambda;i;0l}(t)) z_l + g_{\lambda,i;00}(t)$$

$$(1 \leq \lambda \leq r\,;\, 1 \leq i \leq n).$$

Assume the following relations

$$1 \leq K, \ \operatorname*{Max}_{1\leq i\leq n} |\varphi_i(0)| \leqq 1$$

and

$$\left| \frac{1}{l_1! \cdots l_r!} \frac{\partial^{l_1+\cdots+l_r}}{\partial t_t^{l_1} \cdots \partial t_r^{l_r}} g_{\lambda;i;lh}(0) \right| < K^{l_1+\cdots+l_r+1}$$

$$(l_1, \cdots, l_r = 0, 1, 2, \cdots),$$

where K is a real number. Then the formal power series $\varphi_i(t)$ $(1 \leq i \leq n)$ *converge absolutely in the polydisk*

$$\sup_{1\leq\lambda\leq r} |t_\lambda| < 2^{-r}(n+1)^{-2}K^{-2}.$$

Proof. For the sake of convenience $\varphi_0(t)$ denote the constant 1. Putting

$$\varphi_i(t) = \sum_{l_1,\cdots,l_r=0} c_{i;l_1,\ldots,l_r} t_1^{l_1} \cdots t_r^{l_r} \qquad (0 \leq i \leq n),$$

we shall estimate $|c_{i;l_1,\ldots,l_r}|$ by the induction on $l_1, \cdots, l_r$:

$$\begin{aligned}
&|c_{i;l_1}, \cdots, l_{\lambda-1}, l_\lambda+1, l_{\lambda+1}, \cdots, l_r| \\
&= \frac{1}{l_\lambda+1} \left| \left\{ \frac{1}{l_1! \cdots l_r!} \frac{\partial^{l_1+\cdots+l_r}}{\partial t_1^{l_1} \cdots \partial t_r^{l_r}} \left(\frac{\partial \varphi_i}{\partial t_\lambda} \right) \right\}_{t=0} \right| \\
&= \frac{1}{l_\lambda+1} \left| \left\{ \frac{1}{l_1! \cdots l_r!} \frac{\partial^{l_1+\cdots+l_r}}{\partial t_1^{l_1} \cdots \partial t_r^{l_r}} \left(\sum_{l,h=0}^{n} g_{\lambda;i;lh} \varphi_l \varphi_h \right) \right\}_{t=0} \right| \\
&= \frac{1}{l_\lambda+1} \left| \left\{ \sum_{l,h=0}^{n} \sum_{0\leq p_\mu\leq l_\mu} \sum_{0\leq q_\mu\leq p_\mu} \frac{1}{(l_1-p_1)! \cdots (l_r-p_r)!} \frac{\partial^{l_1+\cdots+l_r-p_1-\cdots-p_r}}{\partial t_1^{l_1-p_1} \cdots \partial t_r^{l_r-p_r}} g_{\lambda;i;lh} \right.\right. \\
&\quad \frac{1}{(p_1-q_1)! \cdots (p_r-q_r)!} \frac{\partial^{p_1+\cdots+p_r-q_1-\cdots-q_r}}{\partial t_1^{p_1-q_1} \cdots \partial t_r^{p_r-q_r}} \varphi_l \\
&\quad \left.\left. \frac{1}{q_1! \cdots q_r!} \frac{\partial^{q_1+\cdots+q_r}}{\partial t_1^{q_1} \cdots \partial t_r^{q_r}} \varphi_h \right\}_{t=0} \right| \\
&\leq \frac{(n+1)^2}{l_\lambda+1} \sum_{0\leq p_\mu\leq l_\mu} \sum_{0\leq q_\mu\leq p_\mu} K^{l_1+\cdots+l_r-p_1-\cdots-p_r+1} \\
&\qquad \operatorname*{Max}_{0\leq j\leq n} |c_{j;p_1-q_1,\ldots,p_r-q_r}| \operatorname*{Max}_{0\leq j\leq n} |c_{j;q_1,\ldots,q_r}|.
\end{aligned}$$

Put

$$\gamma_{l_1,\ldots,l_r} = ((n+1)^2K^2)^{-l_1-\cdots-l_r}K^{-1}\max_{0\leq i\leq n}|c_{i\cdot l_1,\ldots,l_r}|$$
$$(l_1,\cdots,l_r=0,1,2,\cdots)$$

and

$$\rho = ((n+1)^2K)^{-1}.$$

Let us prove the relations in Lemma 1

$$\gamma_{0,\ldots,0} \leqq 1$$

and

$$(l_\lambda+1)\gamma_{l,\ldots,l_\lambda+1,l_{\lambda+1},\ldots,l_r}$$
$$\leq \sum_{0\leq p_\mu\leq l_\mu}\sum_{0\leq q_\mu\leq p_\mu}\rho^{l_1+\cdots+l_r-p_1-\cdots-p_r}\gamma_{p_1-q_1,\ldots,p_r-q_r}\gamma_{q_1,\ldots,q_r}$$
$$(l_1,\cdots,l_r=0,1,2,\cdots).$$

Since $c_{i;0\ldots0}=\varphi_i(0)$ $(0\leq i\leq n)$ and $\max_{0\leq i\leq n}|\varphi_i(0)|=1$, we have the first relation $\gamma_{0,\ldots,0}=K^{-1}\leq 1$ From the above inductive estimation the second inequality is obtained as follows

$$(l_\lambda+1)\gamma_{l_1,\ldots,l_{\lambda-1},l_\lambda+1,l_{\lambda+1},\ldots,l_r}$$
$$=(l_\lambda+1)((n+1)^2K^2)^{-l_1-\cdots-l_r-1}K^{-1}\max_{0\leq j\leq n}|c_{j;l_1,\ldots,l_\lambda+1,\ldots,l_r}|$$
$$\leq((n+1)^2K^2)^{-l_r-\cdots-l_r-1}K^{-1}(n+1)^2\sum_{0\leq p_\mu\leq l_\mu}\sum_{0\leq q_\mu\leq p_\mu}K^{l_1+\cdots+l_r-p_1-\cdots-p_r+1}$$
$$\max_{0\leq j\leq n}|c_{j;p_1-q_1,\ldots,p_r-q_r}|\max_{0\leq j\leq n}|c_{j;q_1,\ldots,q_r}|$$
$$=((n+1)^2K^2)^{-l_1-\cdots-l_r-1}K^{-1}(n+1)^2\sum_{0\leq p_\mu\leq l_\mu}\sum_{0\leq q_\mu\leq p_\mu}K^{l_1+\cdots+l_r-p_1-\cdots-p_r+1}$$
$$((n+1)^2K^2)^{(p_r-q_r)+\cdots+(p_r-q_r)}K\gamma_{p_1-q_1,\ldots,p_r-q_r}((n+1)^2K^2)^{q_1+\cdots+q_r}K\gamma_{q_1,\ldots,q_r}$$
$$=\sum_{0\leq p_\mu\leq l_\mu}\sum_{0\leq q_\mu\leq p_\mu}((n+1)^2K)^{-(l_1+\cdots+l_r-p_1-\cdots-p_r)}\gamma_{p_1-q_1,\ldots,p_r-q_r}\gamma_{q_1,\ldots,q_r}$$
$$=\sum_{0\leq p_\mu\leq l_\mu}\sum_{0\leq q_\mu\leq p_\mu}\rho^{l_1+\cdots+l_r-p_1-\cdots-p_r}\gamma_{p_1-q_1,\ldots,p_r-q_r}\gamma_{q_1,\ldots,q_r}$$

Hence we can apply Lemma 1 and conclude that

$$|c_{i;l_1,\ldots,l_r}|\leq((n+1)^2K^2)^{l_1+\cdots+l_r}K\gamma_{l,\ldots,l_r}$$
$$\leq((n+1)^2K^2)^{l_1+\cdots+l_r}K\Big(\frac{r}{(1-\rho)^r}\Big)^{l_1+\cdots+l_r}$$

$$= ((n+1)^2K^2)^{l_1+\cdots+l_r}K\left\{\frac{r}{(1-(n+1)^{-2}K^{-1})^r}\right\}^{l_1+\cdots+l_r}$$

$$= K\left\{\frac{r(n+1)^2K^2}{(1-(n+1)^{-2}K^{-1})^r}\right\}^{l_1+\cdots+l_r}$$

$$\leq K(2^r(n+1)^2K^2)^{l_1+\cdots+l_r}$$

$$(l_1,\cdots,l_r=0,1,2,\cdots).$$

Since $\lim_{l\to\infty} K^{\frac{1}{l}}=1$, the above estimation implies

$$\overline{\lim}\,\{[|c_{i;l_1,\dots,l_r}|(2^r(n+1)^2K^2)^{-l_1-\cdots-l_r}]^{\frac{1}{l_1+\cdots+l_r}}\}\leq 1.$$

By virtue of Cauchy-Hadamard formula this means that $\varphi_i(t)=\sum_{l_1,\cdots,l_r} c_{i;l_1,\dots,l_r}t_1^{l_1}\cdots t_r^{l_r}$ $(1\leq i\leq n)$ converge absolutely in the polydisk

$$\sup_{1\leq\lambda\leq r}|t_\lambda|<(2^r(n+1)^2K^2)^{-1}.$$

PROPOSITION 4. *Let $y\wedge dy-\frac{1}{2}\omega(y)=0$ be a holomorphic projective differential equation on M. Then for each point u_0 on M there exists a neighboruhood U_{u_0} such that the projective solutions $[\varphi(u|u_0,w)]$ $(w\in W_{u_0})$ are holomorphic in U_{u_0}.*

Proof. Choose a system of local coordinates $(t_1,\cdots t_r)$ of M with the origin at u_0 and express explicitely

$$\omega(y)_{ij}=\sum_{\lambda=1}^{r}\sum_{l,h=0}^{n} g_{\lambda;ij;lh}(t)y_ly_hdt_\lambda \qquad (0\leq i,\ j\leq n).$$

with holomorphic functions $g_{\lambda;ij;lh}(t)$ $(1\leq\lambda\leq r\,;\,0\leq i,\ j,\ l,\ h\leq n)$ in a certain polydisk

$$\sup_{1\leq\lambda\leq r}|t_\lambda|<\eta$$

By virtue of Cauchy-Hadamard formula we have the estimation

$$\overline{\lim}\left\{\left(\left|\frac{1}{l_1!\cdots l_r!}\frac{\partial^{l_1+\cdots+l_r}}{\partial t_1^{l_1}\cdots\delta t_r^{l_r}}g_{\lambda;ij;lh}(0)\right|\eta^{l_1+\cdots+l_r}\right)^{\frac{1}{l_1+\cdots+l_r}}\leq 1.\right.$$

Hence we can choose a real positive number K such that

$$K>1$$

and

$$\left|\frac{1}{l_1!\cdots l_r!}\frac{\partial^{l_1+\cdots+l_r}}{\partial t_1^{l_1}\cdots\partial t_r^{l_r}}g_{\lambda;ij;lh}(0)\right|<K^{l_1+\cdots+l_r+1}$$

Let $w=[w_0:\cdots:w]$ be any point in the initial variety W_{u_0}. We may assume without loss of generality that $w_{i_0}=1$ and $|w_j|\leq 1$ $(0\leq j\leq n)$. Let $(\varphi_0(t),\cdots,\varphi_n(t))$ be a formal power series solution such that

$$\varphi_i(0)=w_j \qquad (0\leq j\leq n)$$

and

$$\varphi_{i_0}(t)\equiv 1.$$

Then $\varphi_{i_0}(t)$ $(0\leq i\leq n)$ satisfy the conditions:

$$\operatorname*{Max}_{0\leq i\leq n}|\varphi_i(0)|=1$$

and

$$\frac{\partial\varphi_i(t)}{\partial t_\lambda}=\sum_{l,h=0}^{n}g_{\lambda;i_0,i;lh}(t)\varphi_l(t)\varphi_h(t) \qquad (0\leq i\leq n\ ;\ i\neq i_0).$$

Hence by virtue of Proposition 3 we can conclude that $\varphi_i(t)$ $(0\leq i\leq n)$ converge absolutely in the polydisk

$$\sup_{1\leq\lambda\leq r}|t_\lambda|<(2^rK^2(n+1)^2)^{-1}.$$

This proves Proposition 4.

THEOREM 2. (*Analyticity of projective solutions*).

Let $y\wedge dy-\frac{1}{2}\omega(y)=0$ *be a holomorphic projective differential equation on a connected complex analytic manifold M. Then the projective solutions* $[\varphi(u|u_0,w)]$ $(w\in W_{u_0},\ u_0\in M)$ *are analytic everywhere on M, i.e. they can be freely analytically continuated on M.*

Proof. This is an immediate consequence of the previous proposition. Let $\sigma:[0,1]\to M$ be any path starting at u_0. Since the image $\sigma([0,1])$ is compact, we can choose real numbers $\xi_1,\cdots,\xi_m$ such that $0=\xi_1<\xi_2<\cdots<\xi_m=1$ and the neighbourhoods $U_{\sigma(\xi_1)},\cdots,U_{\sigma(\xi_m)}$ given in Proposition 4 cover the image $\sigma([0,1])$, where we may assume that $U_{\sigma(\xi_i)}$ $(1\leq i\leq m)$ are simply connected open sets. We may assume that $\sigma(\xi_{i+1})\in U_{\sigma(\xi_i)}$. Then for each point x_i in the projective solution $[\varphi(u|\sigma(\xi_i),x_i)]$ is analytic in $U_{\sigma(\xi_i)}$. Let us define a system of points $(v_1, \cdots, v_m)$ inductively by

$$v_1 = w, \ \ v_{i+1} = [\varphi(\sigma((\xi_{i+1}) | \sigma(\xi_i), \ v_i)] \qquad (1 \leq i \leq m-1).$$

This make sense, because $[\varphi(\sigma(\xi_{i+1}) | \sigma(\xi_i), \ v_i)]$ $(1 \leq i \leq m-1)$ are points in $W_{\sigma(\xi_{i+1})}$ respectively. This means that the projective solution $[\varphi(u | \sigma(\xi_{i+1}), v_{i+1})]$ is the immediate analytic continuation of $[\varphi(u | \sigma(\xi_i), v_i)]$ along the path σ, therefore we can conclude that the projective solution $[\varphi(u | u_0, w)]$ is analytic everywhere on M.

COROLLARY. *Let $(\varphi_0(u), \cdots, \varphi_n(u))$ be a formal power series solution at u_0 such that $(\varphi_0(u_0), \cdots, \varphi_n(u_0)) \neq (0, \cdots, 0)$. Then the ratio $[\varphi_0(u) : \cdots : \varphi_n(u)]$ is analytic and $[\varphi_0(u) : \cdots : \varphi_n(u)] \neq [0 : \cdots : 0]$ everywhere on M.*

Proof. Put $w = [\varphi_0(u_0) : \cdots : \varphi_n(u_0)]$. Then the ratio $[\varphi_0(u) : \cdots : \varphi_n(u)]$ is the projective solution $[\varphi(u | u_0, w)]$.

§ 3. Initial varieties.

We shall show that, if an initial variety W_{u_0} is not empty, all the initial varieties $W_u (u \in M)$ are projective algebraic varieties which are bi-regularly and birationally equivalent each other and the equivalence are given by mean of projective solutions.

PROPOSITION 5. *Let $y \wedge dy - \frac{1}{2} \omega(y) = 0$ be a holomorphic projective differential equation of rank n on M and let u_0 be a point on M such that W_{u_0} is not empty, then W_{u_0} is a projective algebraic variety in $\boldsymbol{P}_n$.*

Proof. We shall construct the homogeneous ideal associated with W_{u_0}. Choosing a system of local coordinates $(t_1, \cdots, t_r)$ of M with the origin at u_0, we may consider the projective differential equation as the following system of partial differential equations

$$\begin{vmatrix} y_i & y_j \\ \dfrac{\partial y_i}{\partial t_\lambda} & \dfrac{\partial y_j}{\partial t_\lambda} \end{vmatrix} = \sum_{l, h=0}^{n} g^\lambda_{i j l h}(t) y_l y_h \qquad (0 \leq i, \ j \leq n \ ; 1 \leq \lambda \leq r)$$

with holomorphic coefficients $g^\lambda_{i j l h}(t)$. We mean by A the local ring of formal power series in $t_1, \cdots, t_r$ and mean by m the maximal ideal of A. Let $Y_0, \cdots, Y_n$ be indeterminates and $D_{i,1}, \cdots, D_{i,r}$ be the derivations of $A\left[\frac{Y_0}{Y_i}, \cdots, \frac{Y_n}{Y_i}\right]$ defined by

$$D_{i,\lambda}(f(t)) = \frac{\partial f(t)}{\partial t_\lambda} \quad \text{for } f(t) \in A,$$

$$D_{i,\lambda}\left(\frac{Y_j}{Y_i}\right) = \sum_{l,h=0}^{n} g^{\lambda}_{i\,jlh}(t)\,\frac{Y_l Y_h}{Y_i Y_i} \qquad (0 \leq i,\ j \leq n\ ;\ 1 \leq \lambda \leq r(.$$

We define operators $E_{i,}(0 \leq i \leq n\ ;\ 1 \leq \lambda \leq r)$ acting on the polynomial algebra $A[Y_0, \cdots, Y_n]$ as follows:

$$E_{i,\lambda}\left(\sum_{l=0}^{m} F_l(t,Y)\right) = \frac{\partial F_0(t)}{\partial t_\lambda} + \sum_{l=1}^{m} Y_i^{l+1} D_{i,\lambda}(Y_i^{-l} F_l(t,Y)),$$

where $F_l(t,Y)$ means a homogeneous polynomial of degree l in $Y_0, \cdots, Y_n$ $(l = 0, 1, 2, \cdots)$.

Denote

$$H_{ijk,\lambda}(t,Y) = Y_i Y_j^2 D_{j,\lambda}\left(\frac{Y_k}{Y_j}\right) + Y_j Y_k^2 D_{k,\lambda}\left(\frac{Y_l}{Y_k}\right) + Y_k Y_i^2 D_{i,\mu}\left(\frac{Y_j}{Y_i}\right),$$

$$L_{ij,\mu\mu}(t,Y) = Y_i^3(D_{i,\lambda}D_{i,\mu} - D_{i,\mu}D_{i,\lambda})\left(\frac{Y_j}{Y_i}\right)$$

$$(0 \leq i,\ j \leq n\ ;\ 1 \leq \lambda,\ \mu \leq n).$$

Then $H_{ijk,\lambda}(t,Y)$, $L_{ij,\lambda\mu}(t,Y)$ are homogeneous polynomials of degree three with coefficients in A. We mean by $\mathfrak{A}$ the smallest homogeneous ideal of $A[Y_0, \cdots, Y_n]$ such that i) $H_{ijk,\lambda}(t,Y)$, $L_{ij,\lambda\mu}(t,Y)$ $(0 \leq i,\ j,\ k \leq n\ ;\ 1 \leq \lambda, \mu \leq n)$ are contained in $\mathfrak{A}$ and ii) $E_{i,\lambda}\mathfrak{A} \subset \mathfrak{A}$ $(0 \leq i \leq n\ ;\ 1 \leq \lambda \leq r)$. Let $\overline{\mathfrak{A}}$ be the homogeneous ideal of $C[Y_0, \cdots, Y_n]$ given by

$$\overline{\mathfrak{A}} = A/m \otimes_A \mathfrak{A},$$

where A/m is canonically identified with $\boldsymbol{C}$. We denote by V the projective algebraic variety (reducible in general) in $\boldsymbol{P}_n$ associated with the homogeneous ideal $\overline{\mathfrak{A}}$. Then our goal is to show $W_{u_0} = V$. Let $w = [w_0 : \cdots : w_n]$ be any point in W_{u_0} and $(\varphi_0(t|v), \cdots, \varphi_n(t|v))$ be a formal solution of the equation such that $\varphi_0(0|w) : \cdots : \varphi_n(0|w)] = [w_0 : \cdots : w_n]$. Then, since

$$\begin{vmatrix} \varphi_i(t|w) & \varphi_j(t|w) \\ \dfrac{\partial \varphi_i(t|w)}{\partial t_\lambda} & \dfrac{\partial \varphi_j(t|w)}{\partial t_\lambda} \end{vmatrix} = \sum_{l,h=0}^{n} g^{\lambda}_{i\,jlh}(t)\varphi_l(t|w)\varphi_h(t|w)$$

$$(0 \leq i,\ j \leq n\ ;\ 1 \leq \lambda \leq r),$$

it follows

$$H_{jkl,\lambda}(t, \varphi_0(t|w), \cdots, \varphi_n(t|w))$$

$$= \varphi_j(t|w) \begin{vmatrix} \varphi_k(t|w) & \varphi_l(t|w) \\ \dfrac{\partial\varphi_k(t|w)}{\partial t_\lambda} & \dfrac{\partial\varphi_l(t|w)}{\partial t_\lambda} \end{vmatrix} + \varphi_k(t|w) \begin{vmatrix} \varphi_l(t|w) & \varphi_j(t|w) \\ \dfrac{\partial\varphi_l(t|w)}{\partial t_\lambda} & \dfrac{\partial\varphi_j(t|w)}{\partial t_\lambda} \end{vmatrix}$$

$$+ \varphi_l(t|w) \begin{vmatrix} \varphi_j(t|w) & \varphi_k(t|w) \\ \dfrac{\partial\varphi_j(t|w)}{\partial t_\lambda} & \dfrac{\partial\varphi_k(t|w)}{\partial t_\lambda} \end{vmatrix}$$

$$= \begin{vmatrix} \varphi_j(t|w) & \varphi_k(t|w) & \varphi_l(t|w) \\ \varphi_j(t|w) & \varphi_k(t|w) & \varphi_l(t|w) \\ \dfrac{\partial\varphi_j(t|w)}{\partial t_\lambda} & \dfrac{\partial\varphi_k(t|w)}{\partial t_\lambda} & \dfrac{\partial\varphi_l(t|w)}{\partial t_\lambda} \end{vmatrix}$$

$$(0 \leq j,\ k,\ l \leq n\ ;\ 1 \leq \lambda \leq r),$$

and

$$L_{ij,\lambda\mu}(t, \varphi_0(t|w), \cdots, \varphi_n(t|w))$$

$$= \varphi_i(t|w)^3 \Big(\frac{\partial}{\partial t_\lambda}\frac{\partial}{\partial t_\mu} - \frac{\partial}{\partial t_\mu}\frac{\partial}{\partial t_\lambda} \Big)\Big(\frac{\varphi_j(t|w)}{\varphi_i(t|w)} \Big) = 0$$

$$(0 \leq i,\ j \leq n\ ;\ 1 \leq \lambda,\ \mu \leq r).$$

The homogeneous ideal $\mathfrak{A}$ is generated by

$$E_{i_1,\lambda_1} \cdots E_{i_m,\lambda_m} H_{jkl,\lambda}(t,Y),$$

$$E_{i_1,\lambda_1} \cdots E_{i_m,\lambda_m} L_{ij,\lambda\mu}(t,Y)$$

$$(0 \leq i,\ j,\ k,\ i_1, \cdots, i_m \leq n\ ;\ 1 \leq \lambda,\ \mu,\ \lambda_1, \cdots, \lambda_m \leq r\ ;\ m = 0,1,2,\cdots)$$

and for a homogeneous polynomian $F(t,Y)$

$$(E_{i,\lambda}F)(t, \varphi_0(t|w), \cdots, \varphi_n(t|w))$$

$$= \varphi_i(t|w)^{1+\det F} \frac{\partial}{\partial t_\lambda} (\varphi_i(t|w)^{-\deg F} F(t, \varphi_0(t|w), \cdots, \varphi_n(t|w)))$$

$$(0 \leq i \leq n\ ;\ 1 \leq i \leq r)$$

This shows that $F(t, \varphi_0(t|w), \cdots, \varphi_n(t|w)) = 0$ for every F in $\mathfrak{A}$ and thus $F(0, w_0, \cdots, w_n) = F(0, \varphi_0(0|w), \cdots, \varphi_n(0|w)) = 0$ for every F in $\mathfrak{A}$. This means that $w = [w_0 : \cdots : w_n]$ belongs to W, namely $W_{u_0} \subset V$. Let us prove the other direction $W_{u_0} \supset V$. Let $w = [w_0 : \cdots : w_n]$ be a point of V and w_i be a non-vanishing component of $(w_0, \cdots, w_n)$, where we may assume that $w_i = 1$.

We denote by $\mathfrak{A}_i$ the ideal of $A\left[\frac{Y_0}{Y_i}, \cdots, \frac{Y_n}{Y_i}\right]$ $\bigcup_{l=1}^{\infty}\{Y_i^{-l}F_l \mid F_l$ are homogeneous polynomials of degree l in $\mathfrak{A}\}$. Then $\mathfrak{A}_i$ is the smallest ideal such that i) $Y_i^{-3}H_{jkl,\lambda}(t,Y)\ (D_{i,\lambda}D_{i,\mu}-D_{i,\mu}D_{i,\lambda})\left(\frac{Y_j}{Y_i}\right)$ $(0\leq j,\ k,\ l\leq n\ ;1\leq\lambda,\ \mu\leq r)$ are contained in $\mathfrak{A}_i$ and ii) $D_{i,\lambda}\mathfrak{A}_i\subset\mathfrak{A}_i$ $(1\leq\lambda\leq r)$. We define formal power series

$$\phi_j(t)=\sum_{l_1,\cdots,l_r}^{\infty}\frac{1}{l_1!\cdots,l_r!}\left[D_{i,1}^{l_1}\cdots D_{i,r}^{l_r}\frac{Y_j}{Y_i}\right]_{(t,Y)=(0,w)}t_1^{l_1}\cdots t_r^{l_r}$$
$$(0\leq j\leq n).$$

Since

$$[D_{i,1}^{p_1-q_1}\cdots D_{i,r}^{p_r-q_r}D_{i\,1}^{q_1}\cdots D_{i\,r}^{q_r}\frac{Y_j}{Y_i}\Big]_{(t,Y)=(0,w)}$$
$$=\left[D_{i,1}^{p_1}\cdots D_{i,r}^{p_r}\frac{Y_j}{Y_i}\right]_{(t,Y)=(0,w)}\qquad(0\leq j\leq n\ ;0\leq q_1\leq p_1,\cdots,0\leq q_r\leq p_r),$$

we have

$$\left[\frac{\partial^{l_1+\cdots+l_r}}{\partial t_1^{l_1}\cdots\partial t_r^{l_r}}\left(\frac{\partial\phi_j(t)}{\partial t_\lambda}\right)\right]_{t=0}=\left[D_{i,1}^{l_1}\cdots D_{i,r}^{l_r}D_{i,\lambda}\frac{Y_j}{Y_i}\right]_{(t,Y)=(0,w)}$$
$$=\left[D_{i,1}^{l_1}\cdots D_{i,r}^{l_r}\left(\sum g_{i\,jlh}^{\lambda}(t)\frac{Y_lY_h}{Y_iY_j}\right)\right]_{(t,Y)=(0,w)}$$
$$=\sum_{0\leq q_\alpha\leq p_\alpha\leq l_\alpha}\frac{1}{(l_1-p_1)!\cdots(l_r-p_r)!}\left[\frac{\partial^{l_1+\cdots l_r-p_1\cdots p_r}}{\partial t_1^{l_1-p_1}\cdots\partial t_r^{l_r-p_r}}g_{i\,jlh}^{\lambda}(t)\right]_{t=0}\times$$
$$\times\frac{1}{(p_1-q_1)!\cdots(p_r-q_r)!}\left[D_{i,1}^{p_1-q_1}\cdots D_{i,r}^{p_r-q_r}\frac{Y_l}{Y_i}\right]_{(t,Y)=(0,w)}$$
$$\frac{1}{q_1!\cdots q_r!}\left[D_{i,1}^{q_1}\cdots D_{i,r}^{q_r}\frac{Y_h}{Y_i}\right]_{(t,Y)=(0,w)}$$
$$=\sum_{0\leq q_\alpha\leq p_\alpha\leq l_\alpha}\frac{1}{(l_1-p_1)!\cdots(l_r-p_r)!}\left[\frac{\partial^{l_1+\cdots+l_r-p_1-\cdots-p_r}}{\partial_1^{l_1-p_1}\cdots\partial_r^{l_r-p_r}}g_{i\,jlh}^{\lambda}(t)\right]_{t=0}$$
$$\frac{1}{(p_1-q_1)!\cdots(p_r-q_r)!}\left[\frac{\partial^{p_1+\cdots+p_r-q_1-\cdots-q_r}}{\partial t_1^{p_1-q_1}\cdots\partial t_r^{p_r-q_r}}\phi_l(t)\right]_{t=0}$$
$$\frac{1}{q_1!\cdots q_r!}\left[\frac{\partial^{q_1+\cdots+q_r}}{\partial t_1^{q_1}\cdots\partial t_r^{q_r}}\phi_h(t)\right]_{t=0}$$
$$=\left[\frac{\partial^{l_1+\cdots+l_r}}{\partial t_1^{l_1}\cdots\partial t_r^{l_r}}\left(\sum_{l,h=0}^{n}g_{i\,jlk}^{\lambda}(t)\phi_l(t)\phi_h(t)\right)\right]_{t=0}$$
$$(0\leq i\leq n\ ;1\leq\lambda\leq r\ ;l_1,l_2,\cdots,l_r=0,1,2,\cdots).$$

This means that

$$\frac{\partial \phi_j(t)}{\partial t_\lambda} = \sum_{l,h=0}^{n} g^{\lambda}_{i\,jlh}(t)\phi_l(t)\phi_h(t)$$

$$D_{i,\lambda}\phi_j(t) = \frac{\partial \phi_j(t)}{\partial t_\lambda} = \left(D_{i,\lambda}\frac{Y_j}{Y_i}\right)_{(t,Y)=(t,\phi)}$$

$$(0 \leq j \leq n\ ;\ 1 \leq \lambda \leq r).$$

Hence by the induction on $l_1, \cdots, l_r$ we have

$$\frac{\partial^{l_1+\cdots+l_r}\phi_j(t)}{\partial t_1^{l_1}\cdots\partial t_r^{l_r}} = \left[D^{l_1}_{i,1}\cdots D^{l_r}_{i,r}\frac{Y_j}{Y_i}\right]_{(t,Y)=(t,\phi)}$$

$$(0 \leq j \leq n\ ;\ l_1, \cdots, l_r = 0, 1, 2, \cdots).$$

Since $\phi_i(t) \equiv 1$ and $[D^{l_1}_{i,r}\cdots D^{l_r}_{i,r}(Y_i^{-3}H_{ijk,\lambda}(t,Y))]_{(t,Y)=(0,w)} = 0$, it follows

$$\left[\frac{\partial^{l_1+\cdots+l_r}}{\partial t_1^{l_1}\cdots\partial t_r^{l_r}}\begin{vmatrix}\phi_j(t) & \phi_k(t)\\ \dfrac{\partial\phi_j(t)}{\partial t_\lambda} & \dfrac{\partial\phi_k(t)}{\partial t_\lambda}\end{vmatrix}\right]_{t=0}$$

$$= \left[D^{l_1}_{i,1}\cdots D^{l_r}_{i,r}\left(\frac{Y_j}{Y_i}D_{i,\lambda}\frac{Y_k}{Y_i} - \frac{Y_k}{Y_i}D_{i,\lambda}\frac{Y_j}{Y_i}\right)\right]_{(t,Y)=(0,w)}$$

$$= \left[D^{l_1}_{i,1}\cdots D^{l_r}_{i,r}\left(Y_i^{-3}\left(Y_jY_i^2D_{i,\lambda}\frac{Y_k}{Y_i} - Y_kY_i^2D_{i,\lambda}\frac{Y_j}{Y_i}\right)\right)\right]_{(t,Y)=(0,w)}$$

$$= \left[D^{l_1}_{i,1}\cdots D^{l_r}_{i,r}\left(-Y_i^{-3}\left(Y_jY_k^2D_{k,\lambda}\frac{Y_i}{Y_k} + Y_kY_i^2D_{i,\lambda}\frac{Y_j}{Y_i}\right)\right)\right]_{(t,Y)=(0,w)}$$

$$= \left[D^{l_1}_{i,1}\cdots D^{l_r}_{i,r}\left(Y_i^{-3}\left(Y_iY_j^2D_{j,\lambda}\left(\frac{Y_k}{Y_j}\right) - H_{ijk,\lambda}(t,Y)\right)\right)\right]_{(t,Y)=(0,w)}$$

$$= \left[\left[D^{l_1}_{1,1}\cdots D^{l_r}_{i,r}\left(\left(\frac{Y_j}{Y_i}\right)^2D_{j,\lambda}\left(\frac{Y_k}{Y_j}\right)\right)\right]_{(t,Y)=(t,\phi)}\right]_{t=0}$$

$$= \left[\frac{\partial^{l_1+\cdots+l_r}}{\partial t_1^{l_1}\cdots\partial t_r^{l_r}}\left(\sum_{l,h=0}^{n} g^{\lambda}_{jklh}(t)\phi_l(t)\phi_h(t)\right)\right]_{t=0}.$$

This means that

$$\begin{vmatrix}\phi_j(t) & \phi_k(t)\\ \dfrac{\partial\phi_j(t)}{\partial t_\lambda} & \dfrac{\partial\phi_k(t)}{\partial t_\lambda}\end{vmatrix} = \sum_{l,h=0}^{n} g^{\lambda}_{jklh}(t)\phi_l(t)\phi_h(t)$$

$$(0 \leq j,\ k \leq n\ ;\ 1 \leq \lambda \leq r).$$

Namely $(\phi_0(t), \cdots, \phi_n(t))$ is a formal solution of the projective differential equation with the initial value $(\varphi_0(0), \cdots, \varphi_n(0)) = (w_0, \cdots, w_n)$, and thus the point $w = [w_0 : \cdots : w_n]$ belongs to W_{u_0}. This completes the proof.

We shall prove the main theorem of this paragraph which is a direct consequence of the above results and Chow's Lemma[3].

THEOREM 3. *Let $y \wedge dy - \frac{1}{2}\omega(y) = 0$ be a holomorphic projective equation of rank n on a connected complex manifold M. If the initial variety W_{u_0} at a point u_0 is not empty, then initial varieties $W_u (u \in M)$ are projective algebraic varieties which are biregularly and birationally equivalent to W_u. The equivalence of W_{u_0} with W_u is given by means of projective solutions*

$$w \to [\varphi(u \mid u_0, w)] \qquad (w \in W_{u_0}),$$

where the equivalence depends on the path of analytic continuation connecting u_0 with u.

Proof. Let u_0 be point of M such that the initial point W_{u_0} is not empty. Then by virtue of Proposition 4 there exists a simply connected neighbourhood U such that for each point w in W_{u_0} there exists a holomorphic solution $(\varphi_0(u \mid u_0, w), \cdots, \varphi_n(u \mid u_0, w))$ in U with the initial condition $[\varphi_0(u_0, w) : \cdots : \varphi_n(u_0 \mid u_0, w)] = w$. Since the equation is holomorphic and the initial variety W_{u_0} is a compact analytic subvariety in $\boldsymbol{P}_n$, we can choose a finite covering $W_{u_0} = \bigcup_{\alpha=i}^{m} W_\alpha$ and holomorphic solutions $(\varphi_0^{(\alpha)}(u \mid u_0, w), \cdots, \alpha_0^{(\alpha)}(u \mid u_0, w))$ $(1 \leq \alpha \leq m)$ such that $(\varphi_0^{(\alpha)}(u \mid u_0, w), \cdots, \varphi_n^{(\alpha)}(u \mid u_0, w))$ is holomorphic in W_α with respect to w. We denote by $\Phi(\gamma_{u,u_0})$ the map of W_{u_0} into W_u such that $\Phi(\gamma_{u,u_0})(w)$ is the analytic continuation of the projective solution $[\varphi(u \mid u_0, w)]$ along a path γ_{u,u_0} connecting u_0 with u. The above result means that $\Phi(\gamma_{u,u_0})$ is a holomorphic map of W_{u_0} into W_u. Exchanging u_0 with u, we see that i) $\Phi(\gamma_{u,u_0})$ is one-to-one and ii) $\Phi(\gamma_{u,u_0}^{-1}) \circ \Phi(\gamma_{u,u^0}) = id_{W_{u_0}}$, namely $\Phi(\gamma_{u,u_0})$ is a biregular equivalence of W_{u_0} onto W_u. Since the graph of $\Phi(\gamma_{u,u_0})$ is a closed analytic subvariety in $\boldsymbol{P}_n \times \boldsymbol{P}_n$, by virtue of Chow's Lemma it must be a projective algebraic variety. This means that the equivalence $\Phi(\gamma_{u,u_0})$ is also birational.

§ 4. Invariant case.

Let us first recall the simplest linear example: Let $A_1, \cdots, A_r$ be mutually commutative complex $n \times n$-matrices. Then the solutions of the linear

3) See [1].

equation $dy - y\sum_{\lambda=1}^{r} A_\lambda du_\lambda = 0$ are given by

$$\varphi(u|w) = (w_1, \cdots, w_n) \exp\{\sum_{\lambda=1}^{r} A_\lambda u_\lambda\}.$$

The situation is similar for projective equations in the following sense:

THEOREM 4. *Let $y\wedge dy - \frac{1}{2} w(y) = 0$ be a holomorphic projective differential equation of rank n on a simply connected complex analytic manifold M on which a connected complex Lie group G acts transitively on M. Assume further that the equation is leaved invariant by the action of G, i.e. ω_{ijlh} $(0\leq i, j, l, h\leq n)$ are invariant, where*

$$\omega(y)_{ij} = \sum_{l,h=0}^{n} \omega_{ijlh} y_l y_h \qquad (0\leq i, j\leq n).$$

Then if an initial variety W_{u_0} is non-empty, there exists a holomorphic group holomorphism ρ of G into the group of automorphisms of the initial variety W_{u_0} such that the projective solution $[\varphi(u|u_0, w)]$ are given by

$$[\varphi(g^{-1}u_0|u_0, w)] = \rho(g)(w) \qquad (w\in W_{u_0},\ g\in G).$$

Proof. Since the equation $y\wedge dy - \frac{1}{2}\omega(y) = 0$ is invariant by the action of G, the initial varieties $W_u (u\in M)$ coincide with W_{u_0}. This means that, if we denote

$$[\varphi(g^{-1}u_0|u_0, w)] = \rho(g)(w),$$

the maps $w\to\rho(g)w$ are automorphisms of the initial variety W_{u_0}. Therefore it is enough to show that the map $(g, w)\to\rho(g)(w)$ is an action of G on W_{u_0}, namely

$$\rho(gh)(w) = \rho(g)(\rho(h)(w)) \qquad (g, h\in G\ ;\ w\in W_{u_0}).$$

Since G leaves the equation invariant, $[\varphi(g^{-1}u|u_0, w)]$ is also a projective solution with the initial point w at gu_0. Hence by virtue of the uniqueness of projective solutions we can conclude that

$$[\varphi(g^{-1}u|u_0, w)] = [\varphi(u|gu_0, w)] \qquad (g\in G).$$

Since

$$[\varphi(u_0|hu_0, w)] = [\varphi(u_0|u_0, [\varphi(u_0|hu_0, w)])],$$

we have

$$[\varphi(u\,|\,hu_0, w)] = [\varphi(u\,|\,u_0, [\varphi(u_0\,|\,hu_0, w)])] \qquad (h \in G).$$

Hence from these relations it follows

$$\begin{aligned}\rho(gh)(w) &= [\varphi((gh)^{-1}u_0\,|\,u_0, w)] = [\varphi(h^{-1}g^{-1}u_0\,|\,u_0, w)]\\ &= [\varphi(g^{-1}u_0\,|\,hu_0, w)] = [\varphi(g^{-1}u_0\,|\,u_0, [\varphi(u_0\,|\,uu_0, w)])]\\ &= [\varphi(g^{-1}u_0\,|\,u_0, [\varphi(h^{-1}u_0\,|\,u_0, w)])] = [\varphi(g^{-1}u_0\,|\,u_0,\ \rho(h)(w))]\\ &= \rho(g)(\rho\,|\,h)(w))\end{aligned}$$

$$(g, h \in G\ ;\ w \in W_{u_0}).$$

This completes the proof of Theorem.

References

[1] L. Chow, On compact analytic varieties, Amer. Jour. Math. **71** (1947).
[2] Gunning, Rossi, Analytic functions of several complex variables ,Prince-Hall, 1965.
[3] H. Morikawa, On osculating systems of differential equations of type (N, 1, 2), Nagoya Math. Jour. vol **31**, (1968).

Institute of Mathematics
Nagoya University