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SOME INTEGRAL FORMULAS FOR HYPERSURFACES IN EUCLIDEAN SPACES

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1. Introduction

Let M be an oriented hypersurface differentiably immersed in a Euclidean space of $n+1\geqq 3$ dimensions. The r-th mean curvature K_r of M at the point P of M is defined by the following equation:

$$\det(\delta_{ij}+ta_{ij})=\sum_{r=0}^{n}\binom{n}{r}K_r t^r \tag{1}$$

where δ_{ij} denotes the Kronecker delta, $\binom{n}{r}=n!/r!\,(n-r)!$, and a_{ij} are the coefficients of the second fundamental form. Throughout this paper all Latin indices take the values $1,\cdots,n$, Greek indices the values $1,\cdots,n+1$, and we shall also follow the convention that repeated indices imply summation unless otherwise stated. Let p denote the oriented distance from a fixed point 0 in E^{n+1} to the tangent hyperplane of M at the point P, and dV denote the area element of M. Let $e_1,\cdots,e_n$ be an ordered orthonormal frame in the tangent space of the hypersurface M at the point P, and denote by x_i the scalar product of e_i and the position vector $\boldsymbol{X}$ of the point P with respect to the fixed point 0 in E^{n+1}. The main purpose of this paper is to establish the following theorems:

THEOREM 1. *Let M be an oriented hypersurface with regular smooth boundary differentiably immersed in a Euclidean space E^{n+1}. Then we have*

$$\int_M p^{m-1}\boldsymbol{X}\cdot\nabla K_r dV+n\int_M p^{m-1}(K_r-K_1K_rp)dV+(m-1)\int_M p^{m-2}K_rx_ix_ja_{ij}dV \tag{2}$$
$$=\int_{\partial M}p^{m-1}K_r\boldsymbol{X}\cdot *d\boldsymbol{X},\qquad r=0,1,\cdots,n-1,$$

where m is any real number, ∇K_r is the gradient of K_r, ∂M is the boundary of M and $*$ denotes the star operator.

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Formula (2) was obtained by Amur [1] for $m=1$ in an alternating form.

THEOREM 2. *Under the same assumption of Theorem* 1, *we have*

$$m\sum_{i=0}^{r}(-1)^i\binom{n}{r-i}\int_M p^{m-1}K_{r-i}x_j a_{jh_0}\Big(\prod_{k=1}^{i} a_{h_{k-1}h_k}\Big)e_{h_i}dV \tag{3}$$

$$= (n-r)\binom{n}{r}\int_M p^m K_{r+1}\boldsymbol{e}dV - \sum_{i=0}^{r}(-1)^i\binom{n}{r-i}\int_{\partial M} p^m K_{r-i}{}^*\boldsymbol{U}_i,$$

$$r = 0, \cdots, n-1,$$

where $\boldsymbol{e}$ *denotes the unit outer normal vector. In particular, we have*

$$m\int_M p^{m-1}\boldsymbol{X}K_n dV = n\int_M p^m \boldsymbol{e}K_n dV - (1/n!)\int_{\partial M} p^m \sigma_{n-1}, \tag{4}$$

and

$$m\int_M p^{m-1}a_{ij}x_i\boldsymbol{e}_j dV = n\int_M p^m K_1 \boldsymbol{e}dV + \int_{\partial M} p^m {*d\boldsymbol{X}}. \tag{5}$$

Formula (4) was obtained by Flanders [4] for $m=1$ and M is closed.

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2. Preliminaries

In a Euclidean space E^{n+1} of $n+1\geq 3$ dimensions, let us consider a fixed right-handed rectangular frame $\boldsymbol{X}, \boldsymbol{e}_1, \cdots, \boldsymbol{e}_{n+1}$, where $\boldsymbol{X}$ is a point in E^{n+1}, and $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_{n+1}$ is an ordered set of mutually orthogonal unit vectors such that its determinant is

$$[\boldsymbol{e}_1, \cdots, \boldsymbol{e}_{n+1}] = 1, \tag{6}$$

so that $\boldsymbol{e}_\alpha\cdot\boldsymbol{e}_\beta = \delta_{\alpha\beta}$. Let F denote the bundle of all such frames. We also use $\boldsymbol{X}$ to denote the position vector of the point P with respect to a fixed point 0 in E^{n+1}. Then we have

$$d\boldsymbol{X} = \theta_\alpha \boldsymbol{e}_\alpha, \qquad d\boldsymbol{e}_\alpha = \theta_{\alpha\beta}\boldsymbol{e}_\beta \tag{7}$$

where d denotes the exterior differentiation, and θ_α, $\theta_{\alpha\beta}$ are Pfaffian forms. Since $d^2\boldsymbol{X} = d(d\boldsymbol{X}) = d(d\boldsymbol{e}_\alpha) = 0$, exterior differentiation of equations of (7) find that

$$d\theta_\alpha = \theta_\beta \wedge \theta_{\beta\alpha}, \qquad d\theta_{\alpha\beta} = \theta_{\alpha\gamma}\wedge\theta_{\gamma\beta}, \qquad \theta_{\alpha\beta}+\theta_{\beta\alpha} = 0, \tag{8}$$

where $\wedge$ denotes the exterior product.

Let M be a hypersurface twice differentiably immersed in E^{n+1}. Consider the set B consisting of frames $\boldsymbol{X}, \boldsymbol{e}_1, \cdots, \boldsymbol{e}_n, \boldsymbol{e}$ in E^{n+1} satisfying the conditions $\boldsymbol{X} \in M$ and $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n$ are vectors tangent to M at $\boldsymbol{X}$. Then we have a cannonical mapping, said λ, from B into F. Let λ^* denote the dual mapping of λ. By setting

(9) $$\omega_\alpha = \lambda^* \theta_\alpha, \qquad \omega_{\alpha\beta} = \lambda^* \theta_{\alpha\beta},$$

from (8) we have

(10) $$d\omega_\alpha = \omega_\beta \wedge \omega_{\beta\alpha}, \qquad d\omega_{\alpha\beta} = \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta}, \qquad \omega_{\alpha\beta} + \omega_{\beta\alpha} = 0.$$

From the definition of B, it follows that $\omega_{n+1} = 0$ and $\omega_1, \cdots, \omega_n$ are linear independent. Thus the first equation of (10) gives

$$\omega_i \wedge \omega_{i,n+1} = 0.$$

From which we can write

(11) $$\omega_{n+1,i} = a_{ij}\omega_j, \qquad a_{ij} = a_{ji}.$$

Throughout a point in the space E^{n+1}, let $\boldsymbol{V}_1, \cdots, \boldsymbol{V}_n, \boldsymbol{J}$ be $n+1$ vectors in the space E^{n+1}, and $\boldsymbol{V}_1 \times \cdots \times \boldsymbol{V}_n$ denote the vector product of the n vectors $\boldsymbol{V}_1, \cdots, \boldsymbol{V}_n$. Then we have

(12) $$\boldsymbol{J} \cdot (\boldsymbol{V}_1 \times \cdots \times \boldsymbol{V}_n) = (-1)^n [\boldsymbol{J}, \boldsymbol{V}_1, \cdots, \boldsymbol{V}_n],$$

where $\cdot$ denotes the inner product of E^{n+1}, from which it follows that

(13) $$\boldsymbol{e}_1 \times \cdots \times \hat{\boldsymbol{e}}_\alpha \times x \cdots \times \boldsymbol{e}_{n+1} = (-1)^{n+\alpha+1} \boldsymbol{e}_\alpha,$$

where the roof means the omitted term. In the following, we denote the combined operation of inner product and the exterior product by (,), and the combined operation of the vector product and the exterior product by [, $\cdots$,]. We list a few formulas for easy reference. For the relevant details, we refer to Amur [1], Chern [2] and Flanders [4].

(14) $$[\boldsymbol{e}, \underbrace{d\boldsymbol{X}, \cdots, d\boldsymbol{X}}_{n-1}] = -(n-1)! * d\boldsymbol{X},$$

where $*$ denotes the star operator.

(15) $$p = \boldsymbol{X} \cdot \boldsymbol{e}, \qquad (d\boldsymbol{e}, * d\boldsymbol{X}) = nK_1 dV, \qquad (d\boldsymbol{X}, * d\boldsymbol{X}) = n dV,$$

where $dV = \omega_1 \wedge \cdots \wedge \omega_n$ is the area element of M.

(16) $$[\underbrace{d\boldsymbol{e}, \cdots, d\boldsymbol{e}}_{r}, \underbrace{d\boldsymbol{X}, \cdots, d\boldsymbol{X}}_{n-r}] = r!(n-r)!\binom{n}{r} K_r \boldsymbol{e} dV,$$

$$r = 0, 1, \cdots, n-1,$$

(17) $$d{*}dX = -nK_1 e dV.$$

If f is a smooth function defined on M. By ***grad*** f or ∇f we mean $\nabla f = f_i e_i$, where f_i are given by $df = f_i \omega_i$, we have

(18) $$df \wedge {*}dX = (\nabla f)dV.$$

A self adjoint linear transformation A of the tangent space of M at X into itself is defined by

(19) $$Ae_i = a_{ij} e_j,$$

where the symmetric matrix (a_{ij}) is given by (11). It follows that

(20) $$AdX = A\omega_i e_i = \omega_i A e_i = \omega_i a_{ij} e_j = de$$

We look for other intrinsic tangent vectors which are obtained as the result of repeated application of the transformation A to dX. Let $A^j dX$ denote the intrinsic tangent vector obtained from dX by applying A repeatedly j times. For convenience we write

(21) $$U_0 = dX, \quad U_j = A^j dX, \quad j = 1, 2, \cdots, n.$$

As in [1], we have

(22) $$\sigma_r = -r!(n-r-1)! \sum_{i=0}^{r} (-1)^i \binom{n}{r-i} K_{r-i} {*}U_i,$$

$$r = 0, 1, \cdots, n-1,$$

where

(23) $$\sigma_r = [e, \underbrace{de, \cdots, de}_{r}, \underbrace{dX, \cdots, dX}_{n-r-1}].$$

3. Lemmas

LEMMA 1. *Let*

(24) $$\pi_r = (-1)^n dp \wedge (X \cdot \sigma_r) = (-1)^n (X \cdot de) \wedge (X \cdot \sigma_r),$$

then we have

(25) $$\pi_r = (-1)^n r!(n-r-1)! \sum_{i=0}^{r} (-1)^i \binom{n}{r-i} K_{r-i} x_j x_{h_i} a_{jh_0} \Big(\prod_{k=1}^{i} a_{h_{k-1}h_k}\Big) dV$$

and

$$(26)\qquad (dp)\wedge\sigma_r=(-1)r!(n-r-1)!\sum_{i=0}^{r}(-1)^i\binom{n}{r-i}K_{r-i}x_ja_{jh_0}\left(\prod_{k=1}^{i}a_{h_{k-1}h_k}\right)e_{h_i}dV$$

where $r=0, 1, \cdots, n-1$.

Proof. By (19), (20) and (21) we have

$$(27)\qquad U_r=\left(\prod_{k=1}^{r}a_{h_{k-1}h_k}\right)\omega_{h_0}e_{h_r}.$$

Hence we get

$$(28)\qquad *U_r=(-1)^{h_0-1}\left(\prod_{k=1}^{r}a_{h_{k-1}h_k}\right)\omega_1\wedge\cdots\wedge\hat{\omega}_{h_0}\wedge\cdots\wedge\omega_n e_{h_r}.$$

Thus by (22), we get

$$\pi_r=(-1)^{n+1}r!(n-r-1)!\sum_{i=0}^{r}(-1)^{i+h_0}\binom{n}{r-i}K_{r-i}x_jx_{h_i}$$

$$\left(\prod_{k=1}^{i}a_{h_{k-1}h_k}\right)\omega_{n+1,j}\wedge\omega_1\wedge\cdots\wedge\hat{\omega}_{h_0}\wedge\cdots\wedge\omega_n$$

$$=(-1)^n r!(n-r-1)!\sum_{i=0}^{r}(-1)^i\binom{n}{r-i}K_{r-i}x_jx_{h_0}a_{jh}\left(\prod_{k=1}^{i}a_{h_{k-1}h_k}\right)dV$$

This proves (25). Formula (26) follow immediately from (22) and (28).

LEMMA 2. *Let* σ_r *and* π_r *be given by* (23) *and* (24). *Then we have*

$$(29)\qquad n!p^mK_{r+1}dV-n!p^{m-1}K_rdV+(-1)^n(m-1)p^{m-2}\pi_r=d(p^{m-1}X\cdot\sigma_r),$$

$$r=0, 1, \cdots, n-1.$$

Proof. Since

$$d(p^{m-1}X\cdot\sigma_r)=(-1)^{n+1}p^{m-1}[e,\underbrace{de,\cdots,de}_{r},\underbrace{dX,\cdots,dX}_{n-r}]$$

$$+(m-1)p^{m-2}dp\wedge(X\cdot\sigma_r)+(-1)^np^{m-1}[X,\underbrace{de,\cdots,de}_{r+1},\underbrace{dX,\cdots,dX}_{n-r-1}]$$

$$=(m-1)(-1)^np^{m-2}\pi_r-n!p^{m-1}K_rdV+n!p^{m-1}K_{r+1}dV.$$

This gives (29).

LEMMA 3. *Let* U_i *and* σ_r *be given by* (20) *and* (23). *Then we have*

$$(30)\qquad r!(n-r-1)!\binom{n}{r}[(n-r)p^mK_{r+1}-np^{m-1}K_r-(m-1)p^{m-2}K_rx_ix_ja_{ij}]dV$$

$$=d(p^{m-1}X\cdot\sigma_r)+r!(n-r-1)!\sum_{i=1}^{r}(-1)^i\binom{n}{r-i}[d(p^{m-1}K_{r-i}X\cdot *U_i)$$

$$-p^{m-1}X\cdot d(K_{r-i}*U_i)]\qquad r=0, 1, \cdots, n-1.$$

Proof. Since by the identities of Newton for the elementary symmetric functions (see, for instance, [1],) we can easily verify that

$$\sum_{i=1}^{r}(-1)^{i-1}\binom{n}{r-i}K_{r-i}(d\boldsymbol{X}, {}^*\boldsymbol{U}_i) = r\binom{n}{r}K_r dV. \tag{31}$$

Hence we have

$$\sum_{i=1}^{r}(-1)^{i}\binom{n}{r-i}[d(p^{m-1}K_{r-i}\boldsymbol{X}\cdot{}^*\boldsymbol{U}_i) - p^{m-1}\boldsymbol{X}\cdot d(K_{r-i}{}^*\boldsymbol{U}_i)]$$

$$= \sum_{i=1}^{r}(-1)^{i}(m-1)p^{m-2}K_{r-i}\binom{n}{r-i}dp \wedge \boldsymbol{X}\cdot{}^*\boldsymbol{U}_i$$

$$+ \sum_{i=1}^{r}(-1)^{i}\binom{n}{r-i}p^{m-1}K_{r-i}(d\boldsymbol{X}, {}^*\boldsymbol{U}_i)$$

$$= \sum_{i=1}^{r}(-1)^{i}(m-1)p^{m-2}K_{r-i}\binom{n}{r-i}dp \wedge \boldsymbol{X}\cdot{}^*\boldsymbol{U}_i - r\binom{n}{r}p^{m-1}K_r$$

$$= -(m-1)p^{m-2}K_r\binom{n}{r}x_i x_j a_{ij}dV - r\binom{n}{r}p^{m-1}K_r dV$$

$$-(-1)^n\frac{(m-1)}{r!(n-r-1)!}p^{m-2}\pi_r.$$

Hence, by Lemma 2, it equals to

$$= -(m-1)p^{m-2}K_r\binom{n}{r}x_i x_j a_{ij}dV - r\binom{n}{r}p^{m-1}K_r dV + (n-r)\binom{n}{r}p^m K_{r+1}dV$$

$$-(n-r)\binom{n}{r}p^{m-1}K_r dV - (1/r!(n-r-1)!)d(p^{m-1}\boldsymbol{X}\cdot\sigma_r)$$

From this formula we can easily get (30).

4. The Proofs of Theorems 1 and 2

Proof of Theorem 1. By (22), we have

$$\sigma_r = -r!(n-r-1)!\left[\binom{n}{r}K_r{}^*d\boldsymbol{X} + \sum_{i=1}^{r}(-1)^{i}\binom{n}{r-i}K_{r-1}{}^*\boldsymbol{U}_i\right]$$

By taking exterior differentiation, we get

$$(r+1)\binom{n}{r+1}K_{r+1}edV = n\binom{n}{r}K_1K_r edV - \binom{n}{r}X\cdot\nabla K_r dV$$

$$- \sum_{i=1}^{r}(-1)^{i}\binom{n}{r-i}d(K_{r-i}{}^*\boldsymbol{U}_i)$$

Taking scalar product with $\boldsymbol{X}$ and multiplying by p^{m-1}, we get

$$(n-r)\binom{n}{r}K_{r+1}p^m dV - n\binom{n}{r}K_1K_rp^m dV + \binom{n}{r}p^{m-1}\boldsymbol{X}\cdot\nabla K_r dV$$
$$= -\sum_{i=1}^{r}(-1)^i\binom{n}{r-i}p^{m-1}\boldsymbol{X}\cdot d(K_{r-i}{}^*\boldsymbol{U}_i).$$

Thus by Lemma 3,

$$LHS = -\sum_{i=1}^{r}(-1)^i\binom{n}{r-i}d(p^{m-1}K_{r-i}\boldsymbol{X}\cdot{}^*\boldsymbol{U}_i) - (1/r!(n-r-1)!)d(p^{m-1}\boldsymbol{X}\cdot\sigma_r)$$
$$+\binom{n}{r}[(n-r)p^mK_{r+1} - np^{m-1}K_r - (m-1)p^{m-2}K_rx_ix_ja_{ij}]dV.$$

On the other hand, we have

$$\sum_{i=1}^{r}(-1)^{i+1}\binom{n}{r-i}d(p^{m-1}K_{r-i}\boldsymbol{X}\cdot{}^*\boldsymbol{U}_i) - (1/r!(n-r-1)!)d(p^{m-1}\boldsymbol{X}\cdot\sigma_r)$$
$$= \sum_{i=1}^{r}(-1)^{i+1}\binom{n}{r-i}d(p^{m-1}K_{r-i}\boldsymbol{X}\cdot{}^*\boldsymbol{U}_i) + \sum_{i=0}^{r}(-1)^i\binom{n}{r-i}d(p^{m-1}K_{r-i}\boldsymbol{X}\cdot{}^*\boldsymbol{U}_i)$$
$$= \binom{n}{r}d(p^{m-1}K_r\boldsymbol{X}\cdot{}^*d\boldsymbol{X}).$$

Therefore we get

$$p^{m-1}(pK_1K_r - K_r)dV - p^{m-1}\boldsymbol{X}\cdot\nabla K_r dV - (m-1)p^{m-2}K_rx_ix_ja_{ij}dV$$
$$= d(p^{m-1}K_{r-i}\boldsymbol{X}\cdot{}^*d\boldsymbol{X}).$$

By applying the Stokes theorem to this formula, we get formula (2). This completes the proof of Theorem 1.

Proof of Theorem 2. Since we have

$$d(p^m\sigma_r) = p^m[\underbrace{de,\cdots,de}_{r+1},\underbrace{d\boldsymbol{X},\cdots,d\boldsymbol{X}}_{n-r-1}] + mp^{m-1}dp\wedge\sigma_r$$
$$= (r+1)!(n-r-1)!\binom{n}{r+1}K_{r+1}edV + mp^{m-1}dp\wedge\sigma_r$$

Hence, by the Stokes theorem and Lemma 1, we get (3). Furthermore, by setting $r = n-1$ or 0 and applying formula (31), we get (4) and (5). This completes the proof of Theorem 2.

5. Some Applications

COROLLARY 1. *Under the same assumption of Theorem* 1, *we have*

(32) $$n!\int_M p^{m-1}(pK_{r+1} - K_r)dV - \int_{\partial M}p^{m-1}\boldsymbol{X}\cdot\sigma_r$$

$$= r!(n-r-1)!(m-1)\sum_{i=0}^{r}(-1)^{i+1}\binom{n}{r-i}\int_M p^{m-2}K_{r-i}x_j x_{h_i} a_{jh_0}\Big(\prod_{k=1}^{i} a_{h_{k-1}h_k}\Big)p^{m-2}dV,$$

$$r = 0, 1, \cdots, n-1.$$

In particular, by setting $m = 1$, we have the Minkowski formulas:

$$\int_M pK_{r+1}dV = \int_M K_r dV + \int_{\partial M} X\cdot\sigma_r/n! \qquad r = 0, 1, \cdots, n-1. \tag{33}$$

This Corollary follows immediately from (25), Lemma 2 and the Stokes theorem. This Corollary was obtained by Shahin [8] for $r = 0,\ n-1$, and by Yano and Tani [9] for the closed case.

COROLLARY 2. *Under the same assumption of Theorem 1, we have*

$$n!\int_M K_{r+1}edV = \int_{\partial M}\sigma_r, \qquad r = 0, 1, \cdots, n-1. \tag{34}$$

Two applications of Corollary 1 for the case $m = 1$, one to M and the other to $M + c$, c in E^{n+1}, gives us (34).

COROLLARY 3. *Under the same assumption of Theorem 1, we have*

$$\int_M X\cdot\nabla K_r + n\int_M p(K_{r+1} - K_1K_r)dV = \int_{\partial M} K_r X\cdot{*dX} - \int_{\partial M} X\cdot\sigma_r/n!, \tag{35}$$

$$r = 0, 1, \cdots, n-1.$$

This Corollary follows immediately from Theorem 1 and Corollary 1.

COROLLARY 4. *There is no minimal closed hypersurface in E^{n+1}.*

Proof. Set $r = 0$, then by (32), we know that if M is closed, then the volume $v(M)$ of M is given by

$$v(M) = \int_M pK_1 dV. \tag{36}$$

Hence, if M is a minimal hypersurface of E^{n+1}, then $K_1 = 0$, hence $v(M) = 0$. But this is impossible. This Corollary was proved by Chern and Hsiung.

COROLLARY 5. *Under the same assumption of Theorem 1, if M is closed, then*

$$\int_M \nabla K_r dV = n\int_M K_1K_r edV, \qquad r = 0, 1, \cdots, n-1. \tag{37}$$

In particular, if the mean curvature K_1 is constant, then we have

$$\int_M \nabla K_r dV = 0, \qquad r = 0, 1, \cdots, n-1. \tag{38}$$

Two applications of Corollary 3, one to M and the other to $M + c$, gives us (37). Formula (38) follows immediately from (37) if K_1 is constant. This Corollary was obtained by Amur [1].

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