Audrey Terras
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A GENERALIZATION OF EPSTEIN'S ZETA FUNCTION

AUDREY TERRAS*)

§ 0. Introduction.

Koecher defined in [3] the following zeta function associated with the matrix $S^{(n)}$ of a positive quadratic form and one complex variable ρ

$$Z_{n_1}(S, \rho) = \sum_A |{}^tASA|^{-\rho}. \tag{1}$$

Here $n \geq n_1$, and the sum is over a complete set of representatives for the n by n_1 integral rank n_1 matrices $A^{(n,n_1)}$ with respect to the equivalence relation $A \sim B$ if $A = BU$ for some unimodular matrix U. The unimodular group $\mathfrak{U}_{n_1}$ is defined by $\mathfrak{U}_{n_1} = \{U^{(n_1)}: U$ integral, n_1 by n_1 with determinant $|U| = \pm 1\}$. We use the notation $|S|$ = determinant of S and $S[A] = {}^tASA$ throughout. Superscripts in parentheses on matrices denote the number of rows and columns. Thus $A^{(n,n_1)}$ has n rows and n_1 columns.

Koecher shows in [3] that $Z_{n_1}(S, \rho)$ converges for $Re\rho > \frac{n}{2}$. But his proof of the analytic continuation and the functional equation,

$$R_{n_1}(S, \rho) = |S|^{\frac{-n_1}{2}} R_{n_1}\left(S^{-1}, \frac{n}{2} - \rho\right) \tag{2}$$

where $R_{n_1}(S, \rho) = \pi^{n_1\left(\frac{n_1-1}{4} - \rho\right)} \prod_{i=0}^{n_1-1} \Gamma\left(\rho - \frac{i}{2}\right) Z_{n_1}(S, \rho)$,

has a gap. This is remedied neatly using an idea of Selberg. One can annihilate the trouble-making terms of the theta function with an appropriate differential operator. We outline these results in § 1 because they do not appear in the literature.

Selberg has defined in [6] a zeta function associated with a positive matrix S and $n-1$ complex variables $\rho = (\rho_1, \rho_2, \cdots, \rho_{n-1})$. This function can be seen to be essentially the same as

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$$\zeta_{(n)}(S, \rho) = \sum \prod_{i=1}^{n-1} |S[U_i]|^{-\rho_i}, \tag{3}$$

where the sum is over a complete set of representatives for unimodular $U^{(n)} = (U_i^{(n,i)}*)$, with respect to the equivalence relation $U \sim V$ if $U = VP$ for P unimodular and upper triangular. Selberg states that this series converges for $Re\rho_i > 1$ and that $\zeta_{(n)}(S, \rho)$ satisfies $n!$ functional equations and can be continued to a meromorphic function in C^{n-1}. There are various proofs of this fact. See [4] and [9]. Selberg used both approaches. We consider the method used in [9] briefly in § 2 because we generalize this method to obtain our main results. And we make use of the $n-1$ functional equations which generate the rest:

For $i = 2, 3, \cdots, n-2$,

$$\pi^{-\rho_i}\Gamma(\rho_i)\zeta(2\rho_i)\zeta_{(n)}(S, \rho) = \pi^{-(1-\rho_i)}\Gamma(1-\rho_i)\zeta(2(1-\rho_i))\zeta_{(n)}(S, \rho') \tag{4}$$

where $\rho_i' = 1 - \rho_i$, $\rho_{i\pm 1}' = \rho_{i\pm 1} + \rho_i - \frac{1}{2}$, $\rho_j' = \rho_j$ for $j \neq i$, $i \pm 1$.

For $i = 1$,

$$\pi^{-\rho_1}\Gamma(\rho_1)\zeta(2\rho_1)\zeta_{(n)}(S, \rho) = \pi^{-(1-\rho_1)}\Gamma(1-\rho_1)\zeta(2(1-\rho_1))\zeta_{(n)}\left(S, 1-\rho_1, \rho_1 + \rho_2 - \frac{1}{2}, \rho_3, \cdots\right)$$

For $i = n-1$,

$$\pi^{-\rho_{n-1}}\Gamma(\rho_{n-1})\zeta(2\rho_{n-1})\zeta_{(n)}(S, \rho) = \pi^{-(1-\rho_{n-1})}\Gamma(1-\rho_{n-1})\zeta(2(1-\rho_{n-1}))\times$$
$$\times \zeta_{(n)}\left(S, \rho_1, \cdots, \rho_{n-3}, \rho_{n-1} + \rho_{n-2} - \frac{1}{2}, 1 - \rho_{n-1}\right).$$

It should be noted that an easy "proof" of the $n!$ functional equations derives from an integral formula of Harish-Chandra for spherical functions (see [1], Proposition 6.8, page 428). Maass has shown in [4] that a further functional equation is trivial, namely:

$$\zeta_{(n)}(S^{-1}, \rho) = |S|^{\sum_{i=1}^{n-1}\rho_i}\zeta_{(n)}(S, \tilde{\rho}), \tag{5}$$

where $\tilde{\rho} = (\rho_{n-1}, \rho_{n-2}, \cdots, \rho_2, \rho_1)$.

We consider here a generalization of both (1) and (3):

$$\zeta_{n_1, \cdots, n_r}(S, \rho_1, \cdots, \rho_{r-1}) = \sum \prod_{i=1}^{r-1} |S[U_i]|^{-\rho_i}. \tag{6}$$

Here $n=\sum_{j=1}^{\gamma} n_j$, with positive integers n_j, and $N_i=\sum_{j=1}^{i} n_j$, for $i=1,2,\cdots,\gamma$. In the sum, $U^{(n)}=(U_i^{(n,N_i)}*)$ runs over a complete set of representatives for unimodular matrices, with respect to the equivalence relation $U\sim V$ if $U=VP$, with P unimodular and having block form

$$P=\begin{pmatrix} P_1^{(n_1)} & & & * \\ & P_2^{(n_2)} & & \\ 0 & & \ddots & \\ & & & P_\gamma^{(n_\gamma)} \end{pmatrix}.$$

We show in § 3 that (6) converges when $Re\,\rho_i>\frac{n}{2}$, $i=1,2,\cdots,\gamma-1$.

It comes out of the proof of (4) in § 2 that

(7) $\zeta_{(n)}(S,\rho)|_{\rho_i=0}=\zeta_{1,\ldots,1,\underset{i}{2},1,\ldots,1}(S,\rho_1,\cdots\overset{\overset{i}{\vee}}{},\cdots,\rho_{n-1})$, where "$\vee$" denotes omission of the variable ρ_i. In § 3 we generalize this relation to

(8) $$\zeta_{n_1,\ldots,n_\gamma}(S,\rho_{N_1},\cdots,\rho_{N_\gamma-1})|_{\rho_{N_l}=0,\,l\neq i_j}=\zeta_{m_1,\ldots,m_\lambda}(S,\rho_{M_1},\cdots,\rho_{M_\lambda-1}),$$

where $N_{i_j}=\sum_{k=1}^{i_j} n_k=\sum_{k=1}^{j} m_k=M_j$, for $j=1,\cdots,\lambda$. The proof is by induction using results of § 1. Thus Koecher's functions are essentially specializations of Selberg's arrived at by setting all but one variable equal to zero. Similarly the functions (6) are specializations of Selberg's function (3).

In § 4 we use the functional equations in § 2 for Selberg's function and the results of § 3 to obtain relations between Koecher's functions $\zeta_{i,n-i}$ and $\zeta_{n-i,i}$. We have two ways of doing this ((4) and (5)).

§ 1. Koecher's Zeta Function

We first note the relation between Koecher's function (1) and the case $\gamma=2$ of (6).

LEMMA 1.1. $$Z_{n_1}(S,\rho)=\zeta_{n_1,n-n_1}(S,\rho)\prod_{i=0}^{n_1-1}\zeta\Big(2\Big(\rho-\frac{i}{2}\Big)\Big).$$

Proof. Consider the map h defined by $h(A^{(n,n_1)})=(U^{(n,n_1)},B^{(n_1)})$, where B is the greatest right divisor of A and $A=UB$ with U primitive. For the definitions of greatest right divisor and primitive, see Siegel [8], volume I, pages 331 and 332. The map h gives a one-to-one map from integral rank n_1 non right equivalent A to primitive non right equivalent U and integral rank n_1 non right equivalent B. One uses a result of Koecher-formula (1.10) of [3], page 7— to complete the proof.

The main results of this section are the functional equation and analytic continuation of $Z_{n_1}(S, \rho)$. The proof requires a series of lemmas, which we shall simply state. For more details, see [9].

Define the theta function by

$$\theta(S^{(n)}, X^{(n_1)}) = \sum e^{-\pi\sigma(S[A]X)} = \theta^S(X), \tag{9}$$

where $\sigma(S)$ = trace of S, and the sum is over all integral $A^{(n, n_1)}$. If we restrict summation to A of rank r, we denote the result $\theta_r(S, X) = \theta_r^S(X)$. Koecher proves (using the Poisson summation formula, in [3], page 8) that $\theta(S, X)$ converges for $Re\, S > 0$, $X > 0$, $n \geq n_1$ and that it satisfies the transformation formula

$$\theta(S, X) = |S|^{\frac{-n_1}{2}} |X|^{\frac{-n}{2}} \theta(S^{-1}, X^{-1}). \tag{10}$$

Selberg defines differential operators D_a on the Riemannian symmetric space P_n of positive $Y^{(n)}$ as follows:

$$D_a = |Y|^a D_Y |Y|^{1-a}, \tag{11}$$

where $D_Y = \left| \frac{1}{2}(1 + \delta_{ij}) \frac{\partial}{\partial y_{ij}} \right|$, δ_{ij} being the Kronecker delta. It is easy to see that the D_a are invariant differential operators with respect to the diffeomorphisms τ_A of the space P_n defined by $\tau_A(Y) = Y[A]$, for $Y \in P_n$, $A \in GL(n, \boldsymbol{R})$. The general linear group $GL(n, \boldsymbol{R})$ consists of all non-singular n by n real matrices.

Further define L^* as the formal adjoint of the differential operator L on P_n, with respect to the τ_A-invariant measure $d\mu = |Y|^{-\frac{n+1}{2}} \prod_{1 \leq i \leq j \leq n} dy_{ij}$. That is, $\int_{P_n} (Lf) g \, d\mu = \int_{P_n} f \, L^* g d\mu$ for f and g infinitely differentiable on P_n with compact support (*i.e.*, in $C_0^\infty(P_n)$). Let $\alpha(Y) = Y^{-1}$. Then one can show that

$$L^* = L^\alpha, \tag{12}$$

where $L^\alpha(f) = [L(f \circ \alpha)] \circ \alpha^{-1}$ for $f \in C^\infty(P_n)$.

Moreover the algebra of all L which are invariant with respect to the τ_A (*i.e.* $L^{\tau_A} = L$), for $A \in GL(n, \boldsymbol{R})$, is commutative. For a proof, see Selberg [5], page 51.

One easily sees that

$$D_Y e^{-\sigma(WY)} = |W|(-1)^n e^{-\sigma(WY)} \tag{13}$$

Also

$$(D_a)^\alpha = (D_a)^* = (-1)^n D_{1+\frac{n+1}{2}-a} \tag{14}$$

Now define

$$f_\gamma^\rho(X) = \prod_{i=1}^{\gamma-1} |X_i|^{\rho_i}. \tag{15}$$

if $X = \begin{pmatrix} X_i^{(N_i)} & * \\ * & * \end{pmatrix}$, $N_i = \sum_{j=1}^{i} n_j$, $i = 1, 2, \cdots, \gamma$, and $n = \sum_{j=1}^{\gamma} n_j = N_\gamma$. Note that $\zeta_{n_1}, \cdots, {}_{n_\gamma}(S, \rho) = \sum f_\gamma^{-\rho}(S[U])$. The f_γ^ρ are eigenfunctions for the invariant differential operators L on the space P_n of positive symmetric matrices . We compute the eigenvalues for D_a in the case $\gamma = 2$ later in this section. Writing $X = {}^tTT$ for T upper triangular with positive diagonal entries we have, if $n = \gamma$, *i.e.*, $n_1 = n_2 = \cdots = n_\gamma = 1$,

$$f_n^\rho({}^tTT) = \prod_{i=1}^{n-1} t_i^{r_i} = \chi^r(T), \text{ where } T = \begin{pmatrix} t_1 & & * \\ & t_2 & \\ & & \ddots \\ 0 & & & t_n \end{pmatrix}, \tag{16}$$

and $r_i = 2\sum_{j=i}^{n-1} \rho_j$. A similar result holds for arbitrary γ. χ is a character on the triangular group and can be extended to $GL(n, \boldsymbol{R})$ by making it constant on cosets $O_n T$, where T is an upper triangular matrix and O_n is the orthogonal group. The result is a right spherical function.

We need to define the following gamma factor

$$G_{n_1}(\rho) = \prod_{i=0}^{n_1-1} \Gamma\left(\rho - \frac{i}{2}\right). \tag{17}$$

Now we can prove the functional equation and obtain the analytic continuation of $Z_{n_1}(S, \rho)$ to the whole complex plane as a meromorphic function.

THEOREM 1.2. *Let L be the invariant differential operator on P_{n_1}*

$$L = D_a D_1,$$

for $a = \frac{1}{2}(n_1 - n + 1)$. *Let* F_{n_1} *be a fundamental domain for* P_{n_1} *with respect to* τ_U *for* $U \in \mathfrak{U}_{n_1}$ *(recall that* $\tau_U(Y) = {}^tU\,YU = Y[U]$*). Then we have the following integral representation:*

$$(18) \qquad 2Z_{n_1}(S^{(n)}, \rho)g(\rho)G_{n_1}(\rho)\pi^{\frac{1}{4}(n_1^2-n_1)-n_1\rho} = \int_{F_{n_1}} f_2^{\rho}(L\theta_{n_1}^S)d\mu \equiv J(S, \rho)$$

$(f_2^{\rho}(X) = |X|^{\rho},\ X \in P_{n_1})$.

The integral can be analytically continued for all ρ *and*

$$J(S, \rho) = |S|^{\frac{-n_1}{2}} J\Big(S^{-1}, \frac{n}{2} - \rho\Big).$$

Further $g(\rho) = \prod_{i=1}^{n_1}\Big(\rho + \frac{n_1 - i - n}{2}\Big)\Big(\rho - \frac{i-1}{2}\Big) = g\Big(\frac{n}{2} - \rho\Big)$.

Proof. By changing to triangular matrix variables T through $Y = {}^tTT$, it is easy to see that

$$J(S, \rho) = g(\rho)\int_{F_{n_1}} f_2^{\rho}\theta_{n_1}^S\ d\mu = 2g(\rho)Z_{n_1}(S, \rho)\pi^{\frac{1}{\pi}(n_1^2-n_1)-n_1\rho}G_{n_1}(\rho),$$

where $L^*f_2^{\rho} = g(\rho)f_2^{\rho}$, since $f_2^{\rho}(X) = |X|^{\rho}$ is an eigen function for L. We compute $g(\rho)$ later.

Now

$$\begin{aligned} J(S, \rho) &= \int_{\substack{F_{n_1}\\|X|\geq 1}} + \int_{\substack{F_{n_1}\\|X|\leq 1}} \\ &= \int_{\substack{F_{n_1}\\|X|\geq 1}} \{f_2^{\rho}L\theta_{n_1}^S + (f_2^{\rho}\circ\alpha)((L\theta_{n_1}^S)\circ\alpha)\}d\mu \\ &= \int_{\substack{F_{n_1}\\|X|\geq 1}} \{f_2^{\rho}L\theta_{n_1}^S + f_2^{-\rho}L^{\alpha}(|S|^{\frac{-n_1}{2}} f_2^{\frac{n}{2}}\theta_{n_1}^{S^{-1}})\}d\mu \\ &\quad + \sum_{r=0}^{n_1-1}\int_{\substack{F_{n_1}\\|X|\geq 1}} f_2^{-\rho}\{|S|^{\frac{-n_1}{2}}L^{\alpha}(f_2^{\frac{n}{2}}\theta_r^{S^{-1}}) - (L\theta_r^S)\circ\alpha\}d\mu \end{aligned}$$

using the transformation formula (10). We have chosen L so that all the terms of the last integral are zero. For recall that $L = D_aD_1 = D_1D_a$. And (13) implies $D_Y(\theta_r^S) = 0$, for $0 \leqslant r < n_1$. And so $D_1(\theta_r^S) = f_2^1D_Y(\theta_s^S) = 0$, $0 \leqslant r < n_1$.

Now $(-1)^{n_1}D_a^* = D_{\frac{n}{2}+1} = f_2^{\frac{n+2}{2}} D_Y f_2^{\frac{-n}{2}}$ by (14). So $(-1)^{n_1}D_a^*(f_2^{\frac{n}{2}}\theta_r^{S^{-1}}) = f^{\frac{n+2}{2}} D_Y(\theta_r^{S^{-1}}) = 0,\ 0 \leqslant r < n_1$. Therefore

$$J(S, \rho) = \int_{\substack{F_{n_1} \\ |X| \geq 1}} \left\{ f_2 \theta_{n_1}^S + f_2^{-\rho} |S|^{-\frac{1}{2}n_1} L^a(f_2^{\frac{1}{2}n}\ \theta_{n_1}^{S^{-1}}) \right\} d\mu.$$

The functional equation $J(S, \rho) = |S|^{-\frac{1}{2}n_1} J\left(S^{-1}, \frac{1}{2}n - \rho\right)$ will follow immediately from the convergence of the above integral for all ρ, since $L\theta_{n_1}^S = f^{-\frac{1}{2}n} L^a(_2^{\frac{1}{2}n}\ \theta_{n_1}^S)$.

The convergence of $J(S, \rho)$ is easy. For we may assume $Re\,\sigma$ is arbitratily large in the function $f_2^\sigma(X) = |X|^\sigma$, since this only increases the function when $|X| \geq 1$. *Here we use also the fact that* $Z_{n_1}(S, \rho)$ *converges for* $Re\,\rho > \frac{n}{2}$, to complate the proof that $J(S, \rho)$ converges for all ρ. In fact we obtain the inequality (which we shall use in the proof of Theorem 3.3)

$$J(S, \rho) \leqslant g(|\rho| + n) \int_{F_{n_1}} f_2^{|\rho|+n} \theta_{n_1}^S\, d\mu + |S|^{-\frac{1}{2}n_1} g'(-(|\rho| + n)) \int_{F_{n_1}} f_2^{|\rho|+n} \theta_{n_1}^{S^{-1}} d\mu,$$

were $Lf_2^{-\rho} = g'(\rho) f_2^{-\rho}$. Actually $g(\rho) = g'(\rho)$, as we shall see.

Next we compute $g(\rho)$ and $g'(\rho)$. Suppose $D_1^*|Y|^\rho = h(\rho)|Y|^\rho(-1)^{n_1}$. Then

$$(-1)^{n_1} h(\rho) \int_{P_{n_1}} e^{-\sigma(Y)} |Y|^\rho d\mu = \int_{P_{n_1}} e^{-\sigma(Y)} D_1^* |Y|^\rho d\mu$$

$$= \int_{P_{n_1}} (D_1 e^{-\sigma(Y)}) |Y|^\rho d\mu = (-1)^{n_1} \int_{P_{n_1}} |Y| e^{-\sigma(Y)} |Y|^\rho d\mu.$$

It is clear that

$$h(\rho) = \frac{(-1)^{n_1} \int_{P_{n_1}} e^{\sigma-(Y)} |Y|^{1+\rho} d\mu}{(-1)^{n_1} \int_{P_{n_1}} e^{-\sigma(Y)} |Y|^\rho d\mu} = \frac{\prod_{i=0}^{n_1-1} \Gamma\left(1 + \rho - \frac{i}{2}\right)}{\prod_{i=0}^{n_1-1} \Gamma\left(\rho - \frac{i}{2}\right)} = \prod_{i=0}^{n_1-1} \left(\rho - \frac{i}{2}\right).$$

So $|Y|^{\frac{n_1+1}{2}} D_Y |Y|^{1-\frac{n_1+1}{2}+\rho} = h(\rho)|Y|^\rho$ and $D_Y|Y|^\rho = h\left(\rho + \frac{n_1 - 1}{2}\right)|Y|^{\rho-1} = k(\rho)|Y|^{\rho-1}$. It follows that $k(\rho) = \prod_{i=1}^{n_1}\left(\rho + \frac{n_1 - i}{2}\right)$ and $g(\rho) = k\left(\rho - \frac{n}{2}\right) \times k\left(\rho - \frac{n_1 - 1}{2}\right)$, $g'(\rho) = k\left(-\rho + \frac{1}{2}(n - n_1 + 1)\right)k(-\rho) = g\left(\frac{n}{2} - \rho\right) = g(\rho)$. This completes the proof.

It will be necessary later to compute the residue of Koecher's zeta function at $\frac{n}{2}$.

THEOREM 1.3. $\operatorname{Res}_{\rho=\frac{n}{2}} Z_{n_1}(S^{(n)}, \rho) = \frac{1}{2}\pi^{\frac{1}{2}(1+nn_1-n_1^2)} |S|^{\frac{-n_1}{2}} \times$

$$\prod_{k=2}^{n_1} \zeta(k)\Gamma\left(\frac{k}{2}\right) \prod_{j=0}^{n_1-1} \Gamma\left(\frac{n-j}{2}\right)^{-1}$$

Proof. The method of Siegel [8], volume III, pages 328 to 333, can be modified to show this result. One needs also formula (3.18) of Koecher [3], page 14. Siegel's proof is for $n = n_1$, so some work is required to obtain the result. For this, see [9], pages 82–91.

§ 2. Selberg's Zeta Function.

We are considering $\zeta_{(n)}(S^{(n)}, \rho)$ defined by (3). This is the case $\gamma = n$ of (6). Since the results of this section have been stated by many authors (Selberg [6], Maass [4], and Godement in a 1962 lecture at Johns Hopkins University), we shall be brief. We must note some details of the proofs for later use.

Define $\mathfrak{P}(n_1, \cdots, n_r) = \left\{ U : U \in \mathfrak{U}_n,\ U = \begin{pmatrix} U_1^{(n_1)} & & * \\ & \ddots & \\ 0 & & U_r^{(n_r)} \end{pmatrix} \right\}$.

LEMMA 2.1. *(A Decomposition for $\zeta_{(n)}$ with respect to $\mathfrak{P}_i^* = \mathfrak{P}(1, \cdots, 1, \underset{i}{2}, 1, \cdots, 1)$)*

$$\zeta_{(n)}(S, \rho) = \sum_{V \in \mathfrak{U}_n/\mathfrak{P}_i^*} \prod_{\substack{j=1 \\ j \neq i}}^{n-1} |S[V_j]|^{-\rho_j} \zeta_{(2)}(T, \rho_i).$$

Here $V^{(n)} = (V_j^{(n,j)} *)$, $V_{i+1} = (V_{i-1} V'_{i-1})$, *and* $T = |S[V_{i-1}]| \{S[V'_{i-1}] - (S[V_{i-1}])^{-1} [{}^tV_{i-1} S V'_{i-1}]\}$. *Also*

$$|T| = |S[V_{i-1}]|\ |S[V_{i+1}]| \text{ for } i \neq 1,\ n-1,$$
$$|T| = |S[V_2]| \text{ if } i = 1, \text{ and}$$
$$|T| = |S|\ |S[V_{n-1}]| \text{ if } i = n-1.$$

Proof. Note that $\mathfrak{U}_n \supset \mathfrak{P}_i^* \supset \mathfrak{P}_{(n)} = \mathfrak{P}(1, \cdots, 1)$. Therefore any $U \in \mathfrak{U}_n/\mathfrak{P}_{(n)}$ may be written uniquely as $U = VW$, $V \in \mathfrak{U}_n/\mathfrak{P}_i^*$, $W \in \mathfrak{P}_i^*/\mathfrak{P}_{(n)}$. And we can take

$$W=\begin{pmatrix} I^{(i-1)} & 0 & 0 \\ 0 & W^* & 0 \\ 0 & 0 & I^{(n-1-i)} \end{pmatrix} \text{ with } W^*\in\mathfrak{U}_2/\mathfrak{P}_{(2)},\ \mathfrak{P}_{(2)}=\mathfrak{P}(1,1).$$

Now we have $|S[VW_j]| = |S[V_j]|$ for $j \neq i$. And $|S[VW_i]| = |T[W_1^*]|$. This proves the lemma.

The analytic continuation and functional equation of $\zeta_{(2)}(T, \rho_i)$ (an Epstein zeta function) can be used to derive the same for $\zeta_{(n)}$. One obtains the functional equations (4) and formula (7) immediately (after an argument on the convergence of the series, which we omit here, as we must generalize it in § 3).

In order to complete the analytic continuation with this approach it is convenient (and not surprising, considering the integral formula of Harish-Chandra, stated in [1], proposition 6.8, page 428) to introduce new variables z, with $\rho_i = z_{i+1} - z_i + \frac{1}{2}$. One uses (4) to show that for $\zeta'_{(n)}(z) = \zeta_{(n)}(\rho(z))$ we have

$$\Xi(z) = |S|^{z_n-\frac{1}{2}}\pi^{-2\sum_{j=1}^{n-1} jz_j} \prod_{1\leq i<j\leq n} \Gamma\left(z_j - z_i + \frac{1}{2}\right)\zeta\left(2\left(z_j - z_i + \frac{1}{2}\right)\right)\zeta'_{(n)}(z) \tag{19}$$

is invariant under all permutations of $z_1, z_2, \cdots, z_n$.

One obtains the analyticity of $\prod_{1\leq i<j\leq n}\left((z_j - z_i - \frac{1}{2}\right)\zeta'_{(n)}(z)$ in $\Delta^* = \{z\in \boldsymbol{C}^n$: there is a permutation σ such that $Re\,(z_{\sigma(j+1)} - z_{\sigma(j)}) > n,\ j = 2, 3, \cdots, n\}$. The region Δ^* is a connected tube and its convex hull is $\boldsymbol{C}^n$ because it is fairly easy to see that it contains n independent lines. Applying Theorem 2.5.10 of Hörmander [2] one obtains the result that $\zeta_{(n)}$ can be continued to $\boldsymbol{C}^{n-1}$. For more details of the above arguments see § 3 of [9].

§ 3. The General Decomposition of the Zeta Function $\zeta_{n_1,\dots,n_\gamma}$ Corresponding to $\mathfrak{P}=\mathfrak{P}(n_1, \cdots, n_\gamma)$ with respect to $\mathfrak{P}^*=\mathfrak{P}(m_1, \cdots, m_\lambda)\supset\mathfrak{P}$.

We first show the convergence of $\zeta_{n_1,\dots,n_\gamma}(S^{(n)}, \rho_1, \cdots, \rho_{\gamma-1}) = \zeta(S, \rho)$ for $Re\,\rho_i > \frac{n}{2}$, $i = 1, 2, \cdots, \gamma - 1$. This results from the following theorem, since $Z_m(S^{(n)}, \rho)$ converges whenever $Re\,\rho > \frac{n}{2}$, as Koecher proves in [3], page 7. (Koecher bounds $Z_m(S^{(n)}, \rho)$ by $c^{\rho}(Z_1(S, \rho))^m$, c being a positive constant. Now the Epstein zeta function $Z_1(S, \rho)$ converges for $Re\,\rho > \frac{n}{2}$ using methods like those of Hecke for the Dedekind zeta function).

THEOREM 3.1. *(Convergence of the Zeta Function for Arbitrary r). For real ρ_i,*

$$\zeta_{n_1,\dots,n_r}(S^{(n)}, \rho_1, \cdots, \rho_{r-1}) \leq \prod_{i=1}^{r-1} \zeta_{N_i, n-N_i}(S, \rho_i).$$

Here $n = \sum_{j=1}^{r} n_j$ and $N_i = \sum_{j=1}^{i} n_j$.

Proof. Define a map $f: \mathfrak{U}_n/\mathfrak{P}(n_1, \cdots, n_r) \longrightarrow \prod_{i=1}^{r-1} \mathfrak{A}_{n,N_i}/\mathfrak{A}_{N_i}$, where $\mathfrak{U}_{n,N_i} = \{V^{(n,N_i)}: V \text{ primitive}\}$. The map f is defined as follows. Let $U \in \mathfrak{U}_n/\mathfrak{P}(n_1, \cdots, n_r)$ and $U = (U_i^{(n,N_i)} *)$. Suppose $U_i = V_i B_i$, with B_i the greatest right divisor of U_i and V_i primitive. (The notion of greatest right divisor and primitive is defined in Siegel [8], volume I, pp. 331, 332). Define $f(U) = (V_1, \cdots, V_{r-1})$. The map is well-defined.

The map is shown to be one-to-one by an induction process. Suppose $f(U) = f(U')$, where $U_i = V_i B_i$ and $U_i' = V_i B_i'$ as above. Then $U_1 B_1^{-1} B_1' = U_1'$. Let $R_1 = B_1^{-1} B_1' \in \mathfrak{U}_{N_1}$. In general let $R_i = B_i^{-1} B_i'$. We assume as the induction hypothesis that $R_i \in \mathfrak{P}(n_1, \cdots, n_i)$. Then $U_{i+1}\begin{pmatrix} R_i & 0 \\ 0 & I^{(n_{i+1})} \end{pmatrix}$ and U_{i+1}' have the same first N_i columns. So $B_{i+1}\begin{pmatrix} R_i & 0 \\ 0 & I^{(n_{i+1})} \end{pmatrix}$ and B_{i+1}' have the same first N_i columns. (Here take $B_r = U$, $B_r' = U'$). Therefore $R_{i+1} = (B_{i+1})^{-1} B_{i+1}' \in \mathfrak{P}(n_1, \cdots, n_{i+1})$. The result follows that $U^{-1}U' \in \mathfrak{P}(n_1, \cdots, n_r)$, $U = U'$, and f is one-to-one.

Now that we have the convergence of the zeta function for arbitrary r, we generalize Lemma 2.1.

THEOREM 3.2. *Suppose $\mathfrak{P} = \mathfrak{P}(n_1, \cdots, n_r) \subset \mathfrak{P}^* = \mathfrak{P}(m_1, \cdots, m_\lambda)$. Let $N_{i_j} = \sum_{k=1}^{i_j} n_k = M_j = \sum_{k=1}^{j} m_k$. Then we have the following representation:*

$$\zeta_{n_1,\dots,n_r}(S, \rho_{N_1}, \cdots, \rho_{N_{r-1}})$$
$$= \sum_{V \in \mathfrak{U}_n/\mathfrak{P}^*} \prod_{j=1}^{\lambda} |S[V_{j-1}]|^{-\sum_{k=i_{j-1}}^{i_j-1} \rho_{N_k}} \zeta_{n_{i_{j-1}+1},\dots,n_{i_j}}(T_j, \rho_{N_{i_{j-1}+1}}, \cdots, \rho_{N_{i_j-1}}).$$

Here $V = (V_j^{(n,M_j)} *)$, and $T_j = \{S - (S[V_{j-1}])^{-1}[{}^tV_{j-1}S]\}[Y_{j-1}]$, if $V_j = (V_{j-1} Y_{j-1})$. Note that $|T_j| = |S[V_j]|/|S[V_{j-1}]|$.

Proof. Write $U \in \mathfrak{U}/\mathfrak{F}$ in the form $U = VW$, $V \in \mathfrak{U}/\mathfrak{P}^*$, $W \in \mathfrak{P}^*/\mathfrak{P}$. Then $W = \begin{pmatrix} W_1 & & 0 \\ & \ddots & \\ 0 & & W_\lambda \end{pmatrix}$, $W_j \in \mathfrak{U}_{m_j}/\mathfrak{P}\,(n_{i_{j-1}+1}, \cdots, n_{i_j})$. Now if $i_{j-1} < k \leq i_j$, we can write as before:

$$|S[VW_{N_k}]| = \left| S\left[(V_{j-1} Y_{j-1}) \begin{pmatrix} Z_{j-1}^{(M_{j-1})} & 0 \\ 0 & Q_{j-1}^{(m_j, N_k - M_{j-1})} \end{pmatrix} \right] \right|$$

where $W = \begin{pmatrix} Z & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & * \end{pmatrix}$. It follows that

$$\begin{aligned} |S[VW_{N_k}]| &= |S[(V_{j-1}Z_{j-1},\ Y_{j-1}Q_{j-1})]| \\ &= |S[V]|\ |S[YQ] - (S[VZ])^{-1}[{}^t(VZ]SYQ]| \\ &= |S[V]|\ |\{S - (S[V])^{-1}[{}^tVS]\}\,[YQ]| \\ &= |S[V_{j-1}]|\ |T_j[Q_{j-1}]|. \end{aligned}$$

This proves the theorem.

Next we generalize formula (7).

THEOREM 3.3. $\zeta_{n_1,\dots,n_r}(S, \rho_{N_1}, \cdots, \rho_{N_{r-1}})|_{\rho_{N_k}=0, k \neq i_j} = \zeta_{m_1,\dots,m_\lambda}(S, \rho_{M_1}, \cdots, \rho_{M_{\lambda-1}})$, *where* $N_{i_j} = M_j$. *(Here we use the notation of Theorem* 3.2).

Proof. We proceed by induction on r. The case $r = 2$ is needed in the induction step, so we shall prove it in detail. That is, we shall show that $\zeta_{n_1,n_2}(T, 0) = 1$. This is equivalent (by Lemma 1.1) to the proof that $Z_{n_1}(T^{(n)}, 0) = \prod_{k=0}^{n_1-1} \zeta(-k)$.

By Theorem 1.2,

$$\prod_{i=1}^{n_1} \Gamma\left(\rho - \frac{i-1}{2}\right) \pi^{-n_1\rho} Z_{n_1}(S, \rho) = |S|^{-\frac{n_1}{2}} \prod_{i=1}^{n_1} \Gamma\left(\frac{n+1-i}{2} - \rho\right) \pi^{-n_1\left(\frac{n}{2} - \rho\right)} Z_{n_1}\left(S^{-1}, \frac{n}{2} - \rho\right).$$

We take the residue at $\rho = 0$ on both sides, recalling the facts:

$$\operatorname*{Res}_{\rho=0} \Gamma(\rho) = 1 \text{ and } \operatorname*{Res}_{\rho=0} Z_{n_1}\left(S^{-1}, \frac{n}{2} - \rho\right) = -\operatorname*{Res}_{\rho=\frac{n}{2}} Z_{n_1}(S^{-1}, \rho)$$

$$= -\frac{1}{2} \pi^{\frac{1}{2}(-n_1^2 + n_1 n + 1)} |S|^{\frac{n_1}{2}} \prod_{k=2}^{n_1} \zeta(k) \Gamma\left(\frac{k}{2}\right) \prod_{i=1}^{n_1-1} \Gamma\left(\frac{n-i}{2}\right)^{-1}, \text{ by Theorem 1.3.}$$

It follows that

$$Z_{n_1}(S,0)=\frac{-\frac{1}{2}\pi^{-\frac{1}{2}(n_1^2-1)}\prod_{i=2}^{n_1}\zeta(i)\Gamma\left(\frac{i}{2}\right)}{\prod_{i=1}^{n_1-1}\Gamma\left(-\frac{i}{2}\right)}.$$

Let $\phi(x)=\zeta(2x)\Gamma(x)$. Then the functional equation of the Riemann zeta function is $\Phi(1-x)=\pi^{\frac{3}{2}-2x}\phi\left(x-\frac{1}{2}\right)$. This means that $\zeta(-y)\Gamma\left(-\frac{y}{2}\right)=\pi^{-y-\frac{1}{2}}\zeta(y+1)\Gamma\left(\frac{y+1}{2}\right)$. Therefore

$$Z_{n_1}(S,0)=\prod_{i=0}^{n_1-1}\zeta(-i).$$

This completes the case $\gamma=2$.

Now we proceed to the induction step. We suppose for convenience that $i\neq 1$. By Theorem 3.2 for $\mathfrak{P}^*=\mathfrak{P}(n_1+n_2,n_3,\cdots,n_\gamma)$ we have $M_j=N_{j+1}$, $j=1,2,\cdots,\gamma-1$, and

$$\zeta_{n_1,\ldots,n_\gamma}(S,\rho_{N_1},\cdots,\rho_{N_{\gamma-1}})=\sum_{V\in\mathfrak{A}_n/\mathfrak{P}^*}\zeta_{n_1,n_2}(T_1,\rho_{N_1})\prod_{j=1}^{\gamma-2}|S[V_j]|^{-\rho_{N_{j+1}}},$$

where $V=(V_j^{(n,M_j)}*)$ and $T_1=S[V_1]$.

Multiplying by the factor in Theorem 1.2 we obtain:

$$\begin{aligned}2\pi^{\frac{1}{4}(n_1^2-n_1)-n_1\rho_{N_1}}&\left\{\prod_{i=1}^{n_1}\left(\rho_{N_1}+\frac{i-1-n}{2}\right)\left(\rho_{N_1}-\frac{i-1}{2}\right)\right\}\phi\left(\rho_{N_1}-\frac{i-1}{2}\right)\zeta(S,\rho)\\&=\sum_{V\in\mathfrak{U}_n/\mathfrak{P}^*}J(T_1,\rho_{N_i})\prod_{j=1}^{\gamma-2}|S[V_j]|^{-\rho_{N_{j+1}}},\\&(\zeta(S,\rho)=\zeta_{n_1,\ldots,n_\gamma}(S,\rho_{N_1},\cdots,\rho_{N_{\gamma-1}})).\end{aligned}$$

The theorem will follow from the case $\gamma=2$ provided that we can show the convergence of this representation in a domain like $n>Re\,\rho_{N_1}>-1$, $Re\,\rho_{N_j}>L$, $j=2,\cdots,\gamma-1$, for some sufficiently large L.

First one can show that there is a positive constant c, depending only on S, ρ, and n, such that

$$\begin{aligned}&\sum_{V\in\mathfrak{U}/\mathfrak{P}^*}|J(S[V_1],\rho_{N_1})\prod_{j=1}^{\gamma-2}|S[V_j]|^{-\rho_{N_{j+1}}}|\\ &<c\sum_{V\in\mathfrak{U}/\mathfrak{P}^*}|J(I[V_1],\ \rho_{N_1})\prod_{j=1}^{\gamma-2}|I[V_j]|^{-\rho_{N_{j+1}}}|.\end{aligned}\tag{20}$$

To prove this inequality, use the following facts.

a) If $W^{(n)}=(w_{ij})$ is reduced and $W_0=\begin{pmatrix} w_1 & & 0 \\ & \ddots & \\ 0 & & w_n \end{pmatrix}$, where $w_{ii}=w_i$, then there is a positive constant c, depending only on n, such that for all vectors x

$$\frac{1}{c}W_0[x]\leqslant W[x]\leqslant c\,W_0[x] \quad \text{and} \quad \frac{1}{c}W_0^{-1}[x]\leqslant W^{-1}[x]\leqslant cW_0^{-1}[x].$$

b) By the proof of Theorem 1.2, $J(S[V_1], \rho_{N_1})$ is a sum of 2 integrals. And $(S[V_1])^{\pm 1}$ comes into these integrals by means of the trace and a determinant.

c) Recall that $\sigma(W[Y])=\sum_{k=1}^{n}W[y_k]$, where $Y=(y_1y_2\cdots y_n)$. And $W=S[V]$, where $V=(v_1v_2\cdots v_n)$, implies that $w_i=S[v_i]$.

d) If W is reduced there exists a positive constant c depending only on n such that $|W|\geqslant cw_1w_2\cdots w_n$. And if W is positive then $|W|\leqslant w_1w_2\cdots w_n$.

e) To apply the previous, one assumes in the sums on the right hand side of (20) that $S[V_j]$, $j=2,3,\cdots,\gamma-2$ are reduced and that $S[V_1]$ or $(S[V_1])^{-1}$ is reduced. In the sums on the left hand side one assumes that $I[V_1]$ or $(I[V_1])^{-1}$ is reduced.

These facts and a little computation suffice to prove (20).

To show convergence we may assume $S=I$, the identity. Then $T=T_1=I[V_1]$ is integral. From the proof of Theorem 1.2 it follows that for $n>Re\,\rho>-1$,

$$|J(T,\rho)|\leqslant g(n)\int_{F_{n_1}}|X|^n\theta_{n_1}^T d\mu+|g(n)|\,|T|^{-\frac{1}{2}n_1}\int_{F_{n_1}}|X|^n\theta_{n_1}^{T^{-1}}d\mu.$$

If T is a matrix with integer entries, the first integral is less than a bound independent of T. For $\int_{F_{n_1}}|X|^n\theta_{n_1}^T d\mu\leq\xi(n)Z_{n_1}(T,n)$, where $\xi(n)$ is a product of Γ-functions, etc., and $\xi(n($ is independent of T. We may assume T to be Minkowski-reduced since $Z_{n_1}(T[U],n)=Z_{n_1}(T,n)$ for $U\in\mathfrak{U}_{n_1+n_2}$. From Koecher [3], page 7, we have $Z_{n_1}(T,n)\leq cZ_{1n_1+n_2-1}(T,n)^{n_1}$. And $Z_1(T,n)=\frac{1}{2}\sum_{m\neq 0}T[m]^{-n}\leq c^n\sum T_0[m]^{-n}\leq c^n t_1^{-n}Z_1(I,n)\leq c^nZ_1(I,n)$, a bound B independent of T, since $t_1\geq 1$. (Here T_0 is the diagonal matrix with the same entries as the diagonal entries of T and $T = \begin{pmatrix} t_1 & & * \\ & t_2 & \\ & & \ddots \\ * & & & t_n \end{pmatrix}$.)

If T is integral, so is adjoint $T = \operatorname{adj} T$. Thus we are able to use the same estimate for the second integral in the formula for $J(T, \rho_{N_1})$. We have

$$\int_{F_{n_1}} |X|^n \theta_{n_1}^{T^{-1}} d\mu = \int_{F_{n_1}} |X|^n \theta_{n_1}^{|T|^{-1}\operatorname{adj} T} d\mu$$

$$= |T|^n \int_{F_{n_1}} |X|^n \theta_{n_1}^{\operatorname{adj} T} d\mu \leq |T|^n B, \text{ where } B \text{ is a bound independent of } T.$$

Therefore when $n > Re\, \rho_{N_1} > -1$, there is a positive constant B independent of T such that

$$|J(T, \rho_{N_1})| \leq |T|^n B.$$

It follows that for $n > Re\, \rho_{N_1} > -1$:

$$\sum_{V \in \mathfrak{U}_n/\mathfrak{P}^*} |J(T, \rho_{N_1}) \prod_{j=1}^{r-2} |I[V_j]|^{-\rho_{N_{j+1}}}$$

$$\leq B \sum_{V \in \mathfrak{U}_n/\mathfrak{P}^*} |I[V_1]|^n \prod_{j=1}^{r-2} |I[V_j]|^{-Re\, \rho_{N_{j+1}}},$$

$$= B \zeta_{n_1+n_2, n_3, \ldots, n_{r-1}}(I, -n + Re\, \rho_{N_2}, Re \rho_{N_3}, \cdots, Re\, \rho_{N_{r-1}}),$$

which converges for $n > Re\, \rho_{N_1} > -1$ and $Re\, \rho_{N_j} > \frac{3n}{2}$, $j = 2, \cdots, r-1$. This finishes the proof.

§4. Relations Between Koecher's Zeta Functions.

There are two methods of obtaining relations between $\zeta_{i,n-i}$ and $\zeta_{n-i,i}$. First we use (4) and the case $r = n$, $\lambda = 2$ of (8). Then we use (5) and the same case of (8).

THEOREM 4.1. *Let* $F_n(\rho) = |S|^{-\rho_{n-1}+\frac{1}{2}} \pi^{2\sum_{j=1}^{n-1} j\rho_j - \frac{(n-1)n}{2}} \times$

$$\prod_{i=1}^{n-1} \frac{\phi\left(1 - \rho_{n-1} - \cdots - \rho_i + \frac{n-1-i}{2}\right)}{\phi\left(\rho_{n-1} + \cdots + \rho_i - \frac{n-1-i}{2}\right)}.$$ *Then* $\zeta_{i,n-i}(S, \rho) =$

$\prod_{j=0}^{n-i-1} F_n(\rho^j)|_{\rho_j=0, j\neq i}\zeta_{n-i,i}\left(S, \frac{n}{2}-\rho_i\right)$. *Here* $\rho^\sigma = \left(\frac{n}{2}-\sum_{i=1}^{n-1}\rho_i, \rho_1, \rho_2, \cdots, \rho_{n-2}\right)$ *and* $\phi(x) = \zeta(2x)\Gamma(x)$.

Proof. If we apply the functional equations (4) in the order $n-1$, $n-2, \cdots, 2, 1$, or if we use the invariance of formula (19) in §2 under $\sigma = (n\ n-1)\cdots(32)(21) = \begin{pmatrix} 1 & 2 & 3\cdots n-1 & n \\ n & 1 & 2\cdots n-2 & n-1 \end{pmatrix}$ we obtain:

$$\pi^{-\sum_{j=1}^{n-1} j\rho_j + \frac{(n-1)(n-2)}{4}} \prod_{i=1}^{n-1} \phi\left(\rho_{n-1} + \cdots + \rho_i - \frac{n-1-i}{2}\right)\zeta_{(n)}(\rho)$$

$$= |S|^{\frac{1}{2}-\rho_{n-1}} \pi^{\sum_{j=1}^{n-1} j\rho_j - \frac{1}{4}(n+2)(n-1)} \prod_{i=1}^{n-1} \phi\left(1-\rho_{n-1} - \cdots - \rho_i + \frac{n-1-i}{2}\right) \times$$

$$\zeta_{(n)}\left(\frac{n}{2} - \sum_{i=1}^{n-1}\rho_i, \rho_1, \cdots, \rho_{n-2}\right).$$

Therefore $\zeta_{(n)}(\rho) = F_n(\rho)\zeta_{(n)}(\rho^\sigma)$. And $\zeta_{(n)}(\rho) = \prod_{j=0}^{i-1} F(\rho^{\sigma^j})\zeta_{(n)}(\rho^{\sigma^i})$. It is clear that $\rho^{\sigma^j} = (\rho_{n-(i-1)}, \cdots, \rho_{n-1}, \frac{n}{2} - \sum_{j=1}^{n-1}\rho_j, \rho_1, \rho_2, \cdots, \rho_{n-(i+1)})$, where ρ_{n-i} is omitted. The result follows easily using (8). A small computation convinces one that $\prod_{j=1}^{n-i-1} F(\rho^{\sigma^j})|_{\rho_k=0, k\neq i}$ makes sense.

THEOREM 4.2. $\zeta_{i,n-i}(S^{-1}, \rho) = |S|^\rho \zeta_{n-i,i}(S, \rho)$.

Proof. Use (8) and (5).

Theorems 4.1 and 4.2 combine to give once again the functional equation of Koecher's zeta function.

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University of Illinois
Urbana, Illinois