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ON HOLOMORPHIC EXTENSION FROM THE BOUNDARY

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0. Introduction

Let D be a bounded domain of the complex n-space $C^n (n \geq 2)$, or more generally a pair (M, D) a finite manifold (c.f. Definition 2.1), and we assume the boundary ∂D is a smooth and connected submanifold. It is well known by Hartogs-Osgood's theorem that every holomorphic function on a neighbourhood of ∂D can be continued holomorphically to D. Generalizing the above theorem we shall prove that if a differentiable function on ∂D satisfies certain conditions which are satisfied for the trace of a holomorphic function on a neighbourhood of ∂D, then it can be continued holomorphically to D (Theorem 2–5). The above conditions will be called the tangential Cauchy Riemann equations.

Using the above result, we shall determine the condition for a diffeomorphism of ∂D to be continued to a holomorphic automorphism of D (Theorem 3–3). Finally as its corollary the analogy to functions holds for cross-sections of a holomorphic vector bundle. (Theorem 3–5)

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1. Tangential Cauchy-Riemann equations

Let N be an n-dimensional complex manifold. From now on we always assume $n \geq 2$. Let M be a real smooth submanifold of N. We denote by $T_p(M)$ the real tangent space of M at p. Let J be the complex structure of N.

$$C_p = T_p(M) \cap JT_p(M)$$

is the maximum complex subspace of $T_p(M)$, and we denote its complex dimension by $m(p)$ and we assume $m(p)$ is constant on M.

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Then $T_p(M) \otimes \boldsymbol{C}$ is decomposed to

$$T_p(M) \otimes \boldsymbol{C} = H_p + \bar{H}_p + L_p \quad \text{(direct sum)}$$

where $\quad H_p = \{X \in T_p(M) \otimes \boldsymbol{C};\ X \text{ is a } \sqrt{-1} \text{ eigen vector of } J\}$

$$\bar{H}_p = \{X \in T_p(M) \otimes \boldsymbol{C};\ X \text{ is a } -\sqrt{-1} \text{ eigen vector of } J\},$$

and L_p is a complemental subspace of $H_p + \bar{H}_p$. We call an element of H_p, $\bar{H}_p$, holomorphic and anti-holomorphic tangent vector respectively. It is evident that $(\overline{H_p}) = \bar{H}_p$, where the upper bar means complex conjugate with respect to $T_p(M)$, and that $\dim_{\boldsymbol{C}} H_p = \dim_{\boldsymbol{C}} \bar{H}_p = m(p)$. Now we define

DEFINITION 1-1. Let f be a complex valued differentiable function defined on a neighbourhood of $p \in M$. If $Xf = 0$ for every $X \in \bar{H}_p$, we call that f satisfies *the tangential Cauchy-Riemann equations* at p.

If f satisfies the tangential Cauchy-Riemann equations at every point of the domain of f, we call f satisfies the tangential Cauchy-Riemann equations (in short, $T-C-R$ equations).

In the following we consider only the case when M is a real hypersurface of N. In this case we define

DEFINITION 1-2. Let M be a real hypersurface of N. We call a real valued differentiable function φ a *defining function of* M if it satisfies the following conditions.

1). $M = \{z \in N;\ \varphi(z) = 0\}$

2). grad φ does not vanish on M.

Let φ be a defining function of M and p_0 a point of M. Let $(z_1, \cdots, z_n)$ be a local coordinate system at p_0. Since grad φ does not vanish on M, then we can assume $\varphi_{\bar{z}_n} := \dfrac{\partial \varphi}{\partial \bar{z}_n}$ does not vanish on some neighbourhood U of p_0. We can choose a base of H_p, $\bar{H}_p$, and L_p at $p \in U$ as following

$$H_p: \begin{cases} (X_1)_p = (\varphi_{z_n})_p \left(\dfrac{\partial}{\partial z_1}\right)_p - (\varphi_{z_1})_p \left(\dfrac{\partial}{\partial z_n}\right)_p \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ (X_{n-1})_p = (\varphi_{z_n})_p \left(\dfrac{\partial}{\partial z_{n-1}}\right)_p - (\varphi_{z_{n-1}})_p \left(\dfrac{\partial}{\partial z_n}\right)_p \end{cases}$$

$$\bar{H}_p: \begin{cases} (\bar{X}_1)_p = (\varphi_{\bar{z}_n})_p \left(\frac{\partial}{\partial \bar{z}_1}\right)_p - (\varphi_{\bar{z}_1})_p \left(\frac{\partial}{\partial \bar{z}_n}\right)_p \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ (\bar{X}_{n-1})_p = (\varphi_{\bar{z}_n})_p \left(\frac{\partial}{\partial \bar{z}_{n-1}}\right)_p - (\varphi_{\bar{z}_{n-1}})_p \left(\frac{\partial}{\partial \bar{z}_n}\right)_p \end{cases}$$

$$L_p: \qquad Y_p = (\varphi_{\bar{z}_n})_p \left(\frac{\partial}{\partial z_n}\right)_p - (\varphi_{z_n})_p \left(\frac{\partial}{\partial \bar{z}_n}\right)_p$$

It means $H = \bigcup_{p \in M} H_p$, $\bar{H} = \bigcup_{p \in M} \bar{H}_p$ are subbundles of $T(M) \otimes \boldsymbol{C}$.

2. Holomorphic extension of functions.

Let M be a Stein manifold and D be a domain of M. Now we introduce the following definition.

DEFINITION 2-1. A pair (M, D) is called a *finite manifold*, if the following conditions are satisfied.

0). M is a Stein manifold and $\dim M \geq 2$

1). D is a connected relatively compact domain of M.

2). the boundary of D, denoted by ∂D, is a connected smooth real hypersurface of M.

Let (M, D) be a finite manifold. We use the following notations.

$$C^\infty(\bar{D}) = \{\text{a differentiable function on } \bar{D}\}$$

$$H(\bar{D}) = \{f \in C^\infty(\bar{D});\ f|_D \text{ is a holomorphic function}\}$$

where $f|_D$ is the restriction of f to D.

We choose a defining function φ of ∂D such that

$$D = \{z; \varphi(z) < 0\} \text{ and } M - D = \{z;\ \varphi(z) \geq 0\}$$

Since φ is a defining function, grad φ does not vanish on ∂D.

It is convenient to express the $T-C-R$ equations in another way. Let f be a differentiable function on ∂D. There exists $F \in C^\infty(\bar{D})$ so that $F|\partial D = f$.

LEMMA 2-2. *A differentiable function f on ∂D satisfies $T-C-R$ equations if and only if $\bar{\partial} F \wedge \bar{\partial} \varphi = 0$ on ∂D, where F is a differentiable function on $\bar{D}$ as above and $\bar{\partial}$ is the Cauchy-Riemann operator.*

Proof is clear from Definition 1-1.

LEMMA 2-3. (Hörmander [1] p. 137) *Let M be a Stein manifold and α a* $(0,1)$ *type 1-form of class C^k. If $\bar\partial\alpha = 0$, there exists a $k-n$ time differentiable function u such that $\bar\partial u = \alpha$.*

We shall prove the following corollary, using the above lemma.

COROLLARY 2-4. *Let α be a $(0,1)$ type 1-form of class C^k on a Stein manifold M. If $\bar\partial\alpha = 0$ and $K = supp\ \alpha$ is compact and $M-K$ is connected, there exists $k-n$ time differentiable function u so that $\bar\partial u = \alpha$ and supp $u \subset K$.*

Proof. There exists a $k-n$ time differentiable function v such that $\bar\partial v = \alpha$ by lemma 2-3. Since $\bar\partial v = 0$ on $M-K$, v is holomorphic on $M-K$. By Hartogs-Osgood's theorem (Kasahara [2]) $v|_{M-K}$ can be continued to a holomorphic function w on M. We put $u=v-w$, it follows that $\bar\partial u=\bar\partial v-\bar\partial w = \bar\partial v = \alpha$, and supp $u\subset K$. Q.E.D.

We shall prove the following theorem by the method of Hörmander [1].

THEOREM 2-5. *Let (M, D) be a finite manifold, and f a differentiable function on ∂D. If f satisfies $T-C-R$ equations, there exists $\tilde f \in H(\bar D)$ such that $\tilde f|\partial D = f$.*

Proof. (1-st step) We construct by induction a differentiable function $U_k \in C^\infty(\bar D)$ for every positive integer k which satisfies the following conditions;

(2-1) $$U_k|_{\partial D} = f \text{ and } \bar\partial U_k = 0(\varphi^k).$$

We extend f to a function on $\bar D$ as an element of $C^\infty(\bar D)$, and we denote it by f also. By lemma 2-2 $\bar\partial f \wedge \bar\partial\varphi = 0$ on ∂D. Then we can decompose $\bar\partial f$ as

$$\bar\partial f = h_1\bar\partial\varphi + \varphi h_2$$

where $h_1 \in C^\infty(\bar D)$ and h_2 is a differentiable $(0,1)$ type 1-form. We write it by $h_2 \in C^\infty_{(0,1)}(\bar D)$ in the following.

By simple calculation we have

$$\begin{aligned}\bar\partial(f - h_1\varphi) &= \bar\partial f - (\bar\partial h_1)\varphi - h_1\bar\partial\varphi \\ &= \varphi h_2 - (\bar\partial h_1)\varphi \\ &= \varphi(h_2 - \bar\partial h_1).\end{aligned}$$

Put $U_1 := f - \varphi h_1$, then $U_1|_{\partial D} = f$ and $\bar\partial U_1 = 0(\varphi)$. We have thus constructed U_1.

Now we assume that U_{k-1} is constructed, i.e.

$$U_{k-1}|_{\partial D} = f, \quad \bar{\partial} U_{k-1} = 0(\varphi^{k-1}).$$

Then we can write $\bar{\partial} U_{k-1} = \varphi^{k-1} h$, $h \in C^{\infty}_{(0,1)}(\bar{D})$. Then

$$\begin{aligned} \bar{\partial}\bar{\partial} U_{k-1} &= 0 \\ &= (k-1)\varphi^{k-2}\bar{\partial}\varphi \wedge h + \varphi^{k-1}\bar{\partial} h \\ &= \varphi^{k-2}((k-1)\bar{\partial}\varphi \wedge h + \varphi \cdot \bar{\partial} h) \end{aligned}$$

Hence $(k-1)\bar{\partial}\varphi \wedge h + \varphi\bar{\partial} h = 0$. However $\varphi\bar{\partial} h$ vanishes on ∂D, so that h must satisfies $\bar{\partial}\varphi \wedge h = 0$ on ∂D.

This imples that $h = \bar{\partial}\varphi \wedge h_{2k-1} + \varphi h_{2k}$, where $h_{2k-1} \in C^{\infty}(\bar{D})$, $h_{2k} \in C^{\infty}_{(0,1)}(\bar{D})$. Put $U_k := U_{k-1} - \left(\frac{1}{k} \cdot \varphi^k\right) h_{2k-1}$. We see that the function U_k satisfies the condition $(2-1)$, because

$$\begin{aligned} \bar{\partial} U_k &= \bar{\partial} U_{k-1} - (\varphi^{k-1}\bar{\partial}\varphi) h_{2k-1} - \left(\frac{1}{k} \cdot \varphi^k\right)\bar{\partial} h_{2k-1} \\ &= \varphi^k \cdot h_{2k} - \left(\frac{1}{k} \cdot \varphi^k\right)\bar{\partial} h_{2k-1} \\ &= \varphi^k\left(h_{2k} - \frac{1}{k}\bar{\partial} h_{2k-1}\right). \end{aligned}$$

(2-nd step) Let $k \geq n+2$. We define $v_k \in C^k_{(0,0)}(M)$ with

$$v_k|_{\bar{D}} = \bar{\partial} U_k \quad \text{and} \quad v_k|_{M-\bar{D}} = 0.$$

Note that supp. $v_k \subset \bar{D}$. By corollary 2–4 there exists $w_k \in C^{k-1-n}(M)$ which satisfies $\bar{\partial} w_k = v_k$ and supp. $w_k \subset \bar{D}$. Put $f_k = U_k - w_k$. Then we have $f_k \in C^{(k-1-n)}(\bar{D})$, $f_k|_{\partial D} = f$ and $\bar{\partial} f_k = \bar{\partial} U_k - \bar{\partial} U_k - \bar{\partial} w_k = 0$. Thus f_k is holomorphic on D and its boundary value is f. Then by the uniqueness of continuation

$$f_k = f_{k+1} = f_{k+2} = \cdots$$

We put $\tilde{f} = f_k = f_{k+1} = f_{k+2} = \cdots$, it is the desired one. Q.E.D.

3. Holomorphic extension of mappings

Let M be a complex manifold and S a real hypersurface of M. As we saw in §1, $T_p(S) \otimes \boldsymbol{C}$ is decomposed at $p \in S$ as follows:

$$T_p(S) \otimes \boldsymbol{C} = H_p + \bar{H}_p + L_p \quad \text{(direct sum)}$$

where H_p, $\bar{H}_p$, are holomorphic and anti-holomorphic tangent space at p, respectively. Here we define the tangential Cauchy-Riemann equations for mapping.

DEFINITION 3-1. Let M, M', be complex manifolds and S, S' real hypersurfaces of M, M', respectively. Let μ be a differentiable mapping from S to S'. The following conditions 1), 1′), 2), 3) are equivalent. If μ satisfies one of the conditions, we say that μ satisfies *the tangential Cauchy-Riemann equations* (in short, $T-C-R$ equations).

1). $\mu_*(H_p(S))\subset H_{\mu(p)}(S')$ for every point $p\in S$

1)′. $\mu_*(\bar{H}_p(S))\subset \bar{H}_{\mu(p)}(S')$ for every point $p\in S$

2). a differentiable function f on an open set of S' satisfies $T-C-R$ equations, then μ^*f satisfies $T-C-R$ equations on its domain.

3). Let $(z_1', \cdots, z_m')$ be a local coordinate system at $q=\mu(p)$ of M. Then $f_i:=\mu^* z_i'$: $(i=1, \cdots, m)$ satisfies $T-C-R$ equations.

We shall prove that four conditions of definition are equivalent.

1) $\Longrightarrow$ 1′). We choose a local coordinate system $(z_1, \cdots, z_n)$ of M at p as follows.

$$H_p=\left\{\left(\frac{\partial}{\partial z_1}\right)_p, \cdots, \left(\frac{\partial}{\partial z_{n-1}}\right)\right\}, \quad \bar{H}_p=\left\{\left(\frac{\partial}{\partial \bar{z}_1}\right)_p, \cdots, \left(\frac{\partial}{\partial \bar{z}_{n-1}}\right)_p\right\}$$

Take some local coordinate system $(z_1', \cdots, z_m')$ of M' at $q=\mu(p)$ and put $f_i=\mu^* z_i'$, then

$$\mu_*\left(\frac{\partial}{\partial z_i}\right)_p=\sum_j\left(\frac{\partial f_j}{\partial z_i}\right)_p\left(\frac{\partial}{\partial z_j'}\right)_{\mu(p)}+\sum_j\left(\frac{\partial \bar{f}_j}{\partial z_i}\right)_p\left(\frac{\partial}{\partial \bar{z}_j'}\right)_{\mu(p)} \qquad i=1, \cdots, n$$

But from the condition 1) $\mu_*\left(\frac{\partial}{\partial z_i}\right)\in H(S')$, so that

$$\left(\frac{\partial \bar{f}_j}{\partial z_i}\right)_p=\left(\overline{\frac{\partial f_j}{\partial \bar{z}_i}}\right)_p=0 \qquad j=1, \cdots, m$$

Hence it follows that

$$\begin{aligned}\mu_*\left(\frac{\partial}{\partial \bar{z}_i}\right)_p &=\sum_{j=1}^m\left(\frac{\partial f_j}{\partial \bar{z}_i}\right)_p\left(\frac{\partial}{\partial z_j'}\right)_{\mu(p)}+\sum_{j=1}^m\left(\frac{\partial \bar{f}_j}{\partial \bar{z}_i}\right)_p\left(\frac{\partial}{\partial \bar{z}_j'}\right)_{\mu(p)}\\ &=\sum_{j=1}^m\left(\frac{\partial \bar{f}_j}{\partial \bar{z}_i}\right)_p\left(\frac{\partial}{\partial \bar{z}_j'}\right)_{\mu(p)}\in \bar{H}_{\mu(p)}(S')\end{aligned}$$

1′) $\Longrightarrow$ 1) is now obvious.

$2) \Longrightarrow 3)$. Since $(z_1', \cdots, z_m')$ is a local coordinate of M' at $\mu(p) = q$, it is trivial that z_i' satisfies $T-C-R$ equations. By condition 2), $f_i = \mu_*(z_i')$ satisfies $T-C-R$ equations.

$1) \Longrightarrow 2)$. Let g be a differentiable function defined on a neighbourhood (in S') of $q = \mu(p)$ which satisfies $T-C-R$ equations. Let X be any element of $\bar{H}_p(S)$. By 1') $\mu_* X \in \bar{H}_{\mu(p)}(S')$, and $X(\mu^* g) = (\mu_* X)g = 0$. Thus g satisfies $T-C-R$ equations.

$3) \Longrightarrow 1)$. We choose a local coordinate system at p as above. We also have

$$\mu_*\left(\frac{\partial}{\partial z_i}\right)_p = \sum_{j=1}^{m}\left(\frac{\partial f_j}{\partial z_i}\right)_p\left(\frac{\partial}{\partial z_i'}\right)_{\mu(p)} + \left(\frac{\partial \bar{f}_j}{\partial z_i}\right)_p\left(\frac{\partial}{\partial \bar{z}_j}\right)_{\mu(p)} \quad (1 \leq i \leq n-1)$$

Since f satisfies $T-C-R$ equations, we have $\left(\frac{\partial f_j}{\partial \bar{z}_i}\right)_p = \overline{\left(\frac{\partial \bar{f}_j}{\partial z_i}\right)} = 0$. Then $\mu_*\left(\frac{\partial}{\partial z_i}\right)_p \in H_{\mu(p)}(S')$. This means $\mu_*(H_p(S)) \subset H_{\mu(p)}(S')$.

LEMMA 3-2. *Let M be a complex manifold and S be a real hypersurface of M. The set of all diffeomorphisms of S which satisfies $T-C-R$ equations is a group.*

Proof is clear by the condition 1) of Definition 3-1. But I don't know the group of lemma 3-2 is a Lie group or not.

Let (M, D) be a finite manifold. We introduce the following notations.

Let $\mathrm{Diff}(\bar{D})$ be the group of all C^∞-diffeomorphisms of $\bar{D}$, and

$$\mathrm{Aut}(\bar{D}) = \{\mu \in \mathrm{Diff}(\bar{D});\ \mu|_D \text{ is a holomorphic automorphism of } D\}$$

Now we shall prove the following

THEOREM 3-3. *If a diffeomorphism $\mu: \partial D \to \partial D$ satisfies $T-C-R$ equations, there exists $\tilde{\mu} \in \mathrm{Aut}(\bar{D})$ such that $\tilde{\mu}|_{\partial D} = \mu$.*

Proof. Let p be any point of ∂D. Since M is a Stein manifold, there is a local coordinate system $(f_1, \cdots, f_n)$ of M at $q = \mu(p)$, where $f_1, \cdots, f_n$ are holomorphic functions on M. By definition 3-1 $\mu^* f_i$ satisfies $T-C-R$ equations. Then by theorem 2-5 there exist $\tilde{f}_i \in H(\bar{D})$ such that $\tilde{f}_i|_{\partial D} = \mu^* f_i$. We take a sufficiently small neighbourhood U_p of p, and define the mapping μ_{U_p}: $U_p \cap \bar{D} \to M$, using the above local coordinate system $(f_1, \cdots, f_n)$ at q, by

$$\mu_{U_p}(p') = (\tilde{f}_1(p'), \cdots, \tilde{f}_n(p)), \quad p' \in U_p \cap \bar{D}$$

By the uniqueness of the holomorphic continuation of functions, there exist a small neighbourhood U of ∂D, so that $U \cap \bar{D}$ is connected, and there exists a holomorphic mapping

$$\mu_U: \bar{D} \cap U \to M \text{ with } \mu_U|_{U_p \cap \bar{D}} = \mu_{U_p}$$

Since $D - D \cap U$ is compact, there exists a holomorphic mapping $\mu: D \to M$ so that $\tilde{\mu}|_{D \cap U} = \tilde{\mu}_U$ by Hartogs-Osgood's theorem (K. Kasahara [2]). We shall prove that the mapping $\tilde{\mu}$ is the desired one.

By the construction of $\tilde{\mu}$, $\tilde{\mu}$ is holomorphic on D and $\tilde{\mu}|_{\partial D} = \mu$. First we shall prove the rank of $\tilde{\mu}$ is $2n$ at each point of a neighbourhood of ∂D in $\bar{D}$. Here we may assume that there exist real vector fields $X_1, \cdots, X_n$, $JX_1, \cdots, JX_{n-1}$ on a small neighbourhood V_{p_0} of p_0 in ∂D, such that they form a base of $T_p(\partial D)$ at every point p of V_{p_0}. We can construct them taking real parts of the base of H and a real vector contained in L given in § 1.

We extend $X_1, \cdots, X_n$ to a neighbourhood W_{p_0} of V_{p_0} and we denote them $\tilde{X}_1, \cdots, \tilde{X}_n$ and we can assume $\tilde{X}_1, \cdots, \tilde{X}_n, J\tilde{X}_1, \cdots, J\tilde{X}_{n-1}$ are linearly independent at each point of W_{p_0}, taking W_{p_0} sufficiently small. Since μ is a diffeomorphisms, $\mu_*(X_1)$, $\mu_*(X_2)$, $\cdots$, $\mu_*(X_n)$, $\mu_*(JX_1)$, $\cdots$, $\mu_*(JX_{n-1})$ are linearly independent at each point of $\mu(V_p)$, and hence $\tilde{\mu}_*(\tilde{X}_1), \cdots, \tilde{\mu}_*(\tilde{X}_n)$, $\tilde{\mu}_*(J\tilde{X}_1), \cdots, \tilde{\mu}_*(J\tilde{X}_{n-1})$ are independent at every point of $\tilde{\mu}(W_p \cap D)$, changing W_{p_0} smaller if necessary. Since $\tilde{\mu}$ is holomorphic on D,

$$\tilde{\mu}_*(J\tilde{X}_i) = J\tilde{\mu}_*(\tilde{X}_i), \qquad 1 \leq i \leq n$$

Then $\tilde{\mu}_*(\tilde{X}_1)$, $\tilde{\mu}_*(\tilde{X}_2)$, $\cdots$, $\tilde{\mu}_*(\tilde{X}_n)$, $\tilde{\mu}_*(J\tilde{X}_1)$, $\cdots$, $\tilde{\mu}_*(J\tilde{X}_n)$ are independent at $\tilde{\mu}(W_p \cap D)$. It means the rank of $\tilde{\mu}$ is $2n$ on $W_{p_0} \cap D$. Since p_0 is an arbitrary point of ∂D, there exists a neighbourhood W of ∂D such taht rank of $\tilde{\mu}$ is $2n$ on $W \cap D$. Hence the set of all ponits of D where rank of $\tilde{\mu}$ is smaller than $2n$ is a compact analytic set of dimension $n-1 \geq 1$ of M. Since M is a Stein manifold, there is no compact analytic set of dimension $n-1 \geq 1$ of M. Then rank $\tilde{\mu}$ is $2n$ at each point of D. Hence $\tilde{\mu}$ is a local diffeomorphism on D.

Next we see that $\tilde{\mu}(\bar{D}) \subset \bar{D}$. In fact, if $\tilde{\mu}(\bar{D}) \not\subset \bar{D}$, there is a boundary point q of $\tilde{\mu}(\bar{D})$ such that $q = \tilde{\mu}(p) \notin \bar{D}$. Since $\tilde{\mu}(\partial D) = \partial D$, we have $p \in D$. This contradicts to the fact $\tilde{\mu}$ is a local diffeomorphism at p.

Since μ^{-1} also satisfies $T-C-R$ equations by Lemma 3-2, theer is $\widetilde{(\mu^{-1})}$ such that $\widetilde{(\mu^{-1})}|_D$ is holomorphic and $\widetilde{(\mu^{-1})}|_{\partial D} = \mu^{-1}$. Since $\tilde{\mu}(\bar{D}) \subset \bar{D}$ and

$(\widetilde{\mu^{-1}})(\bar{D})\subset\bar{D}$, we have $(\tilde{\mu})(\widetilde{\mu^{-1}}) = id = \widetilde{id}$, and $(\widetilde{\mu^{-1}})(\tilde{\mu}) = id = \widetilde{id}$. This means that $\tilde{\mu}$ is a holomorphic automorphism of D. Q.E.D.

By the proof of the above theorem, we conclude the following theorem.

THEOREM 3-4. *Let (M, D) be a finite manifold, N a Stein manifold and S a real hypersurface of N. If a mapping $\mu: \partial D\rightarrow S$ satisfies $T-C-R$ equations, there exists a differentiable mapping $\tilde{\mu}: \bar{D}\rightarrow N$ such that $\tilde{\mu}|_{\partial D} = \mu$ and $\tilde{\mu}|_D$ is holomorphic.*

In the above theorem the conditoin that S is a real hypersurface can be changed to that $\mu: \partial D\rightarrow N$ satisfies the condition 1) of Definition 3-1.

By using the above theorem, we consider the holomorphic extension of a differentiable cross-section of a holomorphic fibre bundle.

Let (M, D) be a finite manifold and E a holomorphic fibre bundle over M. If a differentiable cross-section s over ∂D satisfies $T-C-R$ equations as a mapping $s: \partial D\rightarrow E$, we call s satisfies the tangential Cauchy-Reimann equations, (in short, $T-C-R$ equations).

THEOREM 3-5. *If a differentiable cross-section s over ∂D of a holomorphic fibre bundle whose fibre is a Stein manifold, satisfies $T-C-R$ equations, there exists a differentiable cross-section $\tilde{s}$ over $\bar{D}$ such that $\tilde{s}|_{\partial D} = s$ and $\tilde{s}|_D$ is a holomorphic cross-section.*

Proof. Since M and the fibre of E are Stein manifolds, E is also a Stein manifold by the theorem of Matsushima-Morimoto [3]. Since cross-section s satisfies $T-C-R$ equations, there exists a mapping $\tilde{s}: D\rightarrow E$ such that $\tilde{s}|_{\partial D} = s$ and $\tilde{s}|_D$ is holomprohic by Theorem 3-4.

Then it sufficies to prove $\tilde{s}$ is a cross-section i.e. $\pi\tilde{s} = id$ where π is the projection from E to M.

$\tilde{f} = (\pi\tilde{s})^*f$ is a holomorphic function for every $f\in H(\bar{D})$. It is clear that $\tilde{f}|_{\partial D} = f$ implies $\tilde{f} = (\pi\tilde{s})^*f = f$ on D. By considering coordinate functions, it means $\pi\tilde{s} = id$.

Remark 3.6. If E is a holomorphic vector bundle, E is a Stien manifold since vector space over $\boldsymbol{C}$ is a Stein manifold. In this case if a differentiable cross-section s over ∂D satisfies $T-C-R$ equations, by the local expression, then it satisfies $T-C-R$ equations as cross-section. Then s can be holomorphically extended to the cross-section over $\bar{D}$ by the above theorem.

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