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THE P-HARMONIC BOUNDARY AND ENERGY-FINITE SOLUTIONS OF $\Delta u = Pu$

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The P-harmonic boundary Δ_P and the P-singular point s of a Riemannian manifold R have been shown to play an important role in the study of bounded energy-finite solutions of $\Delta u = Pu$ (Nakai-Sario [7], Kwon-Sario [4], Kwon-Sario-Schiff [5]). The objective of the present paper is to establish, in terms of Δ_P and s, properties of unbounded energy-finite solutions (PE-functions) and of limits of decreasing sequences of positive PE-functions ($\widetilde{PE}$-functions). Also, PE- and $\widetilde{PE}$-minimal functions will be discussed.

For the convenience of the reader we shall briefly review, in **1**, some preliminaries (for details see Kwon-Sario-Schiff [5]).

1. On a connected, separable, oriented, smooth Riemannian manifold of dimension N, consider the P-algebra $M_P(R)$ of bounded Tonelli functions f with finite energy integrals

$$E_R(f) = D_R(f) + \int_R Pf^2 dV < \infty.$$

Here $D_R(f) = \int_R df \wedge * df$ is the Dirichlet integral of f over R, $P(\not\equiv 0)$ a given nonnegative continuous function on R, and $dV = *1$ the volume element of R. It is known that the P-algebra $M_P(R)$ is closed under the lattice operations $f \cup g = \max(f, g)$ and $f \cap g = \min(f, g)$, and that it is complete in the BE-topology: if $\{f_n\}$ is a uniformly bounded sequence in $M_P(R)$, converges to f uniformly on compact subsets of R, and $E_R(f_n - f_m) \to 0$ as $n, m \to \infty$, then $f \in M_P(R)$.

By means of the P-algebra $M_P(R)$ one constructs the P-compactification R_P^* of R, defined by the following properties: R_P^* is a compact Hausdorff

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space and contains R as an open dense subset; every $f \in M_P(R)$ has a continuous extension to R_P^*; and $M_P(R)$ separates the points of R_P^*.

A point $s \in R_P^*$ is called a P-singular point if $f(s) = 0$ for all $f \in M_P(R)$; it exists, and is unique, if and only if $\int_R P dV = \infty$ (Nakai-Sario [7]). It is known that $p \in R_P^*$ is P-singular if and only if $\int_{R \cap U} P dV = \infty$ for every neighborhood U of p in R_P^* (Kwon-Sario [4]). Points of R_P^* which are not P-singular are called P-regular.

Let $M_{P\Delta}(R)$ be the family of BE-limits of functions of $M_P(R)$ with compact supports. The set

$$\Delta_P = \{x \in R_P^* \mid f(x) = 0 \text{ for all } f \in M_{P\Delta}(R)\}$$

is called the P-harmonic boundary and contains the P-singular point s if the latter exists.

2. Consider the family $\tilde{M}_P(R)$ of Tonelli functions on R with finite energy integrals. It is easily seen that $M_P(R) \subset \tilde{M}_P(R)$ and every $f \in \tilde{M}_P(R)$ has a continuous extension to R_P^*.

We write $f = CE\text{-}\lim_n f_n$ on R if $\{f_n\}$ converges to f uniformly on compact subsets of R, and $E_R(f - f_n) \to 0$ as $n \to \infty$. The family $\tilde{M}_P(R)$ is complete with respect to the CE-topology. In fact, let $\{f_n\}$ be a CE-Cauchy sequence in $\tilde{M}_P(R)$. In view of the CD-completeness of Nakai's lattice $\tilde{M}(R)$ (cf. Sario-Nakai [9], Kwon-Sario [2]), $f = CD\text{-}\lim_n f_n$ exists on R and $\int_R P(f - f_{n_i})^2 dV \to 0$ as $i \to \infty$ for some sequence $\{n_i\}$. Since $\{f_n\}$ is CE-Cauchy, we conclude that $f = CE\text{-}\lim_n f_n$ on R (cf. Kwon-Sario-Schiff [5]).

Let $\tilde{M}_{P\Delta}(R)$ be the subfamily of $\tilde{M}_P(R)$ which consists of the CE-limits of functions in $\tilde{M}_P(R)$ with compact supports.

We close this number with the important decomposition theorem (cf. Nakai-Sario [7]): *every* $f \in \tilde{M}_P(R)$ *has the unique decomposition* $f = u + g$, $u \in PE(R)$, $g \in \tilde{M}_{P\Delta}(R)$. *If* $f \geq 0$, *then* $u \geq 0$, *and* $u \leq f$ *for a* P*-superharmonic* f.

The function u is called the *P-harmonic projection* of f, denoted by $u = \pi(f)$.

For the proof take a regular exhaustion $\{R_n\}$ of R, and construct continuous functions u_n^+ (resp. u_n^-) on R such that $u_n^+ = f^+$ (resp. $u_n^- = f^-$) on $R - R_n$ and $u_n^+ \in P(R_n)$ (resp. $u_n^- \in P(R_n)$). Then $E_R(u_n^+) \leq E_R(f^+) \leq E_R(f)$ and $E_R(u_n^-) \leq E_R(f^-) \leq E_R(f)$.

Since by Fatou's lemma

$$\int_R P \varliminf_{n\to\infty} (u_n^+)^2 dV \leq \varliminf_{n\to\infty} E_R(u_n^+) \leq E_R(f) < \infty,$$

we may assume that

$$u^+ = C\text{-}\lim_{n\to\infty} u_n^+, \quad u^- = C\text{-}\lim_{n\to\infty} u_n^-$$

exist on R. Clearly u^+, u^- are solutions of $\Delta u = Pu$.

By virtue of the energy principle (cf. Royden [8])

$$E_R(u_u^+ - u_{n+p}^+) = E_R(u_n^+) - E_R(u_{n+p}^+) \gtrsim 0$$

and hence $d = \lim_{n\to\infty} E_R(u_n^+)$ exists. On letting $p \to \infty$ we obtain

$$E_R(u_n^+ - u^+) \leq E_R(u_n^+) - d \to 0 \text{ as } n \to \infty.$$

Thus $u^+ = CE\text{-}\lim_n u_n^+$ on R and $u^+ \in PE(R)$. Similarly $u^- = CE\text{-}\lim_n u_n^-$ on R and $u^- \in PE(R)$.

Set $u = u^+ - u^- \in PE(R)$ and $g = f - u$ on R. Since $g = CE\text{-}\lim_n g_n$ on R where $g_n = f - (u_n^+ - u_n^-) = 0$ on $R - R_n$, we have the desired decomposition. The uniqueness follows immediately from the energy principle.

The rest of the proof is obvious.

3. As an application of the above orthogonal decomposition theorem we shall prove

THEOREM. *Every (bounded or unbounded) energy-finite P-harmonic function u on R takes the maximum of its absolute value on the P-harmonic boundary:*

$$|u| \leq \max_{\Delta_P} |u|.$$

Proof. Let $M = \max_{\Delta_P}|u|$. If $M = \infty$, there is nothing to prove; we suppose in the sequel that $M < \infty$. If $\sup_R |u| < \infty$, then $M \pm u$ is a P-superharmonic function on R, bounded from below and nonnegative on Δ_P. Therefore

$$M \pm u \gtrsim 0$$

as desired (cf. Kwon-Sario-Schiff [5]). It remains to show that u is bounded.

Suppose $\sup_R |u| = \infty$. Without loss of generality we may assume that $\sup_R u = \infty$. Since $u^+ = u \cup 0 \in \tilde{M}_P(R)$, the orthogonal decomposition yields

$$u^+ = v + g$$

with $v \in PE(R)$ and $g \in \tilde{M}_{P\Delta}(R)$. Moreover, $v \geq u^+ \geq 0$ by virtue of the P-subharmonicity of u^+. Thus

$$\sup_R v \geq \sup_R u^+ = \infty.$$

On the other hand $v = u^+ \leq |u| \leq M < \infty$ on Δ_P.

For $n > M$, $v \cap n \in M_P(R)$, and we have

$$v \cap n = w + g_n$$

with $w \in PBE(R)$ and $g_n \in M_{P\Delta}(R)$. Note that w is independent of n for $n > M \geq \max_{\Delta_P} v$. It follows that

$$\begin{aligned} E_R(v-w) &= E_R(v - v\cap n + g_n) \\ &= E_R(v - v\cap n) + 2E_R(v - v\cap n, g_n) + E_R(g_n) \\ &= E_R(v - v\cap n) + 2E_R(v - w - g_n, g_n) + E_R(g_n) \\ &= E_R(v - v\cap n) - E_R(g_n) \leq E_R(v - v\cap n) \to 0, \end{aligned}$$

and $v \equiv w \in PBE(R)$. This contradicts $\sup_R v = \infty$.

As a consequence we have the Virtanen identity for P-harmonic functions:

COROLLARY. $O_{PE} = O_{PBE}$.

Proof. Since $PBE \subset PE$, we only have to prove that $O_{PE} \supset O_{PBE}$. Suppose $R \in O_{PBE}$. Then $\Delta_P - s = \phi$ (cf. Kwon-Sario-Schiff [5]). If $\Delta_P = \phi$, the Royden harmonic boundary Δ_R is void and $R \in O_G \subset O_{PE}$. In the case $\Delta_P = \{s\}$ the above theorem yields $|u| \leq \max_{\Delta_P} |u| = 0$ for all $u \in PE(R)$. Thus $R \in O_{PE}$ as desired.

4. For a fixed $x_0 \in R$, let $\mu = \mu_{x_0}$ be the P-harmonic measure on Δ_P with center x_0, and $K(x,t)$ the P-harmonic kernel on $R \times \Delta_P$ with $K(x_0, t) \equiv 1$:

$$\pi(f)(x) = \int_{\Delta_P} f(t) K(x,t) d\mu(t)$$

for all f in the family $B_s(\Delta_P)$ of bounded continuous functions on Δ_P which vanish at the P-singular point s (Kwon-Sario-Schiff [5]). In view of $\pi(f) = f$ on Δ_P for $f \in M_P(R)$ we deduce from the above integral representation that the space $PBE(R)$ is in one-to-one correspondence with $M_P(R)|\Delta_P$ and therefore forms a vector lattice (loc. cit.).

In the case of unbounded energy-finite P-harmonic functions we state (cf. Sario-Nakai [9]):

THEOREM. *Every PE-function u on R has the integral representation along the P-harmonic boundary*

$$u(x) = \int_{\Delta_P} u(t)K(x, t)d\mu(t).$$

Proof. Since every PE-function u is a difference of positive PE-functions, $u = \pi(u \cup 0) - \pi((-u) \cup 0)$, it suffices to consider positive PE-functions.

The function $u \cap n$ is P-superharmonic and belongs to the class $M_P(R)$. Therefore

$$\int_{\Delta_P} (u \cap n)(t)K(x, t)d\mu(t) = \pi(u \cap n)(x) \leq (u \cap n)(x) \leq u(x) < \infty.$$

Set $u_n = \pi(u \cap n) \in PBE(R)$. For $n \geq m$

$$u_n(x) - u_m(x) = \int_{\Delta_P} (u \cap n - u \cap m)(t)K(x, t)d\mu(t) \geq 0.$$

Consequently there exists a P-harmonic function v on R such that $v = C\text{-}\lim_n u_n$ on R.

On the other hand, since $u_n - u_m$ is the P-harmonic projection of $u \cap n - u \cap m \in M_P(R)$,

$$E_R(u_n - u_m) \leq E_R(u \cap n - u \cap m) \to 0 \text{ as } n, m \to \infty.$$

Thus $v = CE\text{-}\lim_n u_n$ on R, $v \in PE(R)$, and $v \equiv u$ on Δ_P. It follows that

$$u(x) = v(x) = \lim_{n\to\infty} \int_{\Delta_P} (u \cap n)(t)K(x, t)d\mu(t)$$

$$= \int_{\Delta_P} u(t)K(x, t)d\mu(t)$$

on R as asserted.

COROLLARY. *Let $u, v \in PE(R)$. The least P-harmonic majorant $u \vee v$ and the greatest P-harmonic minorant $u \wedge v$ belong to the space $PE(R)$ and have the following integral representations along the P-harmonic boundary:*

$$(u \vee v)(x) = \int_{\Delta_P} (u \cup v)(t)K(x, t)d\mu(t),$$

$$(u \wedge v)(x) = \int_{\Delta_P} (u \cap v)(t)K(x, t)d\mu(t).$$

5. Let u be a positive PE-function on R. We shall call u a *PE-minimal function* if for every $v \in PE(R)$ with $0 \leq v \leq u$ there corresponds a constant c_v such that $v = c_v u$ on R.

As is to be expected, the P-singular point enters in the topological characterization of the existence of PE-minimal functions (for HD-minimal functions cf. Sario-Nakai [9]):

THEOREM. *There is a one-to-one correspondence between the PE-minimal functions on R and those isolated points of the P-harmonic boundary Δ_P which are different from the P-singular point s.*

Proof. Suppose that there exists a PE-minimal function u on R. In view of the maximum principle the P-harmonic boundary Δ_p contains a point p for which $u(p) > 0$. Clearly $p \neq s$ since $u(s) = 0$.

We claim that such a point p is unique. On the contrary suppose that $u(q) > 0$ for some $q \in \Delta_P - s$. Choose a function $f \in M_P(R)$ such that $0 \leq f \leq 1$ on R, $f(p) = 1$, and $f(q) = 0$. Then $v = \pi(f \cap u) \in PBE(R)$, and $u \geq v \geq 0$ on Δ_P. Again by the maximum principle, $u \geq v \geq 0$ on R and $v = c_v u$ on R for some constant c_v.

On the other hand $v(q) = f(q) = 0$ and $v(q) = c_v u(q) > 0$, a contradiction. Thus $u \equiv 0$ on $\Delta_P - p$ and p is an isolated point of Δ_P by the continuity of u.

Conversely let p be an isolated point of Δ_P such that $p \in \Delta_P - s$. Then there exists a function $f \in M_P(R)$ such that $0 \leq f \leq 1$ on R, $f(p) = 1$, and $f | \Delta_P - p \equiv 0$. Let $u = \pi(f) \in PBE(R)$. If $v \in PE(R)$ such that $0 \leq v \leq u$ on R, $v \equiv 0$ on $\Delta_P - p$ and $0 \leq v(p) \leq 1$. Thus there exists a constant $c_v = v(p)/u(p)$ such that $v = c_v u$ on Δ_P. By means of the maximum principle we conclude that $v = c_v u$ on R and u is PE-minimal.

From the proof we also deduce:

COROLLARY. *Every PE-minimal function is bounded.*

6. In analogy with $\widetilde{HD}$-functions we introduce: a nonnegative P-harmonic function u on R is called a $\widetilde{PE}$-function if

$$u(x) = \inf \{v(x) \mid v \in PE(R),\ v \geq u \text{ on } R\}$$

for all $x \in R$.

To study $\widetilde{PE}$-functions we consider the class $U(\Delta_P)$ of nonnegative functions f on Δ_P such that

$$f(t) = \inf_{v \in F_f} v(t)$$

on Δ_P, where $F_f = \{v \in PE(R) | v \gtrsim f \text{ on } \Delta_P\}$. Clearly every $f \in U(\Delta_P)$ is upper semicontinuous, μ-integrable, and vanishes at the P-singular point s.

We state:

LEMMA. *The class $U(\Delta_P)$ has the following properties:*

(i) *if E is a compact subset of Δ_P which does not contain the P-singular point s, then its characteristic function χ_E belongs to $U(\Delta_P)$,*

(ii) *if $f \in U(\Delta_P)$, then $f \cap \alpha \in U(\Delta_P)$ for all $\alpha > 0$,*

(iii) *the class $U(\Delta_P)$ forms a lattice under the pointwise maximum and minimum operations.*

Proof. Let $g \in B_s(\Delta_P)$ be such that $g \gtrsim f = \chi_E$ on Δ_P. For each $n \gtrsim 1$ choose an open neighborhood U_n of E in R_P^* such that $g + 1/n > 1$ on U_n. Then there exists a function $h_n \in M_P(R)$ such that $0 \lesssim h_n \lesssim 1$ on R, $h_n | E \equiv 1$, and $h_n | R - U_n \equiv 0$ (Kwon-Sario-Schiff [5]). Clearly $f \lesssim h_n \lesssim g + 1/n$ on Δ_P.

Set $u_n = \pi(h_n) \in PBE(R)$. In view of $f \lesssim u_n \lesssim g + 1/n$ for all n,

$$f(t) \lesssim \inf_{u \in F_f} u(t) \lesssim \varliminf_{n \to \infty} u_n(t) \lesssim g(t)$$

for all $t \in \Delta_P$. Since f is upper semicontinuous on Δ_P,

$$f(t) = \inf \{g(t) | g \in B_s(\Delta_P),\ g \gtrsim f\} \gtrsim \inf_{u \in F_f} u(t) \gtrsim f(t)$$

on Δ_P as asserted. This completes the proof of (i).

Contrary to assertion (ii) suppose that $f \cap \alpha \notin U(\Delta_P)$. Then there exist $p \in \Delta_P$ and $\varepsilon > 0$ such that

$$(f \cap \alpha)(p) < \inf_{v \in F_{f \cap \alpha}} v(p) - \varepsilon.$$

Since $f \in U(\Delta_P)$, there exists a sequence of functions $v_n \in F_f$ with $f(p) = \lim_n v_n(p)$. Clearly $\pi(v_n \cap \alpha) \in F_{f \cap \alpha}$ and therefore

$$\inf_{v \in F_{f \cap \alpha}} v(p) \lesssim \lim_{n \to \infty} \pi(v_n \cap \alpha)(p) = (f \cap \alpha)(p).$$

Thus we have $(f \cap \alpha)(p) < (f \cap \alpha)(p) - \varepsilon$, a contradiction.

Statement (iii) follows immediately from the lattice property of the space $PE(R)$ and the definition of the class $U(\Delta_P)$.

7. We are ready to express $\widetilde{PE}$-functions as integrals of functions in $U(\Delta_P)$ along the P-harmonic boundary (cf. Sario-Nakai [9]):

THEOREM. *A function u belongs to the class $\widetilde{PE}(R)$ if and only if it has the integral representation along the P-harmonic boundary*

$$u(x) = \int_{\Delta_P} f(t)K(x, t)d\mu(t)$$

for some $f \in U(\Delta_P)$.

Proof. Let u be defined by the above integral for some $f \in U(\Delta_P)$. Choose a nonincreasing sequence $\{v_n\}$ of functions $v_n \in F_f$ such that $f = \lim_n v_n$ on Δ_P. Clearly $u = \lim_n v_n$ on R. Therefore for any $v \in F_f$

$$u(x) = \int_{\Delta_P} f(t)K(x, t)d\mu(t) \leq \int_{\Delta_P} v(t)K(x, t)d\mu(t) = v(x)$$

and we conclude that

$$u(x) = \inf_{v \in F_f} v(x).$$

Since

$$u(x) \leq \inf\{v(x) \mid v \in PE(R),\ v \geq u\} \leq \inf\{v(x) \mid v \in F_f\} = u(x),$$

the function u belongs to the class $\widetilde{PE}(R)$ as desired.

Conversely let $u \in \widetilde{PE}(R)$. Then there exists a nonincreasing sequence $\{u_n\}$ of positive PE-functions on R such that $u(x) = \lim_n u_n(x)$ on R.

Set $f(t) = \lim_n u_n(t)$ for $t \in \Delta_P$. Clearly $f \in U(\Delta_P)$ and we have

$$u(x) = \lim_{n\to\infty} \int_{\Delta_P} u_n(t)K(x, t)d\mu(t) = \int_{\Delta_P} f(t)K(x, t)d\mu(t)$$

for each $x \in R$.

LEMMA. *Let E be a compact set in the complement of the P-singular point s with respect to the P-harmonic boundary Δ_P. Then the function $w(x) = \int_E K(x, t)d\mu(t)$ has the properties $0 \leq w(x) \leq 1$ on R and $\lim_{x \in R, x \to t} w(x) = 0$ for all $t \in \Delta_P - E$.*

Proof. Let $q \in \Delta_P - E$. Choose a neighborhood U of E with $q \notin U$ and a function $f \in M_P(R)$ such that $0 \leq f \leq 1$ on R, $f \mid E \equiv 1$, and $f \mid R - U \equiv 0$.

Since $f \geq \chi_E$ on Δ_P,

$$0 \leq w(x) \leq \int_{\Delta_P} f(t) K(x, t) d\mu(t) = \pi(f)(x)$$

and therefore

$$0 \leq \overline{\lim_{x \in R, x \to q}} w(x) \leq \lim_{x \in R, x \to q} \pi(f)(x) = f(q) = 0.$$

By means of the above lemma we shall establish a relation between a $\widetilde{PE}$-function u and the corresponding $f \in U(\Delta_P)$ (cf. Sario-Nakai [9]):

THEOREM. *Let $u \in \widetilde{PE}(R)$ have the integral representation along the P-harmonic boundary*

$$u(x) = \int_{\Delta_P} f(t) K(x, t) d\mu(t)$$

with $f \in U(\Delta_P)$. Then the function $\bar{u}(t) = \limsup_{x \in R, x \to t} u(x)$, $t \in \Delta_P$ satisfies $\bar{u} \leq f$ with equality μ-a.e. on Δ_P.

Proof. In view of $u \leq v$ for $v \in F_f$ and $f = \inf_{v \in F_f} v$, the inequality is obvious.

For the proof of the latter assertion first assume that f is bounded. Let $\varepsilon > 0$ and suppose that $\bar{u} < f - \varepsilon$ on a compact subset E of $\Delta_P - s$. If $\mu(E) > 0$, then the function

$$w(x) = \varepsilon \int_E K(x, t) d\mu(t)$$

is P-harmonic and $0 < w(x) \leq \varepsilon$ on R. By the above lemma

$$\overline{\lim_{x \in R, x \to t}} [u(x) + w(x)] = \bar{u}(t) \leq f(t)$$

for all $t \in \Delta_P - E$. Hence for each $v \in F_f$

$$\underline{\lim_{x \in R, x \to t}} [v(x) - u(x) - w(x)] \geq 0$$

on Δ_P. Since $v - u - w$ is bounded from below, $v \geq u + w$ on R (see Kwon-Sario-Schiff [5]). On taking the infimum over F_f, we obtain $u \geq u + w$. In particular

$$0 \geq w(x_0) = \mu(E) \geq 0.$$

For the P-singular point s, $0 \leq \bar{u}(s) \leq f(s) = 0$. Thus $\bar{u} \geq f$ μ-a.e. on Δ_P.

For an unbounded f, set $u_n(x) = \int_{\Delta_P} (f \cap n)(t) K(x, t) d\mu(t) \in \widetilde{PE}(R)$. Since $\bar{u} \geq \bar{u}_n = f \cap n$ μ-a.e. on Δ_P for all $n \geq 1$, we have the desired conclusion.

8. A function $u \in \widetilde{PE}(R)$ is said to be $\widetilde{PE}$-minimal if $u > 0$ on R and for every $v \in \widetilde{PE}(R)$ with $u \geq v$ there exists a constant c_v such that $v = c_v u$ on R.

We maintain (for $\widetilde{HD}$-functions cf. Sario-Nakai [9]):

THEOREM. *If a function u is $\widetilde{PE}$-minimal, then there exists a point p on the P-harmonic boundary, different from the P-singular point s and with a positive μ-measure. In this case $u(x) = u(x_0) K(x, p)$. Conversely if $p \in \Delta_P - s$ has a positive μ-measure, then $K(x, p)$ is $\widetilde{PE}$-minimal.*

Proof. Let u be $\widetilde{PE}$-minimal. Then

$$u(x) = \int_{\Delta_P} \bar{u}(t) K(x, t) d\mu(t)$$

on R. Set $E_n = \{t \in \Delta_P \mid \bar{u}(t) \geq 1/n\}$. Clearly E_n is a compact subset of $\Delta_P - s$, and therefore $\chi_{E_n} \in U(\Delta_P)$. Since

$$u(x) \geq \int_{E_n} \bar{u}(t) K(x, t) d\mu(t) \geq \frac{1}{n} \int_{\Delta_P} \chi_{E_n}(t) K(x, t) d\mu(t) \in \widetilde{PE}(R),$$

there exists a constant c_n, $0 \leq c_n \leq 1$, such that

$$\int_{E_n} K(x, t) d\mu(t) = c_n u(x)$$

on R. For large n, $\mu(E_n) > 0$ and $c_n > 0$. Thus u is bounded by Lemma 7. Set $E = E_n$, and

$$w(x) = \int_E K(x, t) d\mu(t).$$

Then $w = c_n u$ and $\bar{w} = 1$ μ-a.e. on E. In view of $c_n \sup_R u = 1$, $w = cu$ where $c = 1/\sup_R u$. Thus $c\bar{u} = \chi_E$ μ-a.e. on Δ_P.

Let A be a compact subset of E with $\mu(E - A) > 0$. If $\mu(A) > 0$, then $\int_A K(x, t) d\mu(t) = cu(x)$ as above and $c\bar{u} = 0$ μ-a.e. on $\Delta_P - A$. Since $\mu(E - A) > 0$ and $c\bar{u} = 1$ μ-a.e. on E, this is a contradiction. Consequently $\mu(A) = 0$.

On the other hand E is compact and $\mu(E) > 0$. Therefore there exists a point $p \in E$ such that $\mu(E \cap U) > 0$ for all neighborhoods U of p. Suppose $\mu(p) = 0$. Then there exists a sequence $\{U_n\}$ of neighborhoods U_n of p with $0 < \mu(E \cap U_n) < 1/n$. If $\mu(E - U_n) > 0$, then for any compact $K_n \subset U_n$, $\mu(E - K_n) > 0$ and $\mu(E \cap K_n) = 0$ as above. The regularity of μ then implies $\mu(E \cap U_n) = 0$, a contradiction. Thus $\mu(E - U_n) = 0$ for all n. Hence

$$0 < \mu(E) = \mu(E \cap U_n) < \frac{1}{n}$$

for all n, a contradiction, and we conclude that $\mu(p) > 0$ and $cu(x) = K(x, p)\mu(p)$. Since $K(x_0, p) = 1$, $cu(x_0) = \mu(p)$ and therefore $u(x) = u(x_0)K(x, p)$ as asserted.

Conversely let p be a point in $\Delta_P - s$ such that $\mu(p) > 0$. Then

$$K(x, p) = \frac{1}{\mu(p)} \int_{\Delta_P} \chi_p(t) K(x, t) d\mu(t)$$

is a $\widetilde{PE}$-function. If $K(x, p) \geq v(x) \geq 0$ for some $v \in \widetilde{PE}(R)$, then $\bar{K}(t, p) \geq \bar{v}(t) \geq 0$ and $\bar{v}(t) = 0$ μ-a.e. on $\Delta_P - p$. Thus

$$v(x) = \int_{\Delta_P} \bar{v}(t) K(x, t) d\mu(t) = \bar{v}(p)\mu(p)K(x, p)$$

on R and $K(x, p)$ is $\widetilde{PE}$-minimal.

BIBLIOGRAPHY

[1] M. Glasner-R. Katz, *On the behavior of solutions of $\Delta u = Pu$ at the Royden boundary*, J. Analyse Math. 22 (1969), 343–354.

[2] Y.K. Kwon-L. Sario, *A maximum principle for bounded harmonic functions on Riemannian spaces*, Canad. J. Math. 22(1970), 847–854.

[3] ————————, *Harmonic functions on a subregion of a Riemannian manifold*, J. Ind. Math. Soc. (to appear).

[4] ————————, *The P-singular point of the* P-*compactification for $\Delta u = Pu$*, Bull. Amer. Math. Soc. (to appear).

[5] Y.K. Kwon-L. Sario-J. Schiff, *Bounded energy-finite solutions of $\Delta u = Pu$ on a Riemannian manifold*, Nagoya. Math. J. (to appear).

[6] M. Nakai, *A measure on the harmonic boundary of a Riemann surface*, Nagoya Math. J. 17 (1960), 181–218.

[7] M. Nakai-L. Sario, *A new operator for elliptic equations and the P-compactification for $\Delta u = Pu$*, Math. Ann. 189(1970), 242–256.

[8] H.L. Royden, *The equation $\Delta u = Pu$ and the classification of open Riemann surfaces*, Ann. Acad. Sci. Fenn. Ser. A.I. 271 (1959), 27 pp.

[9] L. Sario-M. Nakai, *Classification theory of Riemann surfaces*, Springer, 1970, 446 pp.

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