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ON THE FINITE SUBGROUPS OF GL (3, Z)

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Introduction

We should like to study three dimensional algebraic tori in the same way as Voskresenskiĭ does in [14] and [15]. To do so, it is necessary to determine all finite subgroups of $GL(3,\boldsymbol{Z})$ up to conjugacy.

We find in Serre [11] that the order of any finite subgroup of $GL(3,\boldsymbol{Z})$ is at most $N(n)$, where $N(n)$ is the greatest common divisor of $2^{n^2}(2^n-1)(2^n-2)\cdot\cdot\cdot\cdot(2^n-2^{n-1})$ and $(p^n-1)(p^n-p)\cdot\cdot\cdot\cdot(p^n-p^{n-1})$ for every odd prime p. According to Serre himself*), this estimate was first obtained by Minkowski [16]. This estimate, however, is not the best possible. For example, when $n=2$, the greatest of the orders of all finite subgroups is $2^2\cdot 3=12$ (cf. Serre, ibid.), while $N(n)=48$. We refer the reader to a sharper estimate of the orders of all finite subgroups of $GL(n,\boldsymbol{Z})$ by Minkowski [17]. According to this, the greatest is not larger than $2^4\cdot 3=48$ when $n=3$. In this paper we show that this is the best possible, and further determine all the finite subgroups of $GL(3,\boldsymbol{Z})$ (resp. $SL(3,\boldsymbol{Z})$) up to conjugacy.

First of all, we find all non-conjugate cyclic subgroups of $GL(3,\boldsymbol{Z})$. By Vaidyanathaswamy [12] and [13], any element of $GL(3,\boldsymbol{Z})$ has order 1, 2, 3, 4, 6 or ∞: namely $\varphi(m)\leq 2$ only for $m=1$, 2, 3, 4 or 6, where $\varphi(m)$ is Euler's function. Hence the order of any finite cyclic subgroup of $GL(3,\boldsymbol{Z})$ is 1, 2, 3, 4, or 6. Reiner [10] determined all non-conjugate cyclic subgroups of order m in $GL(3,\boldsymbol{Z})$ for prime numbers $m=2$ and 3. Therefore we must determine all non-conjugate cyclic subgroups of order m in $GL(3,\boldsymbol{Z})$ for $m=4$ and 6.[1)]

Next we determine all non-conjugate non-cyclic subgroups of $GL(3,\boldsymbol{Z})$. Since each element of $GL(3,\boldsymbol{Z})$ has order 1, 2, 3, 4, 6 or ∞, the order of any

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1) For $m=6$, *see* Matuljauskas [7].

finite subgroup of $GL(3, \boldsymbol{Z})$ is of the form $2^i \cdot 3^j$. On the other hand, the structure of abstract groups of small orders are well-known up to isomorphism. By considering the structure of each of them, we show that $i \leqq 4$ and $j \leqq 1$. More explicitely, there exists neither any abelian subgroup of order more than 6, nor any finite subgroup of order more than $2^3 \cdot 3 = 24$ in $SL(3, \boldsymbol{Z})$, hence the order of any finite subgroup of $GL(3, \boldsymbol{Z})$ is at most $2^4 \cdot 3 = 48$. We list in a table below the number of non-conjugate classes of subgroups of a given order in $GL(3, \boldsymbol{Z})$ and $SL(3, \boldsymbol{Z})$.

Finally as an application, we investigate groups of fixed-point-free rational automorphisms of algebraic tori. Here a rational automorphism ϕ of an algebraic torus is called fixed-point-free, when $\phi(x) = x$ if and only if x is the identity element of the torus.

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order	$GL(3, \boldsymbol{Z})$			$SL(3, \boldsymbol{Z})$		
	abel.		non-ab.	abel.		non-ab.
	cyclic	non-cy.		cyclic	non-cy.	
1	1			1		
2	5			2		
3	2			2		
4	4	11		2	4	
6	4		6	1		3
8		6	8			2
12		1	10			4
16			2			
24			11			3
48			3			
sub-total	16	18	40	8	4	12
total	74			24		

0. Notation and conventions

0.0 As usual $\boldsymbol{Z}$ and $\boldsymbol{Q}$ are the domain of rational integers and the field of rational numbers. We use the following notation:

$GL(n, \boldsymbol{Q})$: the general linear group of degree n over $\boldsymbol{Q}$

$GL(n, \boldsymbol{Z})$: the general linear group of degree n over $\boldsymbol{Z}$

$SL(n, \boldsymbol{Z})$: the special linear group of degree n over $\boldsymbol{Z}$

$\{A, B, \cdots, D\}$: the group generated by elements $A, B, \cdots, D$

$\boldsymbol{Z}_m$: the multiplicative cyclic group of order m

${}^tW$: the subgroup of $GL(n, \boldsymbol{Z})$ consisting of the transposed matrices of all matrices of a subgroup W in $GL(n, \boldsymbol{Z})$

$\det(X)$: the determinant of a matrix X in $GL(n, \boldsymbol{Z})$

E_n : the unit matrix in $GL(n, \boldsymbol{Z})$

0.1 Let A and B be matrices in $GL(n, \boldsymbol{Z})$. Then A is called *conjugate to* B *in* $GL(n, \boldsymbol{Z})$ (resp. $SL(n, \boldsymbol{Z})$) if there exists a matrix M in $GL(n, \boldsymbol{Z})$ (resp. $SL(n, \boldsymbol{Z})$) such that $A = M^{-1}BM$. A subgroup V of $GL(n, \boldsymbol{Z})$ is called *conjugate to* another subgroup W *in* $GL(n, \boldsymbol{Z})$ (resp. $SL(n, \boldsymbol{Z})$), if there exists a matrix M in $GL(n, \boldsymbol{Z})$ (resp. $SL(n, \boldsymbol{Z})$) such that $V = M^{-1}WM$. We note that for any odd number n, A (or V) is conjugate to B (or W) in $GL(n, \boldsymbol{Z})$ if and only if they are conjugate to each other in $SL(n, \boldsymbol{Z})$. In this case we merely say *they are conjugate to each other* and denote by $A \sim B$ (or $V \sim W$). Clearly, if V is conjugate to W, V is isomorphic to W.

0.2 According to Coxeter-Moser [1], p. 134, we list, up to isomorphism, all the non-abelian abstract groups of order not more than 24, each element of which has order 1, 2, 3, 4 or 6.

1) Group of order 6

$\mathfrak{S}_3 = \{S, T\}$: the symmetric group of degree 3, i.e.

$$S^3 = T^2 = (ST)^2 = 1$$

2) Groups of order 8

$\mathfrak{Q} = \{i, j, k\}$: the quaternion group, i.e.

$$i^2 = j^2 = k^2 = ijk = -1$$

$\mathfrak{D}_4 = \{S, T\}$: the dihedral group with the following defining relations:

$$S^4 = T^2 = (ST)^2 = 1$$

3) Groups of order 12

$\mathfrak{D}_6 = \{S, T\} \cong \mathfrak{S}_3 \times \boldsymbol{Z}_2$: the dihedral group with the following defining relations:

$$S^6 = T^2 = (ST)^2 = 1$$

$\mathfrak{A}_4 = \{S, T\}$: the alternating group of degree 4, i.e.

$$S^3 = T^2 = (ST)^3 = 1$$

$\langle 2,2,3\rangle = \{S, T\}$: the ZS-metacyclic group with the following defining relations:

$$S^3 = T^2 = (ST)^2$$

4) Groups of order 16

$\mathfrak{D}_4 \times \boldsymbol{Z}_2$: the direct product of the groups $\mathfrak{D}_4$ and $\boldsymbol{Z}_2$

$\mathfrak{Q} \times \boldsymbol{Z}_2$: the direct product of the groups $\mathfrak{Q}$ and $\boldsymbol{Z}_2$

$\langle 2,2|4,2\rangle = \{S, T\}$: the group with the following defining relations:

$$S^4 = T^4 = 1, \quad T^{-1}ST = S^3$$

$(4,4|2,2) = \{S, T\}$: the group with the following defining relations:

$$S^4 = T^4 = (ST)^2 = (S^{-1}T)^2 = 1$$

$\mathfrak{R} = \{R, S, T\}$: the group with the following defining relations:

$$R^2 = S^2 = T^2 = 1, \quad RST = STR = TRS$$

5) Groups of order 24

$\mathfrak{A}_4 \times \boldsymbol{Z}_2$: the direct product of the groups $\mathfrak{A}_4$ and $\boldsymbol{Z}_2$

$\langle 2,2,3\rangle \times \boldsymbol{Z}_2$: the direct product of the groups $\langle 2,2,3\rangle$ and $\boldsymbol{Z}_2$

$\mathfrak{D}_6 \times \boldsymbol{Z}_2$: the direct product of the groups $\mathfrak{D}_6$ and $\boldsymbol{Z}_2$

$\mathfrak{S}_4 = \{S, T\}$: the symmetric group of degree 4, i.e.

$$S^4 = T^2 = (ST)^3 = 1$$

$\langle 2,3,3\rangle = \{S, T\}$: the group with the following defining relations:

$$S^3 = T^3 = (ST)^2$$

$(4,6|2,2) = \{S, T\}$: the group with the following defining relations:

$$S^4 = T^6 = (ST)^2 = (S^{-1}T)^2 = 1$$

1. Finite subgroups of $GL(3, \boldsymbol{Z})$

1.0 First we wish to determine all non-conjugate cyclic subgroups of $GL(3, \boldsymbol{Z})$. To do this we need the following well-known result:[2)]

PROPOSITION 1. *There exist only* 6 *non-conjugate cyclic subgroups of order* 2, 3, 4 *or* 6 *in* $GL(2, \boldsymbol{Z})$:

$$\boldsymbol{Z}_2:\quad W_1 = \left\{\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\right\},\quad W_2 = \left\{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right\},\quad W_3 = \left\{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right\},$$

$$\boldsymbol{Z}_3:\quad W = \left\{\begin{pmatrix}0 & -1\\ 1 & -1\end{pmatrix}\right\},$$

$$\boldsymbol{Z}_4:\quad W = \left\{\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right\},$$

$$\boldsymbol{Z}_6:\quad W = \left\{\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}\right\}.$$

1.1 Groups of order 2

By virtue of Reiner's basic result ([2] Theorem 74.3, p. 508,), it follows that

PROPOSITION 2. *There exist* 5 *non-conjugate subgroups of order* 2 *in* $GL(3, \boldsymbol{Z})$:

$$W_1 = \left\{\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}\right\},\ W_2 = \left\{-\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}\right\},\ W_3 = \left\{\begin{pmatrix}-1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\right\},$$

$$W_4 = \left\{-\begin{pmatrix}-1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\right\},\ W_5 = \left\{-\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right\}.$$

1.2 Groups of order 3

For the same reason as above, we have

PROPOSITION 3. *There exist* 2 *non-conjugate subgroups of order* 3 *in* $GL(3, \boldsymbol{Z})$:

$$W_1 = \left\{\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & -1\end{pmatrix}\right\},\ W_2 = \left\{\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix}\right\}.$$

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2) *See* Voskresenskiĭ [14], p. 192.

Remark. Without Reiner's basic result, we may prove Proposition 2 and 3 by elementary calculations.

1.3 Groups of order 4

We show the following:

PROPOSITION 4. *There exist* 15 *non-conjugate subgroups of order* 4 *in* $GL(3,\boldsymbol{Z})$: *those isomorphic to* $\boldsymbol{Z}_4$

$$W_1=\left\{\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}\right\},\quad W_2=\left\{-\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}\right\},\quad W_3=\left\{\begin{pmatrix}1&0&1\\0&0&-1\\0&1&0\end{pmatrix}\right\},$$

$$W_4=\left\{-\begin{pmatrix}1&0&1\\0&0&-1\\0&1&0\end{pmatrix}\right\},$$

those isomorphic to $\boldsymbol{Z}_2\times\boldsymbol{Z}_2$

$$W_5=\left\{\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix},\ -\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right\},\quad W_6=\left\{\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix},\ \begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\right\},$$

$$W_7=\left\{\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix},\ -\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\right\},\quad W_8=\left\{\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix},\ \begin{pmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}\right\},$$

$$W_9=\left\{\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix},\ -\begin{pmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}\right\},\quad W_{10}=\left\{-\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix},\ -\begin{pmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}\right\},$$

$$W_{11}=\left\{\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix},\ -\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right\},\quad W_{12}=\left\{\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix},\ \begin{pmatrix}-1&0&0\\1&0&-1\\-1&-1&0\end{pmatrix}\right\},$$

$$W_{13}=\left\{\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix},\ -\begin{pmatrix}-1&0&0\\1&0&-1\\-1&-1&0\end{pmatrix}\right\},\quad W_{14}=\left\{\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix},\ \begin{pmatrix}-1&1&-1\\0&0&-1\\0&-1&0\end{pmatrix}\right\},$$

$$W_{15}=\left\{\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix},\ -\begin{pmatrix}-1&1&-1\\0&0&-1\\0&-1&0\end{pmatrix}\right\}.$$

COROLLARY. *In* $SL(3,\boldsymbol{Z})$ *there exist only* 2 *non-conjugate cyclic and* 4 *non-conjugate non-cyclic subgroups of order* 4: W_1, W_3 *and* W_6, W_8, W_{12}, W_{14}.

Proof. We first find all non-conjugate cyclic subgroups of order 4 in $GL(3,\boldsymbol{Z})$. Let $Y\in GL(3,\boldsymbol{Z})$ be of order 4. By Proposition 2 it follows that

$$1)\quad Y^2\sim\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}\quad\text{or}\quad 2)\quad Y^2\sim\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}.$$

Case 1) Assume that $Y^2=M^{-1}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}M$, where $M\in GL(3,\boldsymbol{Z})$. We need an auxiliary result which will often be used later.

LEMMA 1. *Let X be a matrix in $GL(3,\boldsymbol{Z})$. If X^2 is equal to* $\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$, $\begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$ *or* $\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}$, *then X is of the form*

$$\pm\begin{pmatrix}1&0&0\\0&a&b\\0&c&-a\end{pmatrix},\quad\pm\begin{pmatrix}a&0&b\\0&1&0\\c&0&-a\end{pmatrix}\quad\text{or}\quad\pm\begin{pmatrix}a&b&0\\c&-a&0\\0&0&1\end{pmatrix},$$

respectively. Here $a^2+bc+1=0$.

The proof is straightforward.

Hence we have $MYM^{-1}=\pm\begin{pmatrix}1&0&0\\0&a&b\\0&c&-a\end{pmatrix}$ where $a^2+bc+1=0$. Since Y and hence the matrix $\begin{pmatrix}a&b\\c&-a\end{pmatrix}$ have order 4, it follows by Proposition 1 that

$$\begin{pmatrix}1&0&0\\0&a&b\\0&c&-a\end{pmatrix}\sim\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}$$

and so $\{Y\}\sim W_1$ or W_2.

Case 2) Assume now that $Y^2=M^{-1}\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}M$, where $M\in GL(3,\boldsymbol{Z})$.

Lemma 2. *Let X be a matrix in $GL(3, \boldsymbol{Z})$. If X^2 is equal to* $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ *or* $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$, *then X is of the form*

$$\pm \begin{pmatrix} a & b & -b \\ -\dfrac{1+a^2}{2b} & \dfrac{1-a}{2} & \dfrac{1+a}{2} \\ \dfrac{1+a^2}{2b} & \dfrac{1+a}{2} & \dfrac{1-a}{2} \end{pmatrix} \text{ or } \pm \begin{pmatrix} a & b & b \\ -\dfrac{1+a^2}{2b} & \dfrac{1-a}{2} & -\dfrac{1+a}{2} \\ -\dfrac{1+a^2}{2b} & -\dfrac{1+a}{2} & \dfrac{1-a}{2} \end{pmatrix},$$

respectively. Here $b \neq 0$, a and $\dfrac{1+a^2}{2b}$ are all odd integers.

The proof is easy.

By Lemma 2, we have $MYM^{-1} = \pm \begin{pmatrix} a & b & -b \\ -\dfrac{1+a^2}{2b} & \dfrac{1-a}{2} & \dfrac{1+a}{2} \\ \dfrac{1+a^2}{2b} & \dfrac{1+a}{2} & \dfrac{1-a}{2} \end{pmatrix} \equiv \pm N$.

We claim that $Y \sim \pm \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$. It is enough to show that $N \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$.

Easy calculations show that $N \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ if there is a matrix Z in $GL(3, \boldsymbol{Z})$ such that

$$Z = \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ -(1+a)z_{11} + \dfrac{1+a^2}{2b}(z_{12}-z_{13}) & -bz_{11} - \dfrac{1-a}{2}(z_{12}-z_{13}) & bz_{11} + \dfrac{1-a}{2}(z_{12}-z_{13}) \\ -(1-a)z_{11} - \dfrac{1+a^2}{2b}(z_{12}-z_{13}) & bz_{11} - \dfrac{1+a}{2}(z_{12}-z_{13}) & -bz_{11} + \dfrac{1+a}{2}(z_{12}-z_{13}) \end{pmatrix}$$

where $\det(Z) = -(z_{12}+z_{13})\left\{2bz_{11}^2 - 2az_{11}(z_{12}-z_{13}) + \frac{1+a^2}{2b}(z_{12}-z_{13})^2\right\} = \pm 1$, i.e. $z_{12}+z_{13} = \pm 1$ and $2bz_{11}^2 - 2az_{11}(z_{12}-z_{13}) + \frac{1+a^2}{2b}(z_{12}-z_{13})^2 = \pm 1$. Hence N is conjugate to $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$, if $z_{11} \equiv x$ and $2z_{12}+1 \equiv y$ are integers satisfying the following diophantine equation

$$(2|b|x + ay)^2 + y^2 = 2|b|.$$

Theorem $7-4$ ([6], p. 126) shows that the above equation has integral solutions. Therefore N is conjugate to $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$. We can easily see that W_i $(1 \leqq i \leqq 4)$ are not conjugate to each other.

We next find all non-conjugate non-cyclic subgroups, i.e. those isomorphic to $\boldsymbol{Z}_2 \times \boldsymbol{Z}_2$ in $GL(3,\boldsymbol{Z})$. Let S and T be generators of $\boldsymbol{Z}_2 \times \boldsymbol{Z}_2$, then $S^2 = T^2 = E$ and $TS = ST$ where $E = E_3$ is the unit matrix in $GL(3,\boldsymbol{Z})$. By Proposition 2, our proof is divided into three cases.

Case 1) Suppose that $S = \pm M^{-1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} M$, where $M \in GL(3,\boldsymbol{Z})$.

Since $TS = ST$, we have $MTM^{-1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} MTM^{-1}$. The following easy lemma is useful for a characterization of MTM^{-1}.

LEMMA 3. *Let X be a matrix in $GL(3,\boldsymbol{Z})$. If X commutes with* $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, *then X is of the form*

$$\pm \begin{pmatrix} 1 & 0 & 0 \\ 0 & x_{22} & x_{23} \\ 0 & x_{32} & x_{33} \end{pmatrix}$$

where $x_{22}x_{33} - x_{23}x_{32} = 1$.

Therefore we see that $T = \pm M^{-1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & x_{22} & x_{23} \\ 0 & x_{32} & x_{33} \end{pmatrix} M$, where $x_{22}x_{33} - x_{23}x_{32} = 1$.

Since T and so the matrix $T_1 \equiv \begin{pmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{pmatrix}$ have order 2, Proposition 1 implies that T_1 is conjugate to $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Thus $\{S, T\}$ is conjugate to

$$\left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\} \equiv W_5, \quad \left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\} \equiv W_6,$$

$$\left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, -\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\} \equiv W_7, \quad \left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}\right\} \equiv W_8,$$

$$\left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, -\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}\right\} \equiv W_9 \text{ or } \left\{-\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, -\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}\right\} \equiv W_{10}.$$

(Here both $\left\{-\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}$ and $\left\{-\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, -\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}$ are conjugate to W_7, and $\left\{-\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}\right\}$ is conjugate to W_{10}.)

Classifying all elements of W_i $(5 \leqq i \leqq 10)$ of five types of Proposition 2, we easily see that W_i $(5 \leqq i \leqq 10)$ are not conjugate to each other.

Case 2) Suppose now that $S = \pm M^{-1} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} M$, where $M \in GL(3, \boldsymbol{Z})$.

$TS = ST$ implies that $MTM^{-1} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} MTM^{-1}$. The proof of the following is straightforward.

LEMMA 4. *Let X be a matrix in $GL(3, \boldsymbol{Z})$. If X commutes with* $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, *then X is of the form*

$$X = \pm \begin{pmatrix} x_{11} & x_{12} & -x_{12} \\ x_{21} & x_{22} & x_{23} \\ -x_{21} & x_{23} & x_{22} \end{pmatrix}$$

where $(x_{22}+x_{23})\{x_{11}(x_{22}-x_{23})-2x_{12}x_{21}\}=1$. *Furthermore,*

(1) *if* X *has order* 2, *then* $X=\pm\begin{pmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}$, $\pm\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}$, $\pm\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$,

$-\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$, $\pm\begin{pmatrix}-1&0&0\\a&0&-1\\-a&-1&0\end{pmatrix}$, $\pm\begin{pmatrix}1&0&0\\a&-1&0\\-a&0&-1\end{pmatrix}$, $\pm\begin{pmatrix}-1&a&-a\\0&0&-1\\0&-1&0\end{pmatrix}$, $\pm\begin{pmatrix}1&a&-a\\0&-1&0\\0&0&-1\end{pmatrix}$

or $\pm\begin{pmatrix}-(1+2a)&b&-b\\c&a&-(1+a)\\-c&-(1+a)&a\end{pmatrix}$ *where* a, b *and* c *are all integers, and in the last case they satisfy the equation* $2a^2+2a+bc=0$,

(2) *there is no such matrix* X *of order* 3.

First assume that $T=\pm M^{-1}\begin{pmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}M$, $\pm M^{-1}\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}M$,

$\pm M^{-1}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}M$ or $-M^{-1}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}M$. Then $\{S,T\}$ is conjugate to W_8,

W_9, W_{10} or $\left\{\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}, -\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}\right\}=\left\{\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}, -\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right\}$

$\equiv W_{11}$. $\left(\text{Here } \left\{-\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}, \begin{pmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}\right\} \text{ is conjugate to } W_{10}.\right)$ Clearly, W_{11} is not conjugate to W_i $(5\leqq i\leqq 10)$.

Next assume that $T=\pm M^{-1}\begin{pmatrix}-1&0&0\\a&0&-1\\-a&-1&0\end{pmatrix}M$, $\pm M^{-1}\begin{pmatrix}1&0&0\\a&-1&0\\-a&0&-1\end{pmatrix}M$,

$\pm M^{-1}\begin{pmatrix}-1&a&-a\\0&0&-1\\0&-1&0\end{pmatrix}M$ or $\pm M^{-1}\begin{pmatrix}1&a&-a\\0&-1&0\\0&0&-1\end{pmatrix}M$. If S is equal to $M^{-1}\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}M$,

then $\{S,T\}$ is conjugate to $\left\{\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}, \begin{pmatrix}-1&0&0\\a&0&-1\\-a&-1&0\end{pmatrix}\right\}\equiv W$, $\left\{\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix},\right.$

$\left.-\begin{pmatrix}-1&0&0\\a&0&-1\\-a&-1&0\end{pmatrix}\right\}\equiv W'$ $\left\{\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}, \begin{pmatrix}1&0&0\\a&-1&0\\-a&0&-1\end{pmatrix}\right\}=W''$, $\left\{\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix},\right.$

$-\begin{pmatrix} 1 & 0 & 0 \\ a & -1 & 0 \\ -a & 0 & -1 \end{pmatrix}\Bigg\} \equiv W''', {}^tW, {}^tW', {}^tW''$ or ${}^tW'''$. When a is even, we put

$$N = \begin{pmatrix} x_{11} & 0 & 0 \\ \frac{a(x_{22}-x_{23})}{2} & x_{22} & x_{23} \\ -\frac{a(x_{22}-x_{23})}{2} & x_{23} & x_{22} \end{pmatrix}$$

where $x_{22}^2 - x_{23}^2 = \pm 1$. Then $W = N^{-1}W_8N$ and hence W', W'', W''', ${}^tW$, ${}^tW'$, ${}^tW''$ and ${}^tW'''$ are conjugate to W_8, W_{10}, W_8, W_{10}, W_{10}, W_8 and W_{10}, respectively. When a is odd, we consider two non-conjugate subgroups W_{12}, W_{13} isomorphic to $\boldsymbol{Z}_2 \times \boldsymbol{Z}_2$:

$$W_{12} \equiv \left\{\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}\right\}, \quad W_{13} \equiv \left\{\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, -\begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}\right\}.$$

Here W_{12} and so W_{13} are not conjugate to W_i $(5 \leqq i \leqq 11)$. Using Lemma 4 with easy calculations we see that $W = N^{-1}W_{12}N$, where

$$N = \begin{pmatrix} x_{11} & 0 & 0 \\ \frac{a(x_{22}-x_{23})-x_{11}}{2} & x_{22} & x_{23} \\ -\frac{a(x_{22}-x_{23})-x_{11}}{2} & x_{23} & x_{22} \end{pmatrix} \in GL(3, \boldsymbol{Z}).$$

Hence W', W'', W''', ${}^tW$, ${}^tW'$, ${}^tW''$ and ${}^tW'''$ are conjugate to W_{13}, W_{12}, W_{13}, ${}^tW_{12} \equiv W_{14}$, ${}^tW_{13} \equiv W_{15}$, W_{14} and W_{15}, respectively. By calculating one by one, we know that W_{12} is not conjugate to W_{14} and hence W_{13} is not conjugate to W_{15}. Thus W_i $(5 \leqq i \leqq 15)$ are not conjugate to each other. If S is equal to $-M^{-1}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}M$, we see, by replacing S by $-S$ in the above consideration, that $\{S, T\}$ is conjugate to W_8, W_9, W_{13}, W_{15}.

Finally assume that $T = \pm M^{-1}LM$, where

$$L = \begin{pmatrix} -(1+2a) & b & -b \\ c & a & -(1+a) \\ -c & -(1+a) & a \end{pmatrix} \text{ and } 2a^2 + 2a + bc = 0.$$

We need the following three lemmas.

LEMMA 5. *Let a, b and c be integers which satisfy an equation*

$$2a^2+2a+bc=0.$$

Then b is odd if and only if $c=\pm 2(a,c)(a+1,c)$, *and so b is even if and only if* $c=\pm(a,c)(a+1,c)$ *where* (a,c) *is the greatest common divisor of two integers a and c, and so on.*

Proof. Put $c=2^kc'$ where k is a non-negative integer and $(2,c')=1$. Let p be a prime number and suppose p^n divides c'. Since $2a(a+1)=-bc$, p^n divides $(a,c')(a+1,c')$. On the other hand $(a,c')(a+1,c')$ divides c' since $(a,a+1)=1$. Therefore $c'=\pm(a,c')(a+1,c')$. By comparing the exponents of the prime number 2 in these integers a, $a+1$, b and c, we easily get the result. Q.E.D.

LEMMA 6. *L is conjugate to* $\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$ *if and only if a, b and c is odd, even and even integers, respectively.*

Proof. Let $X=(x_{ij})$ be in $GL(3,\boldsymbol{Q})$ and assume that $L=X^{-1}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}X$.

Then $XL=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}X$ and so we obtain

$$X=\begin{pmatrix}x_{11}&\dfrac{(1+a)}{c}x_{11}&-\dfrac{(1+a)}{c}x_{11}\\[2ex] x_{21}&x_{22}&x_{22}-\dfrac{2a}{c}x_{21}\\[2ex] x_{31}&x_{32}&x_{32}-\dfrac{2a}{c}x_{31}\end{pmatrix}$$

where $\det(X)=\dfrac{2x_{11}}{c}(x_{22}x_{31}-x_{21}x_{32})$. Thus L is conjugate to $\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$ if and only if $\dfrac{2c^2}{(a+1,c)(2a,c)}$ divides c. Assume that $\dfrac{2c^2}{(a+1,c)(2a,c)}$ divides c. Then c is even and hence $\dfrac{c^2}{(a+1,c)\left(a,\frac{c}{2}\right)}$ divides c. Therefore a is odd and $\frac{c^2}{(a+1,c)(a,c)}$ divides c, hence $c = \pm(a+1,c)(a,c)$. By Lemma 5, b is even. Conversely we suppose that a, b and c is odd, even, and even, respectively. By Lemma 5, $c=\pm(a+1,c)(a,c)$ and $\frac{2c^2}{(a+1,c)(2a,c)} = \pm(a+1,\ c)(a,c)$ divides c. Thus L is conjugate to $\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$, Q.E.D.

Put $L' \equiv \begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix} L$. In the same way as above, we have

Lemma 7. *L' is conjugate to $\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$ if and only if a, b and c are all even integers.*

By Lemmas 6 and 7, we have to consider the following four cases:

Case i) a, b and c is odd, even and even, respectively,

Case ii) c is odd,

Case iii) a, b and c are all even,

Case iv) b is odd.

We now show that if $S=M^{-1}\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}M$ and $T = M^{-1}LM$, $\{S,T\}$ is conjugate to W_8, W_{12}, W_8 and W_{14} in Case i), ii), iii) and iv), respectively, and so $\{S, -T\}$, $\{-S, T\}$, $\{-S, -T\}$ are conjugate to W_i $(9 \leqq i \leqq 15,\ i \neq 11)$. For example, we show that $W = \{S,T\}$ is conjugate to W_{14} in Case iv). The proof is similar in other cases. In Case iv), by Lemmas 6 and 7, both L and L' are conjugate to $\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}$. Let $X = (x_{ij})$ be in $GL(3,\boldsymbol{Z})$ and assume that $X^{-1}\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}X = \begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}$. Then Lemma 4 implies that

$$X = \pm\begin{pmatrix}x_{11}&x_{12}&-x_{12}\\x_{21}&x_{22}&x_{23}\\-x_{21}&x_{23}&x_{22}\end{pmatrix}$$

where $\det(X) = (x_{22}+x_{23})\{x_{11}(x_{22}-x_{23})-2x_{12}x_{21}\}$. Furthermore assume that $X^{-1}\begin{pmatrix}1 & -1 & 1\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}X = L$. Then we have

$$X = \pm\begin{pmatrix} x_{11} & x_{12} & -x_{12}\\ (1+a)x_{11}-cx_{12} & x_{22} & bx_{11}+2ax_{12}+x_{22}\\ -(1+a)x_{11}+cx_{12} & bx_{11}+2ax_{12}+x_{22} & x_{22}\end{pmatrix}$$

which satisfy $X^{-1}\begin{pmatrix}-1 & 1 & -1\\ 0 & 0 & -1\\ 0 & -1 & 0\end{pmatrix}X = L'$ and $\det(X) = (bx_{11}+2ax_{12}+2x_{22})\times\{-bx_{11}^2-2(1+2a)x_{11}x_{12}+2cx_{12}^2\}$. Therefore W is conjugate to W_{14} if and only if two diophantine equations

$$\begin{cases} bx_{11}+2ax_{12}+2x_{22} = \pm 1 & (1)\\ bx_{11}^2+2(1+2a)x_{11}x_{12}-2cx_{12}^2 = \pm 1 & (2)\end{cases}$$

have at least one integral solution simultaneously. Since b is an odd integer, if the equation (2) has an integral solution, the equation (1) has an integral one. Since $2a(a+1) = -bc$, (2) can be arranged as follows:

$$\frac{\{bx_{11}+2(1+a)x_{12}\}\{bx_{11}+2ax_{12}\}}{b} = \pm 1$$

b being odd, i.e. $c = \pm 2(a,c)(a+1,c)$, we have $a(a+1) = \pm b(a,c)(a+1,c)$. Hence we may put $b = b_1b_2$ where b_1 and b_2 divide a and $a+1$, respectively. The equation

$$\left\{b_1x_{11}+\frac{2(1+a)}{b_2}x_{12}\right\}\left\{b_2x_{11}+\frac{2a}{b_1}x_{12}\right\} = \pm 1$$

has an integral solution $x_{11} = \dfrac{1+a}{b_2}-\dfrac{a}{b_1}$, $x_{12} = \dfrac{b_2-b_1}{2}$. Thus W is conjugate to W_{14} and hence $\{S,-T\}$, $\{-S,T\}$ and $\{-S,-T\}$ are all conjugate to W_{15}.

Case 3) Suppose that $S = -M^{-1}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}M = -\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$, where $M \in GL(3,\boldsymbol{Z})$. Then clearly $\{S,T\}$ is conjugate to W_5 or W_{11}.

Thus we complete the proof of Proposition 4, Q.E.D.

1.4 Groups of order 6

There are two non-isomorphic abstract groups of order 6, i.e. $\boldsymbol{Z}_6$ and $\mathfrak{S}_3$.
we obtain the following:

PROPOSITION 5. *There exist* 10 *non-conjugate subgroups of order* 6 *in* $GL(3,\boldsymbol{Z})$:
those isomorphic to $\boldsymbol{Z}_6$

$$W_1=\left\{\begin{pmatrix}1&0&0\\0&0&-1\\0&1&1\end{pmatrix}\right\},\ W_2=\left\{-\begin{pmatrix}1&0&0\\0&0&-1\\0&1&1\end{pmatrix}\right\},\ W_3=\left\{-\begin{pmatrix}1&0&0\\0&0&-1\\0&1&-1\end{pmatrix}\right\},$$

$$W_4=\left\{-\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\right\},$$

those isomorphic to $\mathfrak{S}_3$

$$W_5=\left\{\begin{pmatrix}1&0&0\\0&0&-1\\0&1&-1\end{pmatrix},\begin{pmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}\right\},\ W_6=\left\{\begin{pmatrix}1&0&0\\0&0&-1\\0&1&-1\end{pmatrix},-\begin{pmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}\right\},$$

$$W_7=\left\{\begin{pmatrix}1&0&0\\0&0&-1\\0&1&-1\end{pmatrix},\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}\right\},\ W_8=\left\{\begin{pmatrix}1&0&0\\0&0&-1\\0&1&-1\end{pmatrix},-\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}\right\},$$

$$W_9=\left\{\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix},\begin{pmatrix}0&0&-1\\0&-1&0\\-1&0&0\end{pmatrix}\right\},\ W_{10}=\left\{\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix},-\begin{pmatrix}0&0&-1\\0&-1&0\\-1&0&0\end{pmatrix}\right\}.$$

COROLLARY. *In* $SL(3,\boldsymbol{Z})$ *there exist only* 4 *non-conjugate subgroups of order* 6: W_1, W_5, W_7, W_9.

Proof. For cyclic subgroups, we refer the reader to Matuljauskas's result [7].[3] We determine all non-conjugate ones isomorphic to $\mathfrak{S}_3$.[4] Let S and T be generators of such a subgroup W. Then $S^3=T^2=(ST)^2=E$. By Proposition 3, it follows that

[3] It is not hard to find all of them by our method.

[4] *See* Nazarova-Roiter [9].

$$1)\quad S\sim\begin{pmatrix}1&0&0\\0&0&-1\\0&1&-1\end{pmatrix}\quad\text{or}\quad 2)\quad S\sim\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}.$$

Case 1) Assume that $S=M^{-1}\begin{pmatrix}1&0&0\\0&0&-1\\0&1&-1\end{pmatrix}M$, where $M\in GL(3,\boldsymbol{Z})$. Since $TS=S^2T$, we get $MTM^{-1}\begin{pmatrix}1&0&0\\0&0&-1\\0&1&-1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&-1&1\\0&-1&0\end{pmatrix}MTM^{-1}$. The following lemma can be proved immediately.

LEMMA 8. *Let X be a matrix in $GL(3,\boldsymbol{Z})$.*

(1) *Assume that* $X\begin{pmatrix}1&0&0\\0&0&-1\\0&1&-1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&-1&1\\0&-1&0\end{pmatrix}X$. *Then* $X=\pm\begin{pmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}$, $\pm\begin{pmatrix}-1&0&0\\0&-1&1\\0&0&1\end{pmatrix}$, $\pm\begin{pmatrix}-1&0&0\\0&1&0\\0&1&-1\end{pmatrix}$, $\pm\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}$, $\pm\begin{pmatrix}-1&0&0\\0&-1&0\\0&-1&1\end{pmatrix}$ *or* $\pm\begin{pmatrix}-1&0&0\\0&1&-1\\0&0&-1\end{pmatrix}$, *all of which have order* 2.

(2) *If X commutes with* $\begin{pmatrix}1&0&0\\0&0&-1\\0&1&-1\end{pmatrix}$ *or* $\begin{pmatrix}1&0&0\\0&-1&1\\0&-1&0\end{pmatrix}$, *then* $X=\pm\begin{pmatrix}1&0&0\\0&0&1\\0&-1&1\end{pmatrix}$, $\pm\begin{pmatrix}1&0&0\\0&-1&1\\0&-1&0\end{pmatrix}$, $\pm\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$, $\pm\begin{pmatrix}1&0&0\\0&1&-1\\0&1&0\end{pmatrix}$ $\pm\begin{pmatrix}1&0&0\\0&0&-1\\0&1&-1\end{pmatrix}$ *or* $\pm\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$.

Hence $\{S,T\}$ is conjugate to $\left\{\begin{pmatrix}1&0&0\\0&0&-1\\0&1&-1\end{pmatrix},\begin{pmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}\right\}\equiv W_5$, $\left\{\begin{pmatrix}1&0&0\\0&0&-1\\0&1&-1\end{pmatrix},-\begin{pmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}\right\}\equiv W_6$, $\left\{\begin{pmatrix}1&0&0\\0&0&-1\\0&1&-1\end{pmatrix},\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}\right\}\equiv W_7$ or $\left\{\begin{pmatrix}1&0&0\\0&0&-1\\0&1&-1\end{pmatrix},-\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}\right\}\equiv W_8$. Using Lemma 8, we see that W_5 is not conjugate to W_7 and so W_6 is not to W_8.

Case 2) Assume now that $S = M^{-1}\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}M$, where $M \in GL(3, \boldsymbol{Z})$.

Since $TS = S^2T$, $MTM^{-1}\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix} = \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}MTM^{-1}$. MTM^{-1} is characterized by the easy lemma:

LEMMA 9. *Let X be a matrix in* $GL(3, \boldsymbol{Z})$.

(1) *Assume that* $X\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix} = \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}X$. *Then* $X = \pm\begin{pmatrix}0&0&-1\\0&-1&0\\-1&0&0\end{pmatrix}$, $\pm\begin{pmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}$ *or* $\pm\begin{pmatrix}0&-1&0\\-1&0&0\\0&0&-1\end{pmatrix}$, *all of which have order* 2.

(2) *If X commutes with* $\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$ *or* $\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$, *then* $X = \pm\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$, $\pm\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$ *or* $\pm\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$.

Lemma 9 states that $\{S, T\}$ is conjugate to $\left\{\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}, \begin{pmatrix}0&0&-1\\0&-1&0\\-1&0&0\end{pmatrix}\right\} \equiv W_9$ or $\left\{\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}, -\begin{pmatrix}0&0&-1\\0&-1&0\\-1&0&0\end{pmatrix}\right\} \equiv W_{10}$. Clearly W_9 is not conjugate to any one of W_i $(5 \leqq i \leqq 8)$ and so W_i $(1 \leqq i \leqq 10)$ are not conjugate to each other, Q.E.D.

1.5 Groups of order 8

By Vaidyanathaswamy [12] and [13], there is no cyclic subgroup of order 8 in $GL(3, \boldsymbol{Z})$, and clearly there is no quaternion subgroup in $GL(3, \boldsymbol{Z})$. Hence any subgroup of order 8 in $GL(3, \boldsymbol{Z})$ is isomorphic to I) $\boldsymbol{Z}_4 \times \boldsymbol{Z}_2$, II) $\boldsymbol{Z}_2 \times \boldsymbol{Z}_2 \times \boldsymbol{Z}_2$ or III) $\mathfrak{D}_4$.

Proposition 6. *There exist* 6 *non-conjugate abelian and* 8 *non-conjugate non-abelian subgroups of order* 8 *in* $GL(3,\boldsymbol{Z})$:

those isomorphic to $\boldsymbol{Z}_4\times\boldsymbol{Z}_2$

$$W_1=\left\{\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix},\ -\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right\},\ W_2=\left\{\begin{pmatrix}1&0&1\\0&0&-1\\0&1&0\end{pmatrix},\ -\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right\},$$

those isomorphic to $\boldsymbol{Z}_2\times\boldsymbol{Z}_2\times\boldsymbol{Z}_2$

$$W_3=\left\{\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix},\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix},\ -\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right\},$$

$$W_4=\left\{\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix},\begin{pmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{pmatrix},\ -\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right\},$$

$$W_5=\left\{\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix},\begin{pmatrix}-1&0&0\\1&0&-1\\-1&-1&0\end{pmatrix},\ -\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right\},$$

$$W_6=\left\{\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix},\begin{pmatrix}-1&1&-1\\0&0&-1\\0&-1&0\end{pmatrix},\ -\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right\},$$

those isomorphic to $\mathfrak{D}_4$

$$W_7=\left\{\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix},\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}\right\},\ W_8=\left\{\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix},\ -\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}\right\},$$

$$W_9=\left\{-\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix},\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}\right\},\ W_{10}=\left\{-\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix},\ -\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}\right\},$$

$$W_{11}=\left\{\begin{pmatrix}1&0&1\\0&0&-1\\0&1&0\end{pmatrix},\begin{pmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}\right\},\ W_{12}=\left\{\begin{pmatrix}1&0&1\\0&0&-1\\0&1&0\end{pmatrix},\ -\begin{pmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}\right\},$$

$$W_{13}=\left\{-\begin{pmatrix}1&0&1\\0&0&-1\\0&1&0\end{pmatrix},\begin{pmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}\right\},\ W_{14}=\left\{-\begin{pmatrix}1&0&1\\0&0&-1\\0&1&0\end{pmatrix},\ -\begin{pmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}\right\}.$$

Corollary. *In* $SL(3,\boldsymbol{Z})$ *there exist only* 2 *non-conjugate dihedral subgroups of order* 8: W_7, W_{11}, *and there is no abelian subgroup of order* 8.

Proof. Case I) Let $W=\{S,T\}$ be an abelian subgroup of the type $\boldsymbol{Z}_4\times\boldsymbol{Z}_2$ i.e. $S^4=T^2=E,\ ST=TS$.

Case I-1) Suppose that $S=\pm M^{-1}\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}M$, where $M\in GL(3,\boldsymbol{Z})$.

Since $TS=ST$, $MTM^{-1}\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}=\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}MTM^{-1}$. The following lemma can be easily obtained.

LEMMA 10. *Let X be a matrix in $GL(3,\boldsymbol{Z})$.*

(1) *If X commutes with* $\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}$ *or* $\begin{pmatrix}1&0&0\\0&0&1\\0&-1&0\end{pmatrix}$, *then* $X=\pm\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}$, $\pm\begin{pmatrix}1&0&0\\0&0&1\\0&-1&0\end{pmatrix}$, $\pm\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$ *or* $\pm\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$.

(2) *Assume that* $X\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}=\begin{pmatrix}1&0&0\\0&0&1\\0&-1&0\end{pmatrix}X$. *Then* $X=\pm\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}$, $\pm\begin{pmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}$, $\pm\begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$ *or* $\pm\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}$, *all of which have order* 2.

T having order 2, by Lemma 10, $\{S,T\}$ is conjugate to $\left\{\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix},\ -\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}\right\}=\left\{\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix},\ -\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right\}\equiv W_1$.

Case I-2) Suppose now that $S=\pm M^{-1}\begin{pmatrix}1&0&1\\0&0&-1\\0&1&0\end{pmatrix}M$, where $M\in GL(3,\boldsymbol{Z})$. Similarly we have $MTM^{-1}\begin{pmatrix}1&0&1\\0&0&-1\\0&1&0\end{pmatrix}=\begin{pmatrix}1&0&1\\0&0&-1\\0&1&0\end{pmatrix}MTM^{-1}$.

LEMMA 11. *Let X be a matrix in $GL(3,\boldsymbol{Z})$.*

(1) *If X commutes with* $\begin{pmatrix}1&0&1\\0&0&-1\\0&1&0\end{pmatrix}$ *or* $\begin{pmatrix}1&1&0\\0&0&1\\0&-1&0\end{pmatrix}$ *then* $X=\pm\begin{pmatrix}1&0&1\\0&0&-1\\0&1&0\end{pmatrix}$,

$\pm\begin{pmatrix}1&1&1\\0&-1&0\\0&0&-1\end{pmatrix}$, $\pm\begin{pmatrix}1&1&0\\0&0&1\\0&-1&0\end{pmatrix}$ *or* $\pm\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$.

(2) *Assume that* $X\begin{pmatrix}1&0&1\\0&0&-1\\0&1&0\end{pmatrix}=\begin{pmatrix}1&1&0\\0&0&1\\0&-1&0\end{pmatrix}X$. *Then* $X=\pm\begin{pmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}$,

$\pm\begin{pmatrix}-1&-1&-1\\0&0&1\\0&1&0\end{pmatrix}$, $\pm\begin{pmatrix}-1&-1&0\\0&1&0\\0&0&-1\end{pmatrix}$ *or* $\pm\begin{pmatrix}-1&0&-1\\0&-1&0\\0&0&1\end{pmatrix}$, *all of which have order* 2.

Thus $\{S,T\}$ is conjugate to $\left\{\begin{pmatrix}1&0&1\\0&0&-1\\0&1&0\end{pmatrix}, -\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right\}\equiv W_2$. Clearly W_1 is not conjugate to W_2.

Case II) Let $W=\{S,T,R\}$ be an abelian subgroup of the type $\boldsymbol{Z}_2\times\boldsymbol{Z}_2\times\boldsymbol{Z}_2$, i.e. $S^2=T^2=R^2=E$, $ST=TS$, $SR=RS$ and $TR=RT$. Put $V=\{S,T\}$. By Proposition 4, V is conjugate to one of W_i $(5\leqq i\leqq 15)$ in the notation of Proposition 4. Using Lemmas 3 and 4, two equalities $SR=RS$ and $TR=RT$ determine R and so W is conjugate to one of subgroups $W_i(3\leqq i\leqq 6)$ in the notation of Proposition 6. Here W_i $(3\leqq i\leqq 6)$ are not conjugate to each other.

Case III) We determine all non-conjugate dihedral subgroups of order 8, $\mathfrak{D}_4=\{S,T\}$, i.e. $S^4=T^2=(ST)^2=E$.

Case III-1) Assume that $S=\pm M^{-1}\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}M$, where $M\in GL(3,\boldsymbol{Z})$.

Since $TS=S^3T$, $MTM^{-1}\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}=\begin{pmatrix}1&0&0\\0&0&1\\0&-1&0\end{pmatrix}MTM^{-1}$. By Lemma 10,

$\{S,T\}$ is conjugate to $\left\{\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}, \begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}\right\}\equiv W_7$, $\left\{\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix},\right.$

$-\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}\Bigg\} \equiv W_8$, $\Bigg\{-\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}, \begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}\Bigg\} \equiv W_9$ or $\Bigg\{-\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix},$ $-\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}\Bigg\} \equiv W_{10}$. Clearly W_i $(7 \leqq i \leqq 10)$ are not conjugate to each other.

Case III-2) Assume now that $S = \pm M^{-1}\begin{pmatrix}1&0&1\\0&0&-1\\0&1&0\end{pmatrix}M$, where. $M \in GL(3, \boldsymbol{Z})$. Similarly we have $MTM^{-1}\begin{pmatrix}1&0&1\\0&0&-1\\0&1&0\end{pmatrix} = \begin{pmatrix}1&1&0\\0&0&1\\0&-1&0\end{pmatrix}MTM^{-1}$. By Lemma 11, we see that $\{S, T\}$ is conjugate to $\Bigg\{\begin{pmatrix}1&0&1\\0&0&-1\\0&1&0\end{pmatrix}, \begin{pmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}\Bigg\} \equiv W_{11}$, $\Bigg\{\begin{pmatrix}1&0&1\\0&0&-1\\0&1&0\end{pmatrix},$ $-\begin{pmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}\Bigg\} \equiv W_{12}$, $\Bigg\{-\begin{pmatrix}1&0&1\\0&0&-1\\0&1&0\end{pmatrix}, \begin{pmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}\Bigg\} \equiv W_{13}$ or $\Bigg\{-\begin{pmatrix}1&0&1\\0&0&-1\\0&1&0\end{pmatrix},$ $-\begin{pmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}\Bigg\} \equiv W_{14}$. Here W_i $(11 \leqq i \leqq 14)$ and hence W_i $(7 \leqq i \leqq 14)$ are not conjugate to each other. Thus the proof of Proposition 6 is complete, Q.E.D.

Using Lemmas 8 and 9, we know that there exists no subgroup of order 9 in $GL(3, \boldsymbol{Z})$. Hence the order of any finite subgroup of $GL(3, \boldsymbol{Z})$ is of the form $2^i \cdot 3^j$ and $j \leqq 1$. From now on, we have only to consider finite subgroups of order 2^i or $2^i 3$ in $GL(3, \boldsymbol{Z})$.

1.6 Groups of order 12

Any abstract groups of order 12, all of whose elements have order 1, 2, 3, 4 or 6, is isomorphic to I) $\boldsymbol{Z}_3 \times \boldsymbol{Z}_2 \times \boldsymbol{Z}_2 = \boldsymbol{Z}_6 \times \boldsymbol{Z}_2$, II) $\mathfrak{D}_6 = \mathfrak{S}_3 \times \boldsymbol{Z}_2$, III) $\mathfrak{A}_4$ or IV) $\langle 2, 2, 3 \rangle$.

PROPOSITION 7. *There exist* 11 *non-conjugate subgroups of order* 12 *in* $GL(3, \boldsymbol{Z})$: *those isomorphic to* $\boldsymbol{Z}_6 \times \boldsymbol{Z}_2$

$$W_1=\left\{\begin{pmatrix}1&0&0\\0&0&-1\\0&1&1\end{pmatrix},\ -\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right\},$$

those isomorphic to $\mathfrak{D}_6$

$$W_2=\left\{\begin{pmatrix}1&0&0\\0&0&-1\\0&1&1\end{pmatrix},\ \begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}\right\},\ W_3=\left\{\begin{pmatrix}1&0&0\\0&0&-1\\0&1&1\end{pmatrix},\ -\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}\right\},$$

$$W_4=\left\{-\begin{pmatrix}1&0&0\\0&0&-1\\0&1&1\end{pmatrix},\ \begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}\right\},\ W_5=\left\{-\begin{pmatrix}1&0&0\\0&0&-1\\0&1&1\end{pmatrix},\ -\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}\right\},$$

$$W_6=\left\{-\begin{pmatrix}1&0&0\\0&0&-1\\0&1&-1\end{pmatrix},\ \begin{pmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}\right\},\ W_7=\left\{-\begin{pmatrix}1&0&0\\0&0&-1\\0&1&-1\end{pmatrix},\ \begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}\right\},$$

$$W_8=\left\{-\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix},\ \begin{pmatrix}0&0&-1\\0&-1&0\\-1&0&0\end{pmatrix}\right\},$$

those isomorphic to $\mathfrak{A}_4$

$$W_9=\left\{\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix},\ \begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix}\right\},\ W_{10}=\left\{\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix},\ \begin{pmatrix}0&-1&1\\0&-1&0\\1&-1&0\end{pmatrix}\right\},$$

$$W_{11}=\left\{\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix},\ \begin{pmatrix}-1&-1&-1\\0&0&1\\0&1&0\end{pmatrix}\right\}.$$

COROLLARY. *In* $SL(3,\boldsymbol{Z})$ *there exist only* 4 *non-conjugate subgroups of order* 12: W_2, W_9, W_{10}, W_{11}, *and there is no abelian subgroup of order* 12.[5)]

Proof. Case I) Let $W=\{S,T\}$ be an abelian subgroup of the type $\boldsymbol{Z}_6\times\boldsymbol{Z}_2$ i.e. $S^6=T^2=E$ and $ST=TS$. Denote by V the subgroup generated by S. By Proposition 5, V is conjugate to W_1, W_2, W_3 or W_4 in the notation of Proposition 5.

Case I-1) Assume that $V=M^{-1}\left\{\pm\begin{pmatrix}1&0&0\\0&0&-1\\0&1&1\end{pmatrix}\right\}M$, where $M\in GL(3,\boldsymbol{Z})$.

5) *See* Dade [3] Theorem 3, p. 27.

Since W is commutative, $MTM^{-1}\begin{pmatrix}1&0&0\\0&0&-1\\0&1&1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&0&-1\\0&1&1\end{pmatrix}MTM^{-1}$. The proof of the following is immediate.

LEMMA 12. *Let X be a matrix in $GL(3,\boldsymbol{Z})$.*

(1) *If X commutes with* $\begin{pmatrix}1&0&0\\0&0&-1\\0&1&1\end{pmatrix}$ *or* $\begin{pmatrix}1&0&0\\0&1&1\\0&-1&0\end{pmatrix}$, *then* $X=\pm\begin{pmatrix}1&0&0\\0&0&-1\\0&1&1\end{pmatrix}$, $\pm\begin{pmatrix}1&0&0\\0&-1&-1\\0&1&0\end{pmatrix}$, $\pm\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$, $\pm\begin{pmatrix}1&0&0\\0&0&1\\0&-1&-1\end{pmatrix}$, $\pm\begin{pmatrix}1&0&0\\0&1&1\\0&-1&0\end{pmatrix}$ *or* $\pm\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$.

(2) *Assume that* $X\begin{pmatrix}1&0&0\\0&0&-1\\0&1&1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&1\\0&-1&0\end{pmatrix}X$. *Then* $X=\pm\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}$, $\pm\begin{pmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}$, $\pm\begin{pmatrix}-1&0&0\\0&1&0\\0&-1&-1\end{pmatrix}$, $\pm\begin{pmatrix}-1&0&0\\0&-1&0\\0&1&1\end{pmatrix}$, $\pm\begin{pmatrix}-1&0&0\\0&1&1\\0&0&-1\end{pmatrix}$ *or* $\pm\begin{pmatrix}-1&0&0\\0&-1&-1\\0&0&1\end{pmatrix}$, *all of which have order* 2.

By the above lemma, W is conjugate to $\left\{\begin{pmatrix}1&0&0\\0&0&-1\\0&1&1\end{pmatrix}, -\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right\}\equiv W_1$.

Case I-2) Assume now that $V=M^{-1}\left\{-\begin{pmatrix}1&0&0\\0&0&-1\\0&1&-1\end{pmatrix}\right\}M$, where $M\in GL(3,\boldsymbol{Z})$. Similarly we have $MTM^{-1}\begin{pmatrix}1&0&0\\0&0&-1\\0&1&-1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&0&-1\\0&1&-1\end{pmatrix}MTM^{-1}$. By Lemma 8, W is conjugate to $\left\{-\begin{pmatrix}1&0&0\\0&0&-1\\0&1&-1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}\right\}\sim W_1$.

Case I-3) Assume that $V=M^{-1}\left\{-\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\right\}M$, where $M\in GL(3,\boldsymbol{Z})$.

Then $MTM^{-1}\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}MTM^{-1}$. By Lemma 9, there is no such subgroup W in $GL(3,\boldsymbol{Z})$.

Case II) We determine all non-conjugate dihedral subgroups of the type $\mathfrak{D}_6$ in $GL(3,\boldsymbol{Z})$. Let S and T be generators of such a subgroup. Then $S^6=T^2=(ST)^2=E$.

Case II-1) Assume that $S=\pm M^{-1}\begin{pmatrix}1&0&0\\0&0&-1\\0&1&1\end{pmatrix}M$, where $M\in GL(3,\boldsymbol{Z})$.

Since $TS=S^5T$, it follows that $MTM^{-1}\begin{pmatrix}1&0&0\\0&0&-1\\0&1&1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&1\\0&-1&0\end{pmatrix}MTM^{-1}$. By Lemma 12, $\{S,T\}$ is conjugate to $\left\{\begin{pmatrix}1&0&0\\0&0&-1\\0&1&1\end{pmatrix},\ \begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}\right\}\equiv W_2$,
$\left\{\begin{pmatrix}1&0&0\\0&0&-1\\0&1&1\end{pmatrix},\ -\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}\right\}\equiv W_3$, $\left\{-\begin{pmatrix}1&0&0\\0&0&-1\\0&1&1\end{pmatrix},\ \begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}\right\}\equiv W_4$ or
$\left\{-\begin{pmatrix}1&0&0\\0&0&-1\\0&1&1\end{pmatrix},\ -\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}\right\}\equiv W_5$. Clearly W_3 is not conjugate to W_4 or W_5, and using Lemma 12, we can show that W_4 is not conjugate to W_5 and hence W_i $(2\leqq i\leqq 5)$ are not conjugate to each other.

Case II-2) Assume that $S=-M^{-1}\begin{pmatrix}1&0&0\\0&0&-1\\0&1&-1\end{pmatrix}M$, where $M\in GL(3,\boldsymbol{Z})$.

Similarly $MTM^{-1}\begin{pmatrix}1&0&0\\0&0&-1\\0&1&-1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&-1&1\\0&-1&0\end{pmatrix}MTM^{-1}$. By Lemma 8, $\{S,T\}$ is conjugate to $\left\{-\begin{pmatrix}1&0&0\\0&0&-1\\0&1&-1\end{pmatrix},\ \begin{pmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}\right\}\equiv W_6$, $\left\{-\begin{pmatrix}1&0&0\\0&0&-1\\0&1&-1\end{pmatrix},\ \begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}\right\}\equiv W_7$.
Using Lemma 8, we see that W_6 is not conjugate to W_7.

Case II-3) Assume that $S=-M^{-1}\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}M$, where $M\in GL(3,\boldsymbol{Z})$.

Then we have $MTM^{-1}\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix}=\begin{pmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}MTM^{-1}$. By Lemma 9, $\{S,T\}$ is conjugate to $\left\{-\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & -1\\ 0 & -1 & 0\\ -1 & 0 & 0\end{pmatrix}\right\}\equiv W_8$. Here W_i $(2\leqq i\leqq 8)$ are not conjugate to each other.

Case III) There are 3 non-conjugate subgroups W_9, W_{10} and W_{11} isomorphic to $\mathfrak{A}_4$. We refer the reader to Nazarova [8].

Case IV) We show that there is no subgroup of the type $\langle 2, 2, 3\rangle$ in $GL(3,\boldsymbol{Z})$. Let W be such a subgroup and let S, T be generators of this subgroup. Then $S^3=T^2=(ST)^2$ and so $S^6=T^4=E$. Hence by Proposition 5, $S=M^{-1}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 1\end{pmatrix}M$, where $M\in GL(3,\boldsymbol{Z})$. Since $TS=S^5T$, this implies that

$$MTM^{-1}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 1\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & -1 & 0\end{pmatrix}MTM^{-1},$$

Then by Lemma 12, there is no such matrix T in $GL(3,\boldsymbol{Z})$. This establishes the proof of this proposition, Q.E.D.

1.7 Groups of order 16

By Corollary to Proposition 6, there is no abelian subgroup of order 16 in $GL(3,\boldsymbol{Z})$. We now show that there exists no non-abelian subgroup of order 16 in $SL(3,\boldsymbol{Z})$.

An abstract non-abelian group of order 16, all of whose elements are of order 1, 2, 3, 4 or 6, is isomorphic to I) $\mathfrak{D}_4\times\boldsymbol{Z}_2$, II) $\mathfrak{Q}\times\boldsymbol{Z}_2$, III) $\langle 2,2|4,2\rangle$, IV) $(4,4|2,2)$ or V) $\mathfrak{R}$. We have the following:

Proposition 8. *There exist* 2 *non-conjugate subgroups of order* 16 *in* $GL(3,\boldsymbol{Z})$, *which are isomorphic to* $\mathfrak{D}_4\times\boldsymbol{Z}_2$.

$$W_1=\left\{\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}, \begin{pmatrix}-1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}, -\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right\},$$

$$W_2 = \left\{\begin{pmatrix}1 & 0 & 1\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}, \begin{pmatrix}-1 & 0 & 0\\ 0 & 0 & -1\\ 0 & -1 & 0\end{pmatrix}, -\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right\}.$$

COROLLARY. *In $SL(3,\boldsymbol{Z})$ there is no subgroup of order* 16.

Proof. Case I) Let W be a subgroup of the type $\mathfrak{D}_4\times\boldsymbol{Z}_2$. By Proposition 6, $\mathfrak{D}_4$ is conjugate to W_i ($7\leqq i\leqq 14$) in the notation of Proposition 6. Let T be a generator of $\boldsymbol{Z}_2$. Suppose $\mathfrak{D}_4 = M^{-1}W_iM$ ($7\leqq i\leqq 10$), where $M\in GL(3,\boldsymbol{Z})$. Then MTM^{-1} commutes with $\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}$ and $\begin{pmatrix}-1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$.

By Lemmas 4 and 10, W is conjugate to $\left\{\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}, \begin{pmatrix}-1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix},\right.$
$\left.-\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}\right\} = \left\{\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}, \begin{pmatrix}-1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} - \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right\} \equiv W_1$. Suppose $\mathfrak{D}_4=M^{-1}W_iM$ ($11\leqq i\leqq 14$), where $M\in GL(3,\boldsymbol{Z})$. Similarly using Lemma 11, we see that W is conjugate to $\left\{\begin{pmatrix}1 & 0 & 1\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}, \begin{pmatrix}-1 & 0 & 0\\ 0 & 0 & -1\\ 0 & -1 & 0\end{pmatrix}, -\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right\} \equiv W_2$.
Here W_2 is not conjugate to W_1.

Case II) Since there is no quaternion subgroup of order 8 in $GL(3,\boldsymbol{Z})$, there exists no subgroup of the type $\mathfrak{Q}\times\boldsymbol{Z}_2$.

Case III) Let $W = \{S,T\}$ be a subgroup of the type $\langle 2, 2|4, 2\rangle$, then $S^4 = T^4 = E$ and $T^{-1}ST = S^3$. First assume that $S = \pm M^{-1}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}M$, where $M\in GL(3,\boldsymbol{Z})$. $T^{-1}ST = S^3$ implies that $MT^{-1}M^{-1}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix} =$ $\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{pmatrix}MT^{-1}M^{-1}$. By Lemma 10, these matrices have all order 2 and so there is no such matrix T. Secondly assume that $S=\pm M^{-1}\begin{pmatrix}1 & 0 & 1\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}M$, where $M \in GL(3, \boldsymbol{Z})$. Similarly there is no such matrix T that S and T generate this subgroup. Thus there exists no subgroup of the type $\langle 2,2|4,2\rangle$ in $GL(3, \boldsymbol{Z})$.

Case IV) Let $W = \{S, T\}$ be a subgroup of the type $(4,4|2,2)$, then $S^4 = T^4 = (ST)^2 = (S^{-1}T)^2 = E$.

Case IV-1) Assume that $T = \pm M^{-1}\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}M$, where $M \in GL(3, \boldsymbol{Z})$.

Since $S^2T^3 = T^3S^2$, it follows that $MS^2M^{-1}\begin{pmatrix}1&0&0\\0&0&1\\0&-1&0\end{pmatrix} = \begin{pmatrix}1&0&0\\0&0&1\\0&-1&0\end{pmatrix} MS^2M^{-1}$.

By Lemma 10, $(MSM^{-1})^2 = \begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$ since $S^4 = E$. Further by Lemma 1, $S = \pm M^{-1}\begin{pmatrix}1&0&0\\0&a&b\\0&c&-a\end{pmatrix}M$, where $a^2 + bc + 1 = 0$. On the other hand, $ST = \pm M^{-1}\begin{pmatrix}1&0&0\\0&b&-a\\0&-a&-c\end{pmatrix}M$ has order 2 and so $S = \pm M^{-1}\begin{pmatrix}1&0&0\\0&0&1\\0&-1&0\end{pmatrix}M$ or $\pm M^{-1}\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}M$. Thus such a subgroup $\{S, T\}$ does not have order 16.

Case IV-2) Assume that $T = \pm M^{-1}\begin{pmatrix}1&0&1\\0&0&-1\\0&1&0\end{pmatrix}M$, where $M \in GL(3, \boldsymbol{Z})$.

In the same way as above, $MS^2M^{-1}\begin{pmatrix}1&1&0\\0&0&1\\0&-1&0\end{pmatrix} = \begin{pmatrix}1&1&0\\0&0&1\\0&-1&0\end{pmatrix} = MS^2M^{-1}$.

By Lemma 11, $(MSM^{-1})^2 = \begin{pmatrix}1&1&1\\0&-1&0\\0&0&-1\end{pmatrix}$. By easy calculations $(ST)^2 = E$ implies $S = \pm M^{-1}\begin{pmatrix}1&0&1\\0&0&-1\\0&1&0\end{pmatrix}M$. Hence $\{S, T\}$ does not have order 16. Thus there exists no subgroup of the type $(4,4|2,2)$ in $GL(3, \boldsymbol{Z})$.

Case V) Let $W = \{R, S, T\}$ be a subgroup of the type $\mathfrak{R}$, i.e. $R^2 = S^2 = T^2 = E$ and $RST = STR = TRS$.

Case V-1) Assume that $R = \pm M^{-1}\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}M$, where $M \in GL(3,\boldsymbol{Z})$.

Since $(ST)R = R(ST)$, it follows that

$$M(ST)M^{-1}\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}M(ST)M^{-1}.$$

By Lemma 1,

$$M(ST)M^{-1} = \pm\begin{pmatrix}1 & 0 & 0\\ 0 & x_{22} & x_{23}\\ 0 & x_{32} & x_{33}\end{pmatrix}$$

where $x_{22}x_{33} - x_{23}x_{32} = 1$. Since ST has order 4, $x_{33} = -x_{22}$. $RST = TRS$ implies that

$$MSM^{-1}\begin{pmatrix}1 & 0 & 0\\ 0 & -x_{22} & -x_{23}\\ 0 & -x_{32} & x_{22}\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & -x_{22} & -x_{23}\\ 0 & -x_{32} & x_{22}\end{pmatrix}MSM^{-1}.$$

On the other hand $T^2 = E$ implies that

$$MSM^{-1}\begin{pmatrix}1 & 0 & 0\\ 0 & x_{22} & x_{23}\\ 0 & x_{32} & -x_{22}\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & -x_{22} & -x_{23}\\ 0 & -x_{32} & x_{22}\end{pmatrix}MSM^{-1},$$

which is a contradiction.

Case V-2) Assume that $R = \pm M^{-1}\begin{pmatrix}-1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}M$, where $M \in GL(3,\boldsymbol{Z})$.

Similarly using Lemma 2 we have a contradiction.

Thus there is no subgroup of this type in $GL(3,\boldsymbol{Z})$. We complete the proof of Proposition 8 and its Corollary, Q.E.D.

By Corollary to Proposition 8, the order of any finite subgroup of $GL(3,\boldsymbol{Z})$ (resp. $SL(3,\boldsymbol{Z})$) is of the form $2^i \cdot 3^j$ and $j \leqq 1$ and $i \leqq 4$ (resp. $i \leqq 3$).

1.8 Groups of order 24

By Corollary to Proposition 6 and Corollary to Proposition 7, there is no abelian subgroup of order 24 in $GL(3,\boldsymbol{Z})$. A non-abelian abstract group of order 24, all of whose elements are of order 1, 2, 3, 4 or 6, is isomorphic to I) $\mathfrak{A}_4\times\boldsymbol{Z}_2$, II) $\langle 2,2,3\rangle\times\boldsymbol{Z}_2$, III) $\mathfrak{D}_6\times\boldsymbol{Z}_2$, IV) $\mathfrak{S}_4$, V) $\langle 2, 3, 3\rangle$ or VI) $(4,6\,|\,2,2)$. We have

PROPOSITION 9. *There exist* 11 *non-conjugate subgroups of order* 24 *in* $GL(3,\boldsymbol{Z})$, *all of which are non-abelian:*

those isomorphic to $\mathfrak{A}_4\times\boldsymbol{Z}_2$

$$W_1=\left\{\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix},\begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix},-\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right\},$$

$$W_2=\left\{\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix},\begin{pmatrix}0&-1&1\\0&-1&0\\1&-1&0\end{pmatrix},-\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right\},$$

$$W_3=\left\{\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix},\begin{pmatrix}-1&-1&-1\\0&0&1\\0&1&0\end{pmatrix},-\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right\}.$$

those isomorphic to $\mathfrak{D}_6\times\boldsymbol{Z}_2$

$$W_4=\left\{\begin{pmatrix}1&0&0\\0&0&-1\\0&1&1\end{pmatrix},\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix},-\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right\},$$

$$W_5=\left\{\begin{pmatrix}1&0&0\\0&0&-1\\0&1&-1\end{pmatrix},\begin{pmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{pmatrix},-\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right\},$$

those isomorphic to $\mathfrak{S}_4$

$$W_6=\left\{\begin{pmatrix}0&0&1\\0&1&0\\-1&0&0\end{pmatrix},\begin{pmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}\right\},\quad W_7=\left\{-\begin{pmatrix}0&0&1\\0&1&0\\-1&0&0\end{pmatrix},-\begin{pmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}\right\},$$

$$W_8=\left\{\begin{pmatrix}0&-1&0\\1&1&1\\-1&0&0\end{pmatrix},\begin{pmatrix}-1&-1&0\\0&1&0\\0&0&-1\end{pmatrix}\right\},\quad W_9=\left\{-\begin{pmatrix}0&-1&0\\1&1&1\\-1&0&0\end{pmatrix},-\begin{pmatrix}-1&-1&0\\0&1&0\\0&0&-1\end{pmatrix}\right\},$$

$$W_{10}=\left\{\begin{pmatrix}1&1&0\\-2&-1&-1\\0&0&1\end{pmatrix},\begin{pmatrix}-1&-1&-1\\0&0&1\\0&1&0\end{pmatrix}\right\},\quad W_{11}=\left\{-\begin{pmatrix}1&1&0\\-2&-1&-1\\0&0&1\end{pmatrix},-\begin{pmatrix}1&-1&-1\\0&0&1\\0&1&0\end{pmatrix}\right\}.$$

COROLLARY. *In* $SL(3,\boldsymbol{Z})$ *there are only* 3 *non-conjugate subgroups* W_6, W_8 *and* W_{10}, *all of which are isomorphic to* $\mathfrak{S}_4$.

Proof. Case I) Suppose $W = \mathfrak{A}_4 \times \boldsymbol{Z}_2$, where $\mathfrak{A}_4$ is an alternating subgroup of degree 4 and $\boldsymbol{Z}_2 = \{R\}$ is a subgroup of order 2 in $GL(3,\boldsymbol{Z})$. By Proposition 7, $\mathfrak{A}_4$ is conjugate to W_i $(i = 9, 10, 11)$ in the notation of Proposition 7. Then

$$MRM^{-1}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} MRM^{-1},$$

where $M \in GL(3,\boldsymbol{Z})$. By Lemma 9, $W = \mathfrak{A}_4 \times \boldsymbol{Z}_2$ is conjugate to $\{W_9, -E\} \equiv W_1$, $\{W_{10}, -E\} \equiv W_2$ or $\{W_{11}, -E\} \equiv W_3$. Clearly W_i $(i = 1, 2, 3)$ are not conjugate to each other.

Case II) Since there is no subgroup isomorphic to $\langle 2,2,3 \rangle$ in $GL(3,\boldsymbol{Z})$ by Proposition 7, there is no subgroup isomorphic to $\langle 2,2,3 \rangle \times \boldsymbol{Z}_2$.

Case III) Let $W = \mathfrak{D}_6 \times \boldsymbol{Z}_2$ be the direct product of a dihedral subgroup $\mathfrak{D}_6$ of order 12 and a subgroup $\boldsymbol{Z}_2 = \{R\}$ in $GL(3,\boldsymbol{Z})$. By Proposition 7, $\mathfrak{D}_6$ is conjugate to W_i $(2 \leqq i \leqq 8)$ in the notation of Proposition 7. First assume that $\mathfrak{D}_6 = M^{-1}W_iM$ $(2 \leqq i \leqq 5)$, where $M \in GL(3,\boldsymbol{Z})$. Then it follows that

$$MRM^{-1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} MRM^{-1}.$$

By Lemma 12, $\mathfrak{D}_6 \times \boldsymbol{Z}_2$ is conjugate to

$$\left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\} \equiv W_4.$$

Next assume that $\mathfrak{D}_6 = M^{-1}W_iM$ $(i = 6, 7)$, where $M \in GL(3,\boldsymbol{Z})$. Similarly we see that $W = \mathfrak{D}_6 \times \boldsymbol{Z}_2$ is conjugate to $\left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}$ $\equiv W_5$, and W_5 is not conjugate to W_4. Finally assume that $\mathfrak{D}_6 = M^{-1}W_8M$, where $M \in GL(3,\boldsymbol{Z})$. Using Lemma 9 we see that there is no such subgroup.

Case IV) Let $W = \{S, T\}$ be a symmetric subgroup of degree 4, then $S^4 = T^2 = (ST)^3 = E$. Denote by V the subgroup generated by S^2T and T. Then V is a dihedral subgroup of order 8, and by Proposition 6, V is conjugate to W_i $(7 \leqq i \leqq 14)$ in the notation of Proposition 6. We show that

W is conjugate to W_6, W_7 or W_8, W_9, W_{10}, W_{11} in the notation of Proposition 9, according as $V \sim W_i$ $(7 \leqq i \leqq 10)$ or W_i $(11 \leqq i \leqq 14)$. For example, we prove that if V is conjugate to W_i $(7 \leqq i \leqq 10)$, W is so to W_6, W_7. The other cases can be proved similarly. Suppose that $V = M^{-1}W_iM$ $(7 \leqq i \leqq 10)$. By the structure of these subgroups

$$S^2T = \pm M^{-1}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}M \text{ or } \pm M^{-1}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{pmatrix}M.$$

If $S^2T = \pm M^{-1}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}M$, it follows that $T = \pm M^{-1}\begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}M$, $\pm M^{-1}\begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}M$, $\pm M^{-1}\begin{pmatrix}-1 & 0 & 0\\ 0 & 0 & -1\\ 0 & -1 & 0\end{pmatrix}M$ or $\pm M^{-1}\begin{pmatrix}-1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}M$, hence $S^2 = M^{-1}\begin{pmatrix}-1 & 0 & 0\\ 0 & 0 & -1\\ 0 & -1 & 0\end{pmatrix}M$, $M^{-1}\begin{pmatrix}-1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}M$, $M^{-1}\begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}M$ or $M^{-1}\begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}M$, respectively. By Lemmas 1 and 2,

$$S = \pm M^{-1}\begin{pmatrix}a & b & b\\ -\frac{1+a^2}{2b} & \frac{1-a}{2} & -\frac{1+a}{2}\\ -\frac{1+a^2}{2b} & -\frac{1+a}{2} & \frac{1-a}{2}\end{pmatrix}M,\ \pm M^{-1}\begin{pmatrix}a & b & -b\\ -\frac{1+a^2}{2b} & \frac{1-a}{2} & \frac{1+a}{2}\\ \frac{1+a^2}{2b} & \frac{1+a}{2} & \frac{1-a}{2}\end{pmatrix}M,$$

$$\pm M^{-1}\begin{pmatrix}a & 0 & b\\ 0 & 1 & 0\\ c & 0 & -a\end{pmatrix}M \text{ or } \pm M^{-1}\begin{pmatrix}a & b & 0\\ c & -a & 0\\ 0 & 0 & 1\end{pmatrix}M.$$

$(ST)^3 = E$ implies that $S = \pm M^{-1}\begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ -1 & 0 & 0\end{pmatrix}M$, $\pm M^{-1}\begin{pmatrix}0 & 0 & -1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix}M$ or $\pm M^{-1}\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}M$, $\pm M^{-1}\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}M$, only if $T = \pm M^{-1}\begin{pmatrix}-1 & 0 & 0\\ 0 & 0 & -1\\ 0 & -1 & 0\end{pmatrix}M$ or $\pm M^{-1}\begin{pmatrix}-1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}M$. Thus $\{S, T\}$ is conjugate to $\left\{\begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ -1 & 0 & 0\end{pmatrix}\right.$,

$$\left.\begin{pmatrix}-1 & 0 & 0\\ 0 & 0 & -1\\ 0 & -1 & 0\end{pmatrix}\right\}\equiv W_6 \text{ or } \left\{\begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ -1 & 0 & 0\end{pmatrix}, -\begin{pmatrix}-1 & 0 & 0\\ 0 & 0 & -1\\ 0 & -1 & 0\end{pmatrix}\right\}\equiv W_7.$$ If $S^2T = M^{-1} \times \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{pmatrix}M$, similarly $\{S, T\}$ is conjugate to W_6 or W_7. Secondly Suppose that $V = M^{-1}W_iM$ $(11 \leqq i \leqq 14)$. In the same way as above, $\{S, T\}$ is conjugate to

$$\left\{\begin{pmatrix}0 & -1 & 0\\ 1 & 1 & 1\\ -1 & 0 & 0\end{pmatrix}, \begin{pmatrix}-1 & -1 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}\right\}\equiv W_8, \quad \left\{\begin{pmatrix}1 & 1 & 0\\ -2 & -1 & -1\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}-1 & -1 & -1\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\right\}\equiv W_9,$$

$$\left\{-\begin{pmatrix}0 & -1 & 0\\ 1 & 1 & 1\\ -1 & 0 & 0\end{pmatrix}, -\begin{pmatrix}-1 & -1 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}\right\}\equiv W_{10} \text{ or } \left\{-\begin{pmatrix}1 & 1 & 0\\ -2 & -1 & -1\\ 0 & 0 & 1\end{pmatrix}, -\begin{pmatrix}-1 & -1 & -1\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\right\}\equiv W_{11}.$$

Trivially, W_6 is not conjugate to W_8 and easy calculations show that W_8 is not so to W_9. Hence W_i $(6 \leqq i \leqq 11)$ are not conjugate to each other.

Case V) We consider the fifth subgroup i.e. a subgroup of the type $\langle 2,3,3\rangle$. Denote by S, T generators of such a subgroup. Then $S^3=T^3=(ST)^2$ and so $S^6 = T^6 = (ST)^4 = E$. By Proposition 5, $T = M^{-1}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 1\end{pmatrix}M$, where $M\in GL(3, \boldsymbol{Z})$. Since $T^3=(ST)^2$, Lemma 1 implies that $M(ST)M^{-1}=\begin{pmatrix}1 & 0 & 0\\ 0 & a & b\\ 0 & c & -a\end{pmatrix}$, where $a^2 + bc + 1 = 0$ and so $S = M^{-1}\begin{pmatrix}1 & 0 & 0\\ 0 & a-b & a\\ 0 & a+c & c\end{pmatrix}M$, which does not have order 4. Thus there is no subgroup of the type $\langle 2,3,3\rangle$ in $GL(3, \boldsymbol{Z})$.

Case VI) Finally let $W = \{S, T\}$ be a subgroup of the type $(4,6|2,2)$. Then $S^4 = T^6 = (ST)^2 = (S^{-1}T)^2 = E$. By Proposition 5, we have three cases.

Case VI-1) Assume that $T = \pm M^{-1}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 1\end{pmatrix}M$, where $M\in GL(3, \boldsymbol{Z})$.

Since $S^2T^5 = T^5S^2$, MS^2M^{-1} commutes with $\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & -1 & 0\end{pmatrix}$, and so by Lemma

12, $MS^2M^{-1} = (MSM^{-1})^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Moreover by Lemma 1, $S = \pm M^{-1} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & -a \end{pmatrix} M$, where $a^2 + bc + 1 = 0$. But for these S, $(ST)^2 \neq E$.

Case VI-2) Assume that $T = -M^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} M$, where $M \in GL(3, \boldsymbol{Z})$. In the same way as above, MS^2M^{-1} commutes with $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$ and so by Lemma 8, $(MSM^{-1})^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Lemma 1 implies that $S = \pm M^{-1} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & -a \end{pmatrix} M$, where $a^2 + bc + 1 = 0$. But for these, $(ST)^2 \neq E$.

Case VI-3) Assume that $T = -M^{-1} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} M$, where $M \in GL(3, \boldsymbol{Z})$. Then MS^2M^{-1} commutes with $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, and lemma 9 shows that S^2 does not have order 2. Hence there exists no subgroup of the type $(4, 6 | 2, 2)$ in $GL(3, \boldsymbol{Z})$.

Thus the proof of the proposition is complete, Q.E.D.

1.9 Groups of order 48

By Corollary to Proposition 8, there is no subgroup of order 48 in $SL(3, \boldsymbol{Z})$. Hence a subgroup of $GL(3, \boldsymbol{Z})$ of order 48 is generated by a subgroup of order 24 in $SL(3, \boldsymbol{Z})$ and a matrix of determinant -1.

PROPOSITION 10. *There exist 3 non-conjugate subgroups of order 48 in $GL(3, \boldsymbol{Z})$:*

$$W_1 = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\},$$

$$W_2 = \left\{ \begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\},$$

$$W_3 = \left\{ \begin{pmatrix} 1 & 1 & 0 \\ -2 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

And there is no subgroup of order more than 48 *in* $GL(3,\boldsymbol{Z})$.

COROLLARY. *In* $SL(3,\boldsymbol{Z})$ *there is no subgroup of order* 48 *or more.*

Proof. Let W be a subgroup of order 48 in $GL(3,\boldsymbol{Z})$, and let V be the subgroup consisting of all elements with determinant 1. By Corollary to Proposition 9, V is conjugate to W_6, W_8 or W_{10} in the notation of Proposition 9. We see that W is conjugate to $\{W_6, -E\} \equiv W_1$, $\{W_8, -E\} \equiv W_2$ and $\{W_{10}, -E\} \equiv W_3$ according as V is so to W_6, W_8 and W_{10}. For example, we show that, if V is conjugate to W_6, then W is so to W_1. Assume that $V = M^{-1}W_6M$, where $M \in GL(3,\boldsymbol{Z})$, and denote by R such an element that generate W together with V. Suppose that $R(M^{-1}SM) = (M^{-1}S'M)R$, where $S = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$ and $S' \in W_6$. Then $(MRM^{-1})S = S'(MRM^{-1})$. By the structure of the subgroup W_6, $S' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$. MRM^{-1} is determined by the following easy lemma:

LEMMA 13. *Let* X *be a matrix in* $GL(3,\boldsymbol{Z})$.

(1) *If* X *commutes with* $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, *then* $X = \pm\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \pm\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $\pm\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ *or* $\pm\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(2) *Assume that* $X\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} X$. *Then* $X = \pm\begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, $\pm\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $\pm\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *or* $\pm\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

(3) *Assume that* $X\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} X$. *Then* $X = \pm\begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $\pm\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $\pm\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ *or* $\pm\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

(4) *Assume that* $X\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} X$. *Then* $X = \pm\begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$, $\pm\begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$, $\pm\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$ *or* $\pm\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$.

(5) *Assume that* $X\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} X$. *Then* $X = \pm\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}$, $\pm\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $\pm\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *or* $\pm\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

(6) *Assume that* $X\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} X$. *Then* $X = \pm\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$, $\pm\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$, $\pm\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *or* $\pm\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

Hence by the above lemma, in all case R is contained in V and so $W\sim\{W_6, -E\} \equiv W_1$. For W_8 and W_{10}, we need the following two lemmas:

LEMMA 14. *Let X be a matrix in $GL(3, \boldsymbol{Z})$.*

(1) *If X commutes with* $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix}$, *then* $X = \pm\begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix}$, $\pm\begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$, $\pm\begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ *or* $\pm\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(2) *Assume that* $X\begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}X$. *Then* $X = \pm\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix}$, $\pm\begin{pmatrix} 0 & 0 & 1 \\ -1 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix}$, $\pm\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ *or* $\pm\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$.

(3) *Assume that* $X\begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}X$. *Then* $X = \pm\begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$, $\pm\begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & -1 \\ -1 & 0 & 0 \end{pmatrix}$, $\pm\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ *or* $\pm\begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(4) *Assume that* $X\begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}X$. *Then* $X = \pm\begin{pmatrix} -1 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $\pm\begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix}$, $\pm\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$ *or* $\pm\begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

(5) *Assume that* $X\begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}X$. *Then* $X = \pm\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$, $\pm\begin{pmatrix} 0 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, $\pm\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$ *or* $\pm\begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

(6) *Assume that* $X\begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}X$. *Then* $X = \pm\begin{pmatrix} -1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, $\pm\begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$, $\pm\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ *or* $\pm\begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$.

LEMMA 15. *Let X be a matrix in $GL(3,\boldsymbol{Z})$.*

(1) *If X commutes with* $\begin{pmatrix} 1 & 1 & 0 \\ -2 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$, *then* $X=\pm\begin{pmatrix} 1 & 1 & 0 \\ -2 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$, $\pm\begin{pmatrix} -1 & -1 & -1 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $\pm\begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *or* $\pm\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(2) *Assume that* $X\begin{pmatrix} 1 & 1 & 0 \\ -2 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}=\begin{pmatrix} -1 & -1 & -1 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}X$. *Then* $X=\pm\begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $\pm\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $\pm\begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$ *or* $\pm\begin{pmatrix} 1 & 0 & 1 \\ -2 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}$.

(3) *Assume that* $X\begin{pmatrix} 1 & 1 & 0 \\ -2 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -2 & -1 & -1 \end{pmatrix}X$. *Then* $X=\pm\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$, $\pm\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$.

(4) *Assume that* $X\begin{pmatrix} 1 & 1 & 0 \\ -2 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}=\begin{pmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}X$. *Then* $X=\pm\begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix}$, $\pm\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -2 & -1 & -1 \end{pmatrix}$, $\pm\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$ *or* $\pm\begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

(5) *Assume that* $X\begin{pmatrix} 1 & 1 & 0 \\ -2 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}X$. *Then* $X=\pm\begin{pmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}$, $\pm\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ -2 & -1 & -1 \end{pmatrix}$.

(6) *Assume that* $X\begin{pmatrix} 1 & 1 & 0 \\ -2 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}X$. *Then* $X=\pm\begin{pmatrix} -1 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$,

$$\pm\begin{pmatrix} 1 & 1 & 1 \\ -2 & -1 & -1 \\ 0 & -1 & 0 \end{pmatrix},\ \pm\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -2 & -1 & -1 \end{pmatrix}\ or\ \pm\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix}.$$

The rest of the statement was already shown at the end of 1.7.

Appendix: Groups of fixed-point-free rational automorphisms of algebraic tori

Let K be a field with the characteristic exponent p and T be an n-dimensional algebraic torus defined over K. A rational automorphism ϕ of T is said to be *fixed-point-free* if the only element of T left fixed by ϕ is the identity element.

Hertzig [5] has shown that if H is a group of fixed-point-free rational automorphisms of T, then H is a finite p-group and $n\equiv 0$ mod. $(p-1)$.

We determine all groups of fixed-point-free rational automorphisms of algebraic tori in the special cases $n=2$, $n=3$, and in general, when n is odd.

A rational automorphism ϕ over the algebraic closure $\bar{K}$ of K can be identified with an element $(\phi_{i,j})$ of $GL(n,\boldsymbol{Z})$ via $\phi(x_1, x_2, \cdots, x_n)=(y_1, y_2, \cdots, y_n)$, where $y_i=\prod_{1\leqq j\leqq n} x_j^{\phi_{ij}}$. Thus we may speak of the characteristic polynomial $\chi_\phi(X)$ of ϕ.

LEMMA 1. *Let ϕ be a rational automorphism of T. Then ϕ is fixed-point-free if and only if $\chi_\phi(1)$ is a power of p.*

Proof. A fixed-point of ϕ is a solution of the equations

$$x_1^{-\phi_{i,1}}\cdots x_{i-1}^{-\phi_{i,i-1}}x_i^{1-\phi_{i,i}}x_{i+1}^{-\phi_{i,i+1}}\cdots x_n^{-\phi_{i,n}}=1\ (1\leqq i\leqq n).$$

By elimination, these reduce to

$$x_i^\delta=1$$

where $\delta=\det(E_n-\phi)=\chi_\phi(1)$, Q.E.D.

COROLLARY. *Let ϕ and Ψ be two rational automorphisms of T. Assume that ϕ is conjugate to Ψ. Then ϕ is fixed-point-free if and only if Ψ is so.*

In the case $n=2$, there exist 2-subgroups of order 2, 4 or 8, and 3-subgroups of order 3 in $GL(2,\boldsymbol{Z})$. But considering all non-conjugate 2-subgroups and 3-subgroup, we have immediately

PROPOSITION 1. *There exist the following groups of fixed-point-free rational automorphisms of a 2-dimensional algebraic torus defined over K:*

(1) *the subgroup* $\left\{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right\}$ *only if* $p=2$

(2) *all groups of order* 3 *only if* $p=3$

(3) *all cyclic groups of order* 4 *only if* $p=2$.

In the case $n=3$, and in general, when n is odd, we have

PROPOSITION 2. *Let n be odd and H a non-trivial group of fixed-point-free rational automorphisms of an n-dimensional algebraic torus defined over K. Then the characteristic exponent of K is 2, and H is the cyclic group of order 2 generated by $-E_n$, where E_n is the unit matrix of $GL(n, \boldsymbol{Z})$.*

Proof. By Herzig (Theorem 1, p. 1041, [5]), H is a finite p-group and $n \equiv 0$ mod. $(p-1)$. Hence $p=2$. To prove this proposition it is sufficient to show that fixed-point-free rational automorphism of order 2 is only $-E_n$ and H does not contain any subgroup of order 4. Let ϕ be a rational automorphism of order 2 and $\phi \neq -E_n$. Then the characteristic polynomial of ϕ is $(X+1)^k(X-1)^m$ where $m \geqq 1$ and $k+m=n$. Hence ϕ is not fixed-point-free. Next let Ψ be automorphism of order 4 in H. Then Ψ^2 is automorphism of order 2 and $\Psi^2 \neq -E_n$. Therefore $\{\Psi\}$ is not a subgroup of fixed-point-free of rational automorphisms, Q.E.D.

In the case $n=4$ and $p=2$, we guess that groups of fixed-point-free rational automorphisms of T have all order 8 at most and are all cyclic. (In fact there is a cyclic group of order 8 of fixed-point-free rational automorphisms of T). More generally, the following natural question arises; Let H be a group of fixed-point-free rational automorphisms of T, then how is the order of the finite p-group H related to the dimension n of T? By Proposition 2, the question is open only if n is even.

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