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ON PRINCIPAL FUNCTION PROBLEM

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To Professor Katsuji Ono on his sixtieth birthday

Sario's theory of principal functions fully discussed in his research monograph [3] with Rodin stems from the principal function problem which is to find a harmonic function p on an open Riemann surface R imitating the ideal boundary behavior of the given harmonic function s in a neighborhood A of the ideal boundary δ of R. The mode of imitation of p to s is described by a linear operator L of functions on ∂A into harmonic functions on A: p imitates the behavior of s at the ideal boundary if $L((p-s)|\partial A) = p|A - s$. Sario [4] considered normal operators L in his terminology as imitative operators and gave a complete solution to the principal function problem with respect to his class of operators.

Recently Yamaguchi [5] introduced a new class of imitative operators, the class of regular operators in his terminology, and also gave an existence theorem in the principal function problem. Sario's class of imitative operators is neither contained in nor does contain that of Yamaguchi.

Therefore it is desirable to introduce a wider class of imitative operators which contains those considered by Sario and Yamaguchi and also to obtain the complete solution to the principal function problem with respect to this new class of operators, which is the object of the present note. In this context refer also to Nakai [2].

1. Throughout this note we will denote by R an open Riemann surface. However the whole argument in the sequel can be applied without any change to the case where R is replaced by a noncompact Riemannian manifold of an arbitrary dimension whose base manifold is orientable, connected, separable, and of class C^2 and whose metric tensor possesses Hölder continuous first order derivatives.

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We fix a neighborhood A of the ideal boundary δ of R, i.e. A is an open subset of R such that $R-A$ is the closure of a relatively compact regular region in R. Here a region is regular, by definition, if its relative boundary is of class C^1. Set $\alpha=\partial A$, the relative boundary of A, which is oriented positively with respect to the region $R-\alpha\cup A$.

Consider a linear subspace $F(\alpha)$ of $C(\alpha)$, the linear space of all finitely continuous functions on α, with the property that there exists a relatively compact regular region B_F containing $R-A$ such that

$$F(\alpha)\supset\{u|\alpha \,\big|\, u\in H(B_F)\}$$

where $H(B_F)$ is the set of all harmonic functions in B_F. We will call such an $F(\alpha)$ an *admissible* linear subspace of $C(\alpha)$.

2. An *imitative operator* L for A is a linear operator from an admissible linear subspace $F(\alpha)$ of $C(\alpha)$ into $C(\alpha\cup A)\cap H(A)$ with the following four properties:

$(L.\ 1)$ $(Lf)|\alpha=f$ *for each* $f\in F(\alpha)$;

$(L.\ 2)$ If $\{u_n\}\in H(B_F)$ *converges to* 0 *uniformly on each compact subset of* B_F, *then* $\{L(u_n|\alpha)\}$ *converges to* 0 *uniformly on each compact subset of* $\alpha\cup A$;

$(L.\ 3)$ *No* $u\in H(R)$ *satisfies* $L(u|\alpha)=u|\alpha\cup A$ *unless it is constant*;

$(L.\ 4)$ *If* $L1=1$, *then* $\int_\delta *d(Lf)=0$ *for every* $f\in F(\alpha)$.

Here in the last statement $\int_\delta *d(Lf)$ is understood to be $\int_\gamma *d(Lf)$ with a C^1-cycle γ in A separating α from the ideal boundary δ which is independent of the choice of γ.

The *principal function problem* to be discussed is the following: Given an imitative operator L for A and a function $s\in C(\alpha\cup A)\cap H(A)$, find a $p\in H(R)$ with

$$L(p|\alpha-s|\alpha)=p|\alpha\cup A-s. \tag{1}$$

The solution p, if exists, is said to be an (L,s)*-principal function* and s its *singularity*.

3. The complete solution to this problem is now given as follows:

The Main Existence Theorem. *If an imitative operator* L *satisfies* $L1\neq1$, *then there exists a unique* (L,s)*-principal function* p *for any singularity* s. *If*

$L1=1$, *then there exists an* (L, s)-*principal function* p *if and only if* s *satisfies*

(2) $$\int_{\delta} *ds = 0$$

and the (L, s)-*principal function* p *is unique up to an additive constant.*

It is easy to see that Sario's normal operators and Yamaguchi's regular operators are imitative operators in our sense but not vice versa. Therefore our result is a proper generalization of existence theorems of Sario [4] (see also [1], [3]) and Yamaguchi [5]. The proof will be given in **4-3.**

4. For simplicity we denote by B the region B_F and β the relative boundary ∂B which is oriented positively with respect to B. Recall that $R-A\subset B$. Let $D\varphi$ with $\varphi\in C(\beta)$ be the unique function in $C(B\cup\beta)\cap H(B)$ with $(D\varphi)|\beta=\varphi$. Define a linear operator T of $C(\beta)$ into itself by

(3) $$T\varphi = (L((D\varphi)|\alpha))|\beta$$

for $\varphi \in C(\beta)$. In view of $(L.\ 2)$, T is a *compact operator* (completely continuous operator). Let

(4) $$\sigma=(s-L(s|\alpha))|\beta$$

and consider an abstract integral equation of Fredholm type

(5) $$(I-T)\rho=\sigma$$

where I is the identity operator on $C(\beta)$ and ρ is to be seeked in $C(\beta)$.

If (1) has a solution p, then $\rho=(p-L(p|\alpha))|\beta$ is a solution of (5). Conversely from a solution ρ of (5) we can construct a solution p of (1) by

(6) $$p|B\cup\beta = D\rho,\quad p|\alpha\cup A = s-L(s|\alpha)+L((D\rho)|\alpha).$$

Since $D\rho$ and $s-L(s|\alpha)+L((D\rho)|\alpha)$ are identical on $\alpha\cup\beta$, this is well defined on R and belongs to $H(R)$ with (1).

5. Let $N=\{\varphi|\varphi\in C(\beta),\ (I-T)\varphi=0\}$. By virtue of $(L.\ 3)$, the dimension $\dim N$ of N is 0 if $L1\neq 1$ and 1 if $L1=1$. In fact, if $(I-T)\varphi=0$, then the function p in (6) with ρ replaceg by φ and s by 0 is in $H(R)$ and $Lp=p$. Thus $\varphi=p|\beta$ is a constant.

By the *Riesz-Schauder theory* for the Fredholm equation (5) with a compact operator T (see e.g. Yoshida [6]), (5) possesses a unique solution ρ if $\dim N=0$, which implies the first part of our theorem in 3.

6. We turn to the case $L1 = 1$. Since $\dim N = 1$, the Riesz-Schauder theory again implies that $\dim N^* = 1$, where

$$N^* = \{\nu \mid \nu \in C(\beta)^*, \quad (I^* - T^*)\nu = 0\}.$$

Here the conjugate space $C(\beta)^*$ of $C(\beta)$ consists of all Radon neasures on β not necessarily positive, and the conjugate operators I^* and T^* of I and T are characterized by

$$I^*\nu = \nu, \quad \int_\beta (T\varphi)d\nu = \int_\beta \varphi d(T^*\nu)$$

for every $\varphi \in C(\beta)$ and $\nu \in C(\beta)^*$. Therefore N^* is generated by a single $\nu_0 \in C(\beta)^*$, $\nu_0 \neq 0$, with

$$\int_\beta (T\varphi)d\nu_0 = \int_\beta \varphi d\nu_0 \tag{7}$$

for every $\varphi \in C(\beta)$, and again by Riesz-Schauder theory, (5) possesses a solution ρ unique up to an additive constant if and only if

$$\int_\beta \sigma d\nu_0 = 0. \tag{8}$$

7. To obtain a ceoncrte ν_0, we consider $w \in C((\alpha \cup A) \cap (B \cup \beta)) \cap H(A \cap B)$ uniquely determined by

$$w \mid \alpha = 0, \quad w \mid \beta = 1. \tag{6}$$

Take an arbitrary $\varphi \in C^1(\beta)$. The Green formula yields

$$\int_{\beta-\alpha} L((D\varphi) \mid \alpha) * dw - w * d(L((D\varphi) \mid \alpha)) = 0$$

and

$$\int_{\beta-\alpha} D\varphi * dw - w * d(D\varphi) = 0.$$

On using (L. 4), (9), and (3), we deduce

$$\int_\beta T\varphi * dw = \int_\alpha D\varphi * dw$$

and

$$\int_\beta \varphi * dw = \int_\alpha D\varphi * dw.$$

Combining the last two, we conclude

$$\int_\beta (T\varphi)*dw = \int_\beta \varphi*dw \tag{10}$$

for every $\varphi \in C^1(\beta)$ and hence for every $\varphi \in C(\beta)$ by continuity. Moreover since

$$\int_\beta *dw = \int_{\beta-\alpha} w*dw = \int_{A\cap B} dw \wedge *dw > 0,$$

the measure $*dw|\beta$ qualifies to be ν_0 in (7).

8. Finally we transform (8) into (2) for $\nu_0 = *dw|\beta$, which completes the proof for the second part of our theorem in **3**.

Let $\varepsilon \in (0,1)$ and $A_\varepsilon = A - \{w \leq \varepsilon\}$. Give the orientation to $\alpha_\varepsilon = \partial A_\varepsilon = \{w = \varepsilon\}$ positively with respect to $R - \alpha_\varepsilon \cup A_\varepsilon$. By the Green formula we see that

$$\int_{\beta-\alpha_\varepsilon} (s - L(s|\alpha))*dw - w*d(s - L(s|\alpha)) = 0.$$

Again by $(L.\ 4)$ together with (9) and $\int_{\beta-\alpha_\varepsilon} *ds = 0$, we have

$$\int_\beta \sigma d\nu_0 - \int_{\alpha_\varepsilon} (s - L(s|\alpha))*dw - \int_\beta *ds - \varepsilon \int_\beta *ds = 0. \tag{11}$$

Since $*dw > 0$ on α_ε, on putting $k_\varepsilon = \sup_{\alpha_\varepsilon} |s - L(s|\alpha))|$ we obtain

$$\left|\int_{\alpha_\varepsilon} (s - L(s|\alpha))*dw\right| \leq k_\varepsilon \int_{\alpha_\varepsilon} *dw = k_\varepsilon \int_\beta *dw.$$

Clearly $\lim_{\varepsilon\to 0} k_\varepsilon = 0$ and therefore

$$\lim_{\varepsilon\to 0} \int_{\alpha_\varepsilon} (s - L(s|\alpha))*dw = 0.$$

On making $\varepsilon \to 0$ in (11), we finally conclude that

$$\int_\beta \sigma d\nu_0 = \int_\beta *ds \tag{12}$$

The proof of our theorem in **3** is herewith complete.

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