GAUSSIAN MEASURE ON A BANACH SPACE
AND ABSTRACT WiNER MEASURE

HIROSHI SATO

In this paper, we shall show that any Gaussian measure on a separable or reflexive Banach space is an abstract Wiener measure in the sense of L. Gross [1] and, for the proof of that, establish the Radon extensibility of a Gaussian measure on such a Banach space. In addition, we shall give some remarks on the support of an abstract Wiener measure.

An abstract Wiener measure is a $\sigma$-extension in a Banach space $X$ of the canonical Gaussian cylinder measure $\mu_X$ of a real separable Hilbert space $H$ which is contained in $X$ densely. The idea of the abstract Wiener measure coincides with that of the White Noise (T. Hida [13]) and plays an important role not only in the theory of probability but in the theory of functional analysis (T. Hida [13], Y. Umemura [12], I.E. Segal [4,5], L. Gross [3] and Yu. L. Daletskii [16]).

We shall show first that any Gaussian measure on a separable or reflexive Banach space can be extended to a Radon measure on the strong topological $\sigma$-algebra (Theorem 1). With the same idea of the proof of Theorem 1, we can prove that this result is true for any probability measure on a Banach space, the finite dimensional distribution of which is Radon.

Utilizing the above result, we shall restrict the support of a Gaussian measure to a separable subspace which is explicitly constructed. Furthermore, constructing a suitable Hilbert subspace of the support, we shall show that any Gaussian measure on such a Banach space is an abstract Wiener measure (Theorem 2). L. Gross [1] showed that there exists and abstract Wiener measure on any separable Banach space. Our result shows that any given Gaussian measure on a separable or reflexive Banach space is an abstract Wiener measure. This means that the study of a Gaussian measure

Received Oct. 9, 1968.
on such a Banach space can be reduced to that of an abstract Wiener measure on a separable Banach space, and clears a new way for the investigation of a Gaussian measure on a Banach space, and makes the study of an abstract Wiener measure more meaningful.

As a corollary of Theorem 2, we shall show that the canonical Gaussian cylinder measure of a nonseparable Hilbert space cannot be extended to a \( \sigma \)-additive measure in any Banach space.

Before stating the remaining results in this paper, we establish terminology and notation.

Let \( X \) be a real Banach space, \( X^* \) be its topological dual space and \( \xi(x) \), \( (\xi \in X^*, x \in X) \), be the natural linear form. A cylinder set in \( X \) is a set of the form

\[
C = \{ x \in X : (\xi_1(x), \cdots, \xi_n(x)) \in D \}
\]

where \( \xi_1, \xi_2, \cdots, \xi_n \) are in \( X^* \) and \( D \) is a Borel set in the \( n \)-dimensional Euclidean space \( R_n \). \( \mathcal{A}_X \) is the family of all cylinder sets in \( X \) and \( \mathcal{A}_X \) is the minimal \( \sigma \)-algebra including \( \mathcal{A}_X \). \( \tau_X \) is the weak topological \( \sigma \)-algebra in \( X \) and \( \tau_X \) is the strong topological \( \sigma \)-algebra in it. Evidently we have

\[
\mathcal{A}_X \subset \mathcal{A}_X \subset \tau_X \subset \tau_X
\]

and if \( X \) is separable, then \( \mathcal{A}_X = \tau_X \) (E. Mourier [8]).

Let \( \mathfrak{X} \) be a real Hilbert space. The canonical Gaussian cylinder measure \( \mu_{\mathfrak{X}} \) of \( \mathfrak{X} \) is a finitely additive nonnegative set function on \( (\mathfrak{X}, \mathcal{A}_{\mathfrak{X}}) \) such that

\[
\mu_{\mathfrak{X}}[x \in \mathfrak{X} : \xi(x) \leq \alpha] = \frac{1}{\sqrt{2\pi} |\xi|} \int_{-\infty}^{\alpha} \exp \left[ -\frac{u^2}{2|\xi|^2} \right] du, \tag{1.1}
\]

for any \( \xi \in \mathfrak{X}^* \) and real number \( \alpha \), where \( |\xi| \) is the norm in \( \mathfrak{X}^* \). It is well-known that \( \mu_{\mathfrak{X}} \) does not have \( \sigma \)-additive extension to \( (\mathfrak{X}, \mathcal{A}_{\mathfrak{X}}) \), (see Corollary of Lemma 6).

Let \( \|x\| \) be a continuous norm on \( \mathfrak{X} \), and \( X \) be the Banach space obtained by the completion of \( \mathfrak{X} \) in the norm \( \|x\| \). Since we may consider \( X^* \) as a subspace of \( \mathfrak{X}^* \) through the natural imbedding, \( \mu_{\mathfrak{X}} \) induces a Gaussian cylinder measure \( \mu \) on \( (X, \mathcal{A}_X) \) as follows. If \( \xi_1, \xi_2, \cdots, \xi_n \) are in \( X^* \) and \( D \) is a Borel set in \( R_n \), define

\[
\mu[x \in X : (\xi_1(x), \cdots, \xi_n(x)) \in D] = \mu_{\mathfrak{X}}[x \in \mathfrak{X} : (\xi_1(x), \cdots, \xi_n(x)) \in D]. \tag{1.2}
\]
μ is well-defined. Furthermore, if μ has a σ-additive extension on \((X, \mathcal{F}_X)\), then we call it the σ-extension of \(μ_X\) on the Banach space \(X\) and the norm \(\|x\|\) admissible on \(X\). If a norm on \(X\) is induced by an inner product, namely, a continuous symmetric bilinear form on \(X\), then we call it Hilbertian. A measurable norm is defined by L. Gross [1,2] as follows. A norm \(\|x\|_1\) on \(X\) is a measurable norm if for every positive real number \(ε\) there exists a finite dimensional projection \(P_0\) of \(X\) such that for every finite dimensional projection \(P\) orthogonal to \(P_0\) we have

\[ μ_X[x ∈ X: \|Px\| > ε] < ε. \]

L. Gross [1] showed that the measurable norm is admissible.

In the last section, we shall give some remarks on the admissible norm. We shall give a necessary and sufficient condition for a Hilbertian norm to be admissible (Theorem 3) and show that there exists a measurable norm such that there is no Hilbertian admissible norm stronger than it (Example 2). This means that as a support of an abstract Wiener measure we can choose a Banach subspace which includes no Hilbert subspace of full measure. We shall also show that there exists an admissible norm which is not a measurable norm. This means that for a norm to be an admissible norm it is not necessary to be a measurable norm.

2. Gaussian measure and Radon measure.

Let \(X\) be a Banach space with norm \(\|x\|\) and \(X^*\) be the topological dual for \(X\) with norm \(\|ξ\|\). A probability measure \(μ\) on \((X, \mathcal{F}_X)\) is Gaussian if for every \(ξ ∈ X^*\), \(ξ(x)\) is a Gaussian random variable with mean zero on the probability space \((X, \mathcal{F}_X, μ)\). In other words, for every \(ξ ∈ X^*\) and real number \(α\),

\[ μ[x ∈ X: ξ(x) < α] = \frac{1}{\sqrt{2πν(ξ)}} \int_{-∞}^α \exp\left[-\frac{u^2}{2ν(ξ)}\right]du, \tag{2.1} \]

where \(ν(ξ)\) is the variance of \(ξ(x)\).

Theorem 1. Every Gaussian measure \(μ\) on a separable or reflexive Banach space \((X, \mathcal{F}_X)\) can be extended to a Radon measure on \((X, τ_X)\).

Proof. If \(X\) is separable, \(\mathcal{F}_X = τ_X\) and the proof is trivial. Let \(X\) be a reflexive Banach space and let \(X^{**}\) be the topological dual space of \(X^*\). Let \(\mathcal{F}^*\) be the minimal σ-algebra of subsets of \(X^{**}\) with respect to which
all the functions $\xi(x)$, $x \in X^*$, are measurable, where $\xi(x)$ ($x \in X^*$, $x \in X^{**}$) denotes the continuous linear form and $\tau^*$ is the topological $\sigma$-algebra with respect to $X^*$-topology in $X^{**}$ (W. Dunford and J.T. Schwartz [15], p. 419). Define a measure $\mu^*$ on $(X^{**}, \mathcal{B}^*)$ as follows:

$$
\mu^*[x \in X^{**}: (\xi_1(x), \cdots, \xi_n(x)) \in D] = \mu[x \in X: (\xi_1(x), \cdots, \xi_n(x)) \in D].
$$

(2.2)

where $\xi_1, \xi_2, \cdots, \xi_n$ are in $X^*$ and $D$ is a Borel set in $R_n$. The measure $\mu^*$ is well defined and is Gaussian. Since all the open sets in $\mathcal{B}^*$ form an open basis which determines $X^*$-topology and since $X^{**}$ is the topological dual for the Banach space $X^*$, $\mu^*$ can be extended to a Radon measure $\bar{\mu}$ on $(X^{**}, \tau^*)$ uniquely (Yu. V. Prohorov [10], Theorem 1, Lemma 3 and Example 1). Since $X$ is reflexive, we have $X = X^{**}$ and $\tau^* = \tau_X$. Therefore $\bar{\mu}$ is a Gaussian Radon measure on $(X, \tau_X)$. Since $X$ is a Banach space, the weak Radon measure $\bar{\mu}^*$ can be extended to a strong Radon measure $\bar{\mu}$ on $(X, \tau_X)$ and, it is easy to see from (2, 2), that $\bar{\mu}$ is an extension of $\mu$. Thus we have proved the theorem.

Remark. Without any change in the proof, we can prove Theorem 1 not only for a Gaussian measure but for any probability measure on a Banach space, the finite dimensional distribution of which is Radon.

We can therefore consider a Gaussian measure on a Banach space $X$ as a Radon measure on $(X, \tau_X)$.

3. Gaussian measure and abstract Wiener measure.

Let $\mu$ be a Gaussian measure on a separable or reflexive Banach space $X$. We use the same notations used in Section 2. Choose the maximal subset $\{\xi_\alpha; \alpha \in \Lambda\}$ of $X^*$ such that

$$
\xi_\alpha \in X^* \text{ and } \|\xi_\alpha\| = 1, \quad \alpha \in \Lambda
$$

(3.1)

$$
\int_X \xi_\alpha(x)\xi_\beta(x)d\mu(x) = 0 \quad \text{if} \quad \alpha \neq \beta, \; \alpha, \beta \in \Lambda.
$$

Lemma 1. Let $\Lambda_0 = \{\alpha \in \Lambda; \nu(\xi_\alpha) \neq 0\}$, then $\Lambda_0$ is an at most countable subset of $\Lambda$. 

Proof. Let \( \{\alpha_n\}_{n=1,2,\ldots} \) be an arbitrary countable subset of \( A \). Since it holds that
\[
\sup_{n} |\xi_{\alpha_n}(x)| \leq \sup_{\|\xi\| = 1} |\xi(x)| = \|x\| < +\infty, \text{ for every } x \in X,
\]
we can choose a positive number \( M \) such that
\[
\mu\{x \in X : \sup_{n} |\xi_{\alpha_n}(x)| \leq M\} > \frac{1}{2}. \tag{3.3}
\]
On the other hand, we have
\[
\mu\{x \in X : \sup_{n} |\xi_{\alpha_n}(x)| \leq M\}
= \lim_{N \to +\infty} \mu\{x \in X : \sup_{1 \leq n \leq N} |\xi_{\alpha_n}(x)| \leq M\}
= \lim_{N \to +\infty} \mu\{x \in X : |\xi_{\alpha_n}(x)| \leq M\},
\]
Since the collection \( \{\xi_{\alpha_n}(x)\} \) is Gaussian, from (3.1), \( \xi_{\alpha_n}(x) \) and \( \xi_{\alpha_m}(x) \) are mutually independent if \( n \neq m \). Therefore,
\[
\mu\{x \in X : \sup_{n} |\xi_{\alpha_n}(x)| \leq M\}
= \lim_{N \to +\infty} \prod_{1 \leq n \leq N} \mu\{x \in X : |\xi_{\alpha_n}(x)| \leq M\}
= \lim_{N \to +\infty} \prod_{1 \leq n \leq N} \frac{1}{\sqrt{2\pi} v(\xi_{\alpha_n})} \int_{-M}^{M} \exp\left[-\frac{u^2}{2v(\xi_{\alpha_n})}\right]du
\]
Together with (3.3), we have
\[
\lim_{N \to +\infty} v(\xi_{\alpha_n}) = 0. \tag{3.4}
\]
Since the choice of the countable subset \( \{\alpha_n\} \) is arbitrary, the set
\[
A_N = \left\{ \alpha \in A ; \ v(\xi_{\alpha}) \geq \frac{1}{N} \right\}
\]
must be a finite subset of \( A \) for every positive integer \( N \). Otherwise we have a contradiction to (3.4). Therefore,

\[
A_0 = \bigcup_{N=1}^{+\infty} A_N
\]

must be a countable subset of \( A \).

**Lemma 2.** Define \( X_\alpha, \alpha \in \Lambda \), by

\[
X_\alpha = \{ x \in X; \xi_\alpha(x) = 0 \}, \quad \alpha \in \Lambda,
\]

and set \( \bar{X} = \bigcap_{\alpha \in \Lambda - A_0} X_\alpha \). Then we have

\[
\mu[\bar{X}] = 1. \tag{3.5}
\]

**Proof.** Let \( \Gamma \) be the family of all finite subsets of \( \Lambda - A_0 \) and define \( X_J = \bigcap_{\alpha \in J} X_\alpha; J \in \Gamma \). Obviously \( X_J \) is a strongly closed linear subspace of \( X \) and the family \( \{X_J; J \in \Gamma\} \) is directed. Since \( v(\xi_\alpha) = 0 \), \( \xi_\alpha(x) \) is a Dirac measure for every \( \alpha \in \Lambda - A_0 \), we have

\[
\mu[X_J] = 1 \quad \text{for every } J \in \Gamma.
\]

Therefore,

\[
\mu[\bar{X}] = \mu[\bigcap_{J \in \Gamma} X_J] = \inf_{J \in \Gamma} \mu[X_J] = 1,
\]

(L. Schwartz [11]). Thus we have proved the lemma.

This lemma means that the measure \( \mu \) is concentrated in some closed linear subspace \( \bar{X} \). \( \bar{X} \) is also a Banach space with the norm \( \|x\| \). Let \( \mathcal{C} \) be the closed linear manifold spanned by \( \{\xi_\alpha; \alpha \in \Lambda - A_0\} \). Then the topological dual \( \bar{X}^* \) for \( \bar{X} \) is isomorphic to \( X^*/\mathcal{C} \).

It is easy to see that in \( \bar{X}^* \)

\[
v(\xi) = 0 \quad \text{implies } \xi = 0. \tag{3.6}
\]

Let \( \|\xi\| \) be the norm in \( \bar{X}^* \) again.

Hereafter, we restrict the measure \( \mu \) to \( \bar{X} \). For every \( \xi, \eta \in X^* \) define

\[
(\xi, \eta) = \int_{\bar{X}} \xi(x)\eta(x)d\mu(x), \tag{3.7}
\]

\[
|\xi| = \sqrt{(\xi, \xi)} = \sqrt{v(\xi)} \tag{3.8}
\]
Then, according to (3. 6),
\[ |\xi| = 0 \text{ if and only if } \|\xi\| = 0, \quad (3. 9) \]
in $\hat{X}^*$. Therefore the bilinear form $(\xi, \eta)$ is an inner product and $|\xi|$ is a norm on $\hat{X}^*$. Next we shall show that the norm $|\xi|$ is continuous.

**Lemma 3.** There exists a positive constant $C$ such that
\[ |\xi| \leq C \|\xi\| \quad \text{for every } \xi \in \hat{X}^*. \quad (3. 10) \]

**Proof.** It is sufficient to show
\[ C = \sup_{\|\xi\| = 1, \xi \in \hat{X}^*} |\xi| < +\infty. \]
Suppose not, then there exists a sequence $\{\xi_n\}$ in $\hat{X}^*$ such that
\[ \|\xi_n\| = 1, \quad n = 1, 2, 3, \ldots \]
\[ \lim_{n \to +\infty} |\xi_n| = +\infty. \]
By choosing a sufficiently large number $M$, we have
\[ \mu[x \in \hat{X}: \sup_n |\xi_n(x)| \leq M] > \frac{1}{2}, \quad (3. 11) \]
(see the proof of Lemma 1). On the other hand,
\[ \mu[x \in \hat{X}: \sup_n |\xi_n(x)| \leq M] = \lim_{n \to +\infty} \mu[x \in \hat{X}: \sup_{1 \leq \nu \leq n} |\xi_{\nu}(x)| \leq M] \]
\[ \leq \lim_{n \to +\infty} \mu[x \in \hat{X}: |\xi_n(x)| \leq M] \]
\[ = \lim_{n \to +\infty} \frac{1}{\sqrt{2\pi}} \int_{-M}^M \exp \left[-\frac{u^2}{2|\xi_n|^2}\right] du \]
\[ = \lim_{n \to +\infty} \frac{1}{\sqrt{2\pi}} \int_{-M}^{M} \left\{ \frac{M}{|\xi_n|} \right\} \exp \left[-\frac{u^2}{2}\right] du = 0. \]
This contradicts (3. 11) and concludes the proof.

Let $\mathcal{H}^*$ be the Hilbert space obtained by the completion of $\hat{X}^*$ with respect to the inner product $(\xi, \eta)$, and let $\mathcal{H}$ be its topological dual space. By the definition (3. 8) of the norm $|\xi|$, the relation (1. 2) is valid for $\mu$ and the canonical Gaussian cylinder measure $\mu_\mathcal{H}$ of the Hilbert space $\mathcal{H}$. 
This means that $\mu$ is a $\sigma$-extension of $\mu_\mathcal{X}$ in $\mathcal{X}$. On the other hand, it is easy to see that the system $\{\xi_\alpha/|\xi_\alpha|: \alpha \in \Lambda_0\}$ is a C.O.N.S. (complete orthonormal system) in $\mathcal{X}^*$. Since $\Lambda_0$ is at most countable, $\mathcal{X}$ is a separable Hilbert space.

**Lemma 4.** $\mathcal{X}$ is a subspace of $\mathcal{X}$.

*Proof.* The measure $\mu$ extends to a Gaussian measure $\mu^*$ on $\mathcal{X}^{**}$ by (2, 2), where $\mathcal{X}^{**}$ is the topological dual for $\mathcal{X}^*$. Then $\mathcal{X}$ is a measurable subset of $\mathcal{X}^{**}$ and $\mu^*(\mathcal{X}^{**}) = \mu^*(\mathcal{X}) = 1$ is true (see the proof of Theorem 1). Since $\mathcal{X}^*$ is included in $\mathcal{X}$ its dual $\mathcal{X}$ is included in $(\mathcal{X}^*)^* = \mathcal{X}^{**}$. The relation (1. 2) is also valid for $\mu^*$ and $\mu_\mathcal{X}$. Therefore, by identifying $\mathcal{X}^*$ and $\mathcal{X}$, for every $x_0 \in \mathcal{X}$

$$\mu^*[\mathcal{X} + x_0] = \mu^*[\mathcal{X}] = 1,$$

(3. 12)
due to the fact that $\mu^*$ is quasi-invariant. (Y. Umemura [12]). On the other hand, if $\mathcal{X}$ is not a subspace of $\mathcal{X}$, namely, if there exists $x_0$ in $\mathcal{X}$ which is not in $\mathcal{X}$, then we have

$$[\mathcal{X} + x_0] \cap \mathcal{X} = \phi. \quad (3. 13)$$

For, if there exists $y$ in $[\mathcal{X} + x_0] \cap \mathcal{X}$, then there exists $y'$ in $\mathcal{X}$ such that $y = y' + x_0$. Since $\mathcal{X}$ is a linear space, $x_0 = y - y'$ is in $\mathcal{X}$. This is a contradiction to the assumption on $x_0$ and (3. 13) is true. Thus we have

$$1 = \mu^*[\mathcal{X}^{**}] \geq \mu^*[|\mathcal{X} + x_0] \cup \mathcal{X}] = 2.$$ 

This contradicts (3. 12), which proves the lemma.

**Lemma 5.** $\mathcal{X}$ is dense in $\mathcal{X}$.

*Proof.* Let $\mathcal{X}$ be the closure of $\mathcal{X}$ in $\mathcal{X}$. If there exists $x_0$ in $\mathcal{X} - \mathcal{X}$, then, by the Hahn-Banach theorem, there exists $\xi \neq 0$ in $\mathcal{X}^*$ such that $\xi(x) = 0$ on $\mathcal{X}$. On the other hand, let $|x|_0$ be the norm on $\mathcal{X}$. Then we have

$$|\xi| = \sup_{|x|_0 = 1} |\xi(x)| = 0.$$ 

According to (3. 9), this means $\xi = 0$ in $\mathcal{X}^*$ and contradicts the choice of $\xi$. Therefore $\mathcal{X} = \mathcal{X}$, that is, $\mathcal{X}$ is dense in $\mathcal{X}$.
Corollary. $\mathcal{X}$ is separable.

Proof. The space $\mathcal{X}$ is a separable Hilbert space and, by Lemma 5, is dense in $\mathcal{X}$. Furthermore, the norm $|x|_0$ on $\mathcal{X}$ is stronger than that on $\mathcal{X}$. Therefore $\mathcal{X}$ is separable.

Summing up these results, we can derive the following theorem.

Theorem 2. (A). Let $\mu$ be a Gaussian measure on a separable or reflexive Banach space. Then there exists a separable closed linear subspace $\mathcal{X}$ such that $\mu[\mathcal{X}] = 1$ and (3. 6) is valid in $\mathcal{X}^*$.

(B). Let $\mu$ be a Gaussian measure on a separable Banach space $\mathcal{X}$, and assume that (3. 6) is valid in $\mathcal{X}^*$. Then there exists a dense Hilbert subspace $\mathcal{K}$ of $\mathcal{X}$ such that $\mu$ is an abstract Wiener measure, that is, $\mu$ is a $\sigma$-extension in $\mathcal{X}$ of the canonical Gaussian cylinder measure $\mu_\mathcal{K}$ of $\mathcal{K}$. The norm $\|x\|$ is admissible on $\mathcal{K}$.

Corollary. There is no admissible norm on a nonseparable Hilbert space $\mathcal{H}$.

Proof. Suppose that a norm $\|x\|$ on $\mathcal{H}$ is admissible, $X$ be the completion of $\mathcal{H}$ in the norm $\|x\|$, and let $\mu$ be the $\sigma$-extension in $X$ of the canonical Gaussian cylinder measure $\mu_\mathcal{H}$ of $\mathcal{H}$. Since $\mathcal{H}$ is dense in $X$ and $\|x\| = 0$ implies $x = 0$ in $\mathcal{H}$, we can show that $X^*$ is a dense subspace of $\mathcal{H}^*$ and (3. 6) is valid in $X^*$ in the manner similar to that used in the proof of Lemma 5. Therefore, we can choose a C.O.N.S. $\{\xi_\alpha^* : \alpha \in \Lambda\}$ of $\mathcal{H}^*$ from $X^*$. $\Lambda$ is an uncountable set since $\mathcal{H}^*$ is nonseparable. Let $\xi_\alpha = \xi_\alpha^*/\|\xi_\alpha^*\|$; $\alpha \in \Lambda$. Then (3. 1) is valid for $\{\xi_\alpha : \alpha \in \Lambda\}$. On the other hand, considering (3. 6), $v(\xi_\alpha) = \frac{1}{\|\xi_\alpha^*\|} \neq 0$ for every $\alpha \in \Lambda$. This contradicts Lemma 1.

4. Admissible norm.

Let $\mathcal{H}$ be a separable Hilbert space with norm $|x|$ and inner product $(x, y)$. We study the condition under which a Hilbertian norm on $\mathcal{H}$ is admissible.

Lemma 6\(^{(\ast)}\). Let $H$ be a separable Hilbert space and let $\mu$ be a Gaussian cylinder measure on $(H, \mathfrak{M}_H)$, that is, for every $\xi \in H^*$, $\xi(x)$ is a Gaussian random variable on $(H, \mathfrak{M}_H, \mu)$ with mean $m(\xi)$ and variance $v(\xi)$. (In this lemma, we do not assume zero mean.)

\(^{(\ast)}\) This lemma was suggested by Prof. K. Ito.
Then $\mu$ has a $\sigma$-additive extension to $(H, \overline{\mathfrak{A}}_H)$ if and only if the characteristic functional of $\mu$ is of the form

$$\int_H e^{i\xi(x)} d\mu(x) = \exp \left[ i \langle \xi, m \rangle - \frac{1}{2} \|S\xi\|^2 \right], \quad \xi \in H^*, \quad (4.1)$$

where $m$ is an element of $H$, $S$ is a nonnegative self-adjoint Hilbert-Schmidt operator and $\|\xi\|$ is the norm on $H^*$.

**Proof.** The sufficiency is derived from V.V. Sazonov [6].

We have only to prove the necessity. Assume that there exists a $\sigma$-additive extension to $(H, \overline{\mathfrak{A}}_H)$ and denote it by $\mu$ again. Identify $H^*$ and $H$ and let $\langle \cdot, \cdot \rangle$ be its inner product and $\|\cdot\|$ be its norm. Then $\langle \xi, x \rangle$; $\xi \in H^*(= H)$, $x \in H$ denotes the natural linear form.

Let $\{\xi_n\}$ be a sequence in $H$ convergent to zero. Then $\langle \xi_n, x \rangle$ converges to zero for all $x$ in $H$. Since $\{\langle \xi_n, x \rangle\}$ is a Gaussian random sequence on $(H, \overline{\mathfrak{A}}_H, \mu)$,

$$m(\xi_n) = \int_H \langle \xi_n, x \rangle d\mu(x) \quad (4.2)$$

converges to zero (§33, Lemma 1 of K. Ito [14]). Therefore $m(\xi)$ is a continuous linear functional on $H^*$ and there exists $m \in H$ such that

$$m(\xi) = \langle \xi, m \rangle \quad \text{for any} \quad \xi \in H. \quad (4.3)$$

Next, let $\{\varphi_j\}$ be a C.O.N.S. in $H$, and, for $m$ and for every $\xi$, $x$ in $H$, set

$$m_j = \langle \varphi_j, m \rangle, \quad x_j = x_j(x) = \langle \varphi_j, x \rangle, \quad j = 1, 2, 3, \ldots, \quad (4.4)$$

$$\xi_j = \xi_j(\xi) = \langle \varphi_j, \xi \rangle. \quad (4.5)$$

Then obviously

$$\mu[\xi \in H: \sum_{j=1}^{+\infty} x_j(x)^2 < +\infty] = \mu[H] = 1. \quad (4.6)$$

On the other hand, let

$$\xi^N = \sum_{j=1}^{N} \xi_j \varphi_j, \quad N = 1, 2, 3, \ldots$$

$$v_{ij} = \int_H (x_j(x) - m_j)(x_i(x) - m_i)d\mu(x), \quad i, j = 1, 2, 3, \ldots. \quad (4.7)$$
Then
\[
\int_H \exp \left[ i \langle \xi^N, x \rangle \right] d\mu(x)
= \int_H \exp \left[ i \sum_{j=1}^{N} \xi_j x_j(x) \right] d\mu(x)
= \exp \left[ i \sum_{j=1}^{N} m_j \xi_j - \frac{1}{2} \sum_{k,j=1}^{N} v_{kj} \xi_k \xi_j \right].
\] (4.7)

Averaging both sides of (4.7) with respect to the measure
\[
\langle 2\pi \rangle^{-\frac{N}{2}} \exp \left[ -\frac{1}{2} \sum_{j=1}^{N} \xi_j^2 \right] d\xi_1 d\xi_2 \cdots d\xi_N,
\]
we have
\[
\int_H \exp \left[ -\frac{1}{2} \sum_{j=1}^{N} x_j(x)^2 \right] d\mu(x) \leq \frac{1}{\sqrt{1+ \sum_{j=1}^{N} v_{jj}}}.
\] (4.8)

If \( \sum_{j=1}^{+\infty} v_{jj} \) is divergent, then from (4.8) we have
\[
\int_H \exp \left[ -\frac{1}{2} \sum_{j=1}^{+\infty} x_j(x)^2 \right] d\mu(x) = 0,
\]
and
\[
\exp \left[ -\frac{1}{2} \sum_{j=1}^{+\infty} x_j(x)^2 \right] = 0, \text{ a.e. .}
\]
Therefore
\[
\mu \left[ \sum_{j=1}^{+\infty} x_j(x)^2 = +\infty \right] = 1.
\]
This contradicts (4.5) and we have,
\[
\sum_{j=1}^{+\infty} v_{jj} < +\infty.
\] (4.9)

Define a linear operator \( V \) on \( H \) by
\[
\langle V \varphi_i, \varphi_j \rangle = v_{ij}, \quad i, j = 1, 2, 3, \ldots
\] (4.10)
Then \( V \) is a nonnegative self-adjoint operator on \( H \) and further, it is nuclear, since
\[
\sum_{j=1}^{+\infty} \langle V \varphi_j, \varphi_j \rangle = \sum_{j=1}^{+\infty} v_{jj} < +\infty.
\]
Let $S$ be $\sqrt{V}$. Then it is easy to see that $S$ is the required Hilbert-Schmidt operator. Thus we have proved the lemma.

**Corollary 1.** The canonical Gaussian cylinder measure $\mu_{\mathcal{X}}$ on a Hilbert space $\mathcal{X}$ does not have a $\sigma$-additive extension to $(\mathcal{X}, \mathcal{F})$.

**Proof.** The characteristic functional of $\mu_{\mathcal{X}}$ is

$$\int_{\mathcal{X}} \exp [i\xi(x)]d\mu_{\mathcal{X}}(x) = \exp \left[ -\frac{1}{2} |\xi|^2 \right] = \exp \left[ -\frac{1}{2} I|\xi|^2 \right], \quad (4.11)$$

where $|\xi|$ is the norm on $\mathcal{X}^*$ and $I$ is the identity. But $I$ is not of Hilbert-Schmidt type. Therefore, by Lemma 6, $\mu_{\mathcal{X}}$ does not have a $\sigma$-additive extension to $(\mathcal{X}, \mathcal{F})$.

**Corollary 2.** In Lemma 6, if $\mu$ has a $\sigma$-additive extension to $(H, \mathcal{F}_H)$ and mean zero, then for every $\xi, \eta \in H^*(=H)$

$$\int_H \xi(x)\eta(x)d\mu(x) = \langle S\xi, S\eta \rangle, \quad (4.12)$$

where $S$ is the Hilbert-Schmidt operator determined by (4.1).

Utilizing Lemma 6, we have the following theorem.

**Theorem 3.** A Hilbertian norm $\|x\|$ on a separable Hilbert space $\mathcal{X}$ is admissible if and only if there exists a one to one Hilbert-Schmidt operator $S_0$ such that

$$\|x\| = |S_0x|, \quad x \in \mathcal{X}, \quad (4.13)$$

where $|x|$ is the initial norm on $\mathcal{X}$.

**Proof.** The sufficiency is well-known (for example, see Y. Umemura [12]).

We prove the necessity. Let $\|x\|$ be a Hilbertian admissible norm induced by an inner product $\langle x, y \rangle$ on $\mathcal{X}$ and let $H$ be the completion of $\mathcal{X}$ in the norm $\|x\|$. Then $H$ is also a Hilbert space with the inner product $\langle x, y \rangle$. Let $\mu$ be the $\sigma$-extension in $H$ of the canonical Gaussian cylinder measure $\mu_{\mathcal{X}}$ of $\mathcal{X}$. Then $\mu$ is a Gaussian measure on the Hilbert space $H$. Therefore, by Lemma 6, there exists a nonnegative Hilbert-Schmidt opera-
tor $S$ on $H^*$ determined by (4.1). Since we are assuming mean zero, (4.12) is also valid (Corollary 2 of Lemma 6).

Identifying $\mathcal{X}$ and $\mathcal{X}^*$, and remembering $H^*$ is a subspace of $\mathcal{X}^*(=\mathcal{X})$, we have

$$\|S\xi\|^2 = \int_X \xi(x)^2 d\mu(x)$$

$$= \int_X (\xi, x)^2 d\mu(x) = |\xi|^2,$$  \hspace{1cm} (4.14)

for every $\xi$ in $H^*$ where $\|\xi\|$ is the norm on $H^*$. Consequently,

$$\|S\xi\| = |\xi|, \quad \text{for every } \xi \in H^*. \hspace{1cm} (4.15)$$

Since $\|x\| = 0$ implies $x = 0$ in $\mathcal{X}$ and so $|\xi| = 0$ implies $\xi = 0$ in $H^*$. Therefore, by (4.15), $S\xi = 0$ implies $\xi = 0$ in $H^*$ and $S$ is a one to one operator.

Let $\{\lambda_j\}$ and $\{\varphi_j\}$ be eigenvalues and eigenvectors of $S$, respectively. Then $\lambda_j > 0$, $j = 1, 2, \ldots$, and $\sum_{j=1}^{+\infty} \lambda_j^2 < +\infty$ because $S$ is a one to one Hilbert-Schmidt operator.

Further, since $\mu$ is the $\sigma$-extension of the canonical Gaussian cylinder measure $\mu_\mathcal{X}$, we have

$$(\varphi_i, \varphi_j) = \int_\mathcal{X} \varphi_i(x) \varphi_j(x) d\mu_\mathcal{X}(x)$$

$$= \int_\mathcal{X} \langle \varphi_i, x \rangle \langle \varphi_j, x \rangle d\mu(x)$$

$$= \langle S\varphi_i, S\varphi_j \rangle = \lambda_i \lambda_j \delta_{ij},$$

$$i, j = 1, 2, 3, \ldots.$$  \hspace{1cm} (4.15)

Let $\phi_j = \lambda_j^{-1} \varphi_j$, $j = 1, 2, 3, \ldots$. Then $\{\phi_j\}$ is a C.O.N.S. in $\mathcal{X}$ and

$$\sum_{j=1}^{+\infty} |S\phi_j|^2 = \sum_{j=1}^{+\infty} \|S^2 \phi_j\|^2 = \sum_{j=1}^{+\infty} \|\lambda_j \phi_j\|^2$$

$$= \sum_{j=1}^{+\infty} \lambda_j^2 < +\infty.$$  \hspace{1cm} (4.15)

Therefore $S$ can be extended to a Hilbert-Schmidt operator on $\mathcal{X}^*(=\mathcal{X})$ and we denote it by $S$ again. Let $S_0$ be the dual operator of $S$ in $\mathcal{X}$. Then $S_0$ is the required operator. In fact, since $SH^*$ is dense in $H^*$ and $H^*$ is dense in $\mathcal{X}^*(=\mathcal{X})$, for every $x$ in $\mathcal{X}(\subset H)$,
\[ \|x\| = \sup_{\|\xi\| = 1} |\xi(x)| = \sup_{\|\xi\| = 1} |(S\xi)(x)| \]

\[ = \sup_{\|\xi\| = 1} |(S\xi, x)| = \sup_{\|\xi\| = 1} |(\xi, S^*x)| \]

\[ = \sup_{\|\xi\| = 1} |(\xi, S_0x)| = |S_0x|. \]

The proof is now complete.

**Corollary.** Let \(\|x\|\) be an admissible norm on \(\mathfrak{X}\). If there exists a Hilbertian admissible norm stronger than \(\|x\|\) then for any C.O.N.S. \(\{\varphi_j\}\) in \(\mathfrak{X}\) we have

\[ \sum_{j=1}^{+\infty} \|\varphi_j\|^2 < +\infty. \quad (4.16) \]

**Proof.** Suppose that \(\|x\|'\) is a Hilbertian admissible norm stronger than \(\|x\|\), say, \(\|x\| \leq \|x\|'\). By Theorem 3, there exists a Hilbert-Schmidt operator \(S\) such that \(\|x\|' = |Sx|\), \(x \in \mathfrak{X}\). Then for any C.O.N.S. \(\{\varphi_j\}\) in \(\mathfrak{X}\),

\[ \sum_{j=1}^{+\infty} \|\varphi_j\|^2 \leq \sum_j \|\varphi_j\|'^2 \]

\[ = \sum_j |S\varphi_j|^2 < +\infty. \]

This was to be proved.

Next we give some examples of admissible norms on a separable Hilbert space \(\mathfrak{X}\).

**Example 1.** Define

\[ \|x\|_1 = |Sx|, \quad x \in \mathfrak{X}, \]

where \(S\) is a one to one Hilbert-Schmidt operator on \(\mathfrak{X}\). Then \(\|x\|_1\) is a measurable norm (Section 1). Therefore, by Theorem 3, every Hilbertian admissible norm is a measurable norm.

**Example 2.** Define

\[ \|x\|_2 = \sup_n \frac{1}{\gamma_n} |(\varphi_n, x)|, \quad x \in \mathfrak{X} \]

where \(\{\varphi_n\}\) is a C.O.N.S. in \(\mathfrak{X}\). Then \(\|x\|_2\) is a measurable norm but there is no Hilbertian admissible norm stronger than \(\|x\|_2\).
In fact it is evident that \( \|x\|_2 \) is a norm on \( \mathfrak{K} \). To prove that \( \|x\|_2 \) is a measurable norm, we imbed \( \mathfrak{K} \) in a measurable space \((\Omega, \mathcal{F})\) in which \( \mathfrak{K} \) is an \( \mathcal{F} \)-measurable subspace and all functions \( (\varphi_n, x), n = 1, 2, 3, \ldots \) are extended to \( \mathcal{F} \)-measurable functions on \( \Omega \), further, there exists a \( \sigma \)-additive extension \( \mu \) of \( \mu_{\mathfrak{K}} \). As an example of such a space, we can choose the space of all sequences.

Then since \( \mu \) is a \( \sigma \)-extension of \( \mu_{\mathfrak{K}} \), we have

\[
\mu(x \in \Omega: \|x\|_2 < +\infty) \\
= \mu\left(x \in \Omega: \sup_{1 \leq n < N} \frac{1}{\sqrt{n}} |(\varphi_n, x)| < +\infty\right) \\
= \lim_{N \to +\infty} \lim_{M \to +\infty} \mu\left(x \in \Omega: \sup_{1 \leq n < N} \frac{1}{\sqrt{n}} |(\varphi_n, x)| \leq M\right) \\
= \lim_{N \to +\infty} \lim_{M \to +\infty} \mu_{\mathfrak{K}}\left(x \in \mathfrak{K}: \sup_{1 \leq n < N} \frac{1}{\sqrt{n}} |(\varphi_n, x)| \leq M\right) \\
= \lim_{N \to +\infty} \lim_{M \to +\infty} \frac{1}{\sqrt{2\pi}} \int_{-M\sqrt{n}}^{M\sqrt{n}} \exp\left[-\frac{u^2}{2}\right] du \\
= \lim_{N \to +\infty} \lim_{M \to +\infty} \left[1 - \sqrt{\frac{2}{\pi}} \int_{-M\sqrt{n}}^{+\infty} \exp\left[-\frac{u^2}{2}\right] du\right] \\
\geq \lim_{N \to +\infty} \lim_{M \to +\infty} \left[1 - \sqrt{\frac{2}{\pi}} \frac{1}{M\sqrt{n}} \exp\left[-\frac{M^2}{2n}\right]\right] \\
\geq \lim_{N \to +\infty} \lim_{M \to +\infty} \left[1 - \exp\left[-\frac{M^2}{2n}\right]\right] \\
\geq \lim_{N \to +\infty} \lim_{M \to +\infty} \left[1 - \sum_{n=1}^{N} \exp\left[-\frac{M^2}{2n}\right]\right] \\
= \lim_{N \to +\infty} \lim_{M \to +\infty} \left[1 - \frac{1 - \exp\left[-\frac{M^2}{2N}\right]}{1 - \exp\left[-\frac{M^2}{2}\right]} \exp\left[-\frac{M^2}{2}\right]\right] \\
= 1,
\]

and for any positive number \( \varepsilon \)

\[
\mu(x \in \Omega: \|x\|_2 < \varepsilon) \\
= \lim_{N \to +\infty} \lim_{M \to +\infty} \mu_{\mathfrak{K}}\left(x \in \mathfrak{K}: \sup_{1 \leq n < N} \frac{1}{\sqrt{n}} |(\varphi_n, x)| < \varepsilon\right) \\
\geq \lim_{n \to +\infty} \left[1 - \frac{1}{\varepsilon\sqrt{n}} \exp\left[-\frac{\varepsilon^2}{2n}\right]\right] > 0,
\]
because
\[ \sum_{n=1}^{\infty} \frac{1}{\varepsilon \sqrt{n}} \exp \left[ -\frac{\varepsilon^2}{2} n \right] \leq \frac{1}{\varepsilon} \frac{\exp \left[ -\frac{\varepsilon^2}{2} \right]}{1 - \exp \left[ -\frac{\varepsilon^2}{2} \right]} < +\infty. \]

Therefore, by Corollary 4.5 of L. Gross [2], \( \| x \|_2 \) is a measurable norm.

While for the C.O.N.S. \( \{ \varphi_n \} \) in \( E \)
\[ \sum_{n=1}^{\infty} \| \varphi_n \|_2^2 = \sum_{n=1}^{\infty} \left( \sup_{\nu} \frac{1}{\sqrt{\nu}} |(\varphi_\nu, \varphi_n)| \right)^2 \]
\[ = \sum_{n=1}^{\infty} \frac{1}{n} = +\infty. \]

By Corollary of Theorem 3, there is no Hilbertian admissible norm stronger than \( \| x \|_2 \). This means that there is no Hilbert space of full measure which is included in the Banach space obtained by the completion of \( E \) in the norm \( \| x \|_2 \).

**Example 3.** Define
\[ \| x \|_3 = \left( \sup_n \frac{1}{n} \sum_{\nu=1}^{n} |(\varphi_\nu, x)|^2 \right)^{1/2}, \quad x \in E \]
where \( \{ \varphi_n \} \) is a C.O.N.S. in \( E \). Then \( \| x \|_3 \) is an admissible norm on \( E \) but not a measurable norm.

**Proof.** Imbed \( E \) in the measurable space \( (\Omega, \varpi, \mu) \) as in Example 2. Then by the law of large number, we have
\[ \mu[ x \in \Omega: \| x \|_3 < +\infty ] \]
\[ \geq \mu \left[ x \in \Omega: \limsup_n \frac{1}{n} \sum_{\nu=1}^{n} |(\varphi_\nu, x)|^2 = 1 \right] = 1. \]

Therefore \( \| x \|_3 \) is an admissible norm; but, according to Corollary 4.5 of L. Gross [2], it is not a measurable norm. This means that for a norm on a separable Hilbert space to be admissible, it is not necessary to be a measurable norm in the sense of L. Gross [1].
BIBLIOGRAPHY


[16] Далекукий, Ю. Л., Бесконечномерные эллиптические операторы и связанные с ними параболические уравнения. Успехи Матем. Наук, Том. 22, стр. 3-54, (1967).

Tokyo Metropolitan University