HOROCYCLIC CLUSTER SETS OF FUNCTIONS DEFINED IN THE UNIT DISC

STEPHEN DRAGOSH

1. Introduction.

This paper contains in part the author's Ph.D. dissertation written under the supervision of Professor F. Bagemihl at the University of Wisconsin-Milwaukee. This research was supported by grants from the National Science Foundation and the University of Wisconsin Graduate School.

Unless otherwise stated, $f: D \to W$ shall mean $f(z)$ is an arbitrary single-valued function defined in the open unit disc $D: |z| < 1$ and assuming values in the extended complex plane W. The unit circle $|z| = 1$ is denoted by Γ.

We assume the reader to be familiar with the rudiments of the theory of cluster sets. A general reference would be Noshiro [21] or Collingwood and Lohwater [9]. We shall use the following sets defined in [9, p. 207]:

$C(f, \zeta)$, the cluster set of f at ζ;

$C_{\mathscr{A}}(f, \zeta)$, the outer angular cluster set of f at ζ;

$C_{\Delta}(f, \zeta)$, the cluster set of f at ζ on a Stolz angle Δ at ζ;

$F(f)$, the set of Fatou points of f;

$I(f)$, the set of Plessner points of f;

$M(f)$, the set of Meier points of f;

$R(f, \zeta)$, the range of f at ζ.

We denote the cluster set of f at ζ on a chord χ at ζ by $C_{\chi}(f, \zeta)$. The principal chordal cluster set of f at ζ is defined to be

$$\Pi_{\chi}(f, \zeta) = \bigcap_{\chi} C_{\chi}(f, \zeta),$$

where the intersection is taken over all chords χ at ζ; and the inner angular cluster set of f at ζ is defined to be

---

Received July 8, 1968

$$C_{\mathscr{B}}(f, \zeta) = \bigcap_{\varDelta} C_{\varDelta}(f, \zeta),$$

where the intersection is taken over all Stolz angles $\varDelta$ at ζ. In addition we shall define the following sets:

Bagemihl defined and studied the majority of these sets in [3].

If $f: D \to W$, then a point $w \in W$ is a non-tangential cluster value of f at ζ provided there exists a sequence $\{z_n\}$ lying between two chords at ζ such that $\lim z_n = \zeta$ and $\lim f(z_n) = w$.

A circle internally tangent to $\varGamma$ at a point $\zeta \in \varGamma$ is called a horocycle at ζ, and will be denoted by $h_r(\zeta)$, where r $(0 < r < 1)$ is the radius of the horocycle. The point ζ itself is not considered to be part of $h_r(\zeta)$. A point $w \in W$ is a horocyclic cluster value of f at ζ provided there exists a sequence $\{z_n\}$ lying between two horocycles at ζ such that $\lim z_n = \zeta$ and $\lim f(z_n) = w$. Our purpose is to examine the relationships between non-tangential and horocyclic cluster values of a function f in D. In particular, we shall compare (metrically and topologically) the sets of Fatou points, Plessner points and Meier points of f with their horocyclic analogues.

Section 2 deals with arbitrary single-valued functions in D. First it is shown (Theorem 2) that Collingwood's theorem concerning $K(f)$, f meromorphic in D, is true for f arbitrary in D. If one defines $K_{\omega}(f)$ as the horocyclic analogue of $K(f)$, then (Theorem 3) $K_{\omega}(f)$ is both residual and of measure 2π on $\varGamma$; i.e. the horocyclic analogue of Collingwood's theorem is true. Theorem 4 states that there exists a set residual and of measure 2π on $\varGamma$ such that at each point ζ of the set, each non-tangential

cluster value of f at ζ is a horocyclic cluster value of f at ζ relative to every pair of horocycles at ζ. An immediate corollary is that almost every (in the sense of Lebesgue) horocyclic Fatou point of f is a Fatou point of f, and almost every Plessner point of f is a horocyclic Plessner point of f. This had been shown by Bagemihl [3, Theorems 1 and 2] for meromorphic functions.

Littlewood [16] and Lohwater and Piranian [17, Theorem 9] have shown that not almost every Fatou point of f need be a horocyclic Fatou point of f even if f is holomorphic and bounded in D. Theorems 5 and 12 demonstrate the same result. In [10] it has been shown that not almost every horocyclic Plessner point of f need be a Plessner point of f even if f is holomorphic in D. For the function f in [10], each of the sets of Fatou points of f and horocyclic Plessner points of f has measure 2π. In Section 3 some further properties of points which are simultaneously Fatou points of f and horocyclic Plessner points of f are proved for f meromorphic in D.

The results of the preceding paragraph imply the non-existence of the following horocyclic analogues of Fatou's theorem [11] and Plessner's theorem [22]: If f is holomorphic and bounded in D, then almost every point of Γ is a horocyclic Fatou point of f; if f is meromorphic in D, then almost every point of Γ is either a horocyclic Fatou point of f or a horocyclic Plessner point of f. Moreover, in Section 4 a function f is constructed such that f is holomorphic in D, but the union of the sets of horocyclic Fatou points, horocyclic Plessner points and horocyclic Meier points of f has measure zero. The horocyclic behavior of this function is explained by the introduction of what we call the primary-tangential pre-Meier point. The explanation is a consequence of a theorem (Theorem 11) similar to the statement cited as the horocyclic analogue of Plessner's theorem. Specifically, if f is meromorphic in D, then almost every point of Γ is either a primary-tangential pre-Meier point of f or a horocyclic Plessner point of f. A theorem similar to the statement cited as the horocyclic analogue of Fatou's theorem is Theorem 10: If f is a normal meromorphic function in D, then almost every point of Γ is either a primary-tangential pre-Meier point of f or a point ζ at which $\Pi_{T_\omega}(f, \zeta) = W$.

It can be easily shown [3, Theorem 3] that if f is meromorphic in D, then almost every Meier point of f is a horocyclic Meier point of f. Sec-

tion 5 is devoted to proving that not almost every horocyclic Meier point of f need be a Meier point of f even if f is holomorphic and bounded in D.

To conclude the introduction we give a brief description of horocyclic notation and terminology.

Given a horocycle $h_r(\zeta)$ at a point $\zeta \in \Gamma$, the region interior to $h_r(\zeta)$ will be denoted by $\Omega_r(\zeta)$. The half of $h_r(\zeta)$ lying to the right of the radius at ζ as viewed from the origin will be denoted by $h_r^+(\zeta)$, and is called the right horocycle at ζ with radius r. The left horocycle is defined analogously. Also, $\Omega_r^+(\zeta)$ and $\Omega_r^-(\zeta)$ denote the right and left half, respectively, of $\Omega_r(\zeta)$.

Suppose that $0 < r_1 < r_2 < 1$ and that r_3 $(0 < r_3 < 1)$ is so large that the circle $|z| = r_3$ intersects both of the horocycles $h_{r_1}(\zeta)$ and $h_{r_2}(\zeta)$. We define the right horocyclic angle $H_{r_1, r_2, r_3}^+(\zeta)$ at ζ with radii r_1, r_2, r_3 to be

$$H_{r_1, r_2, r_3}^+(\zeta) = \operatorname{comp} \overline{[\Omega_{r_1}^+(\zeta)]} \cap \Omega_{r_2}^+(\zeta) \cap \{z : |z| \geqslant r_3\},$$

where the bar denotes closure and "comp" denotes complement, both relative to the plane. The corresponding left horocyclic angle is denoted $H_{r_1, r_2, r_3}^-(\zeta)$. We write $H_{r_1, r_2, r_3}(\zeta)$ to denote a horocyclic angle at ζ without specifying whether it be right or left, or simply $H(\zeta)$ in the event r_1, r_2, r_3 are arbitrary.

Define the right outer horocyclic angular cluster set of f at ζ to be

$$C_{\mathfrak{A}^+}(f, \zeta) = \bigcup_{H^+} C_{H^+}(f, \zeta),$$

and the right inner horocyclic angular cluster set of f at ζ to be

$$C_{\mathfrak{B}}(f, \zeta) = \bigcap_{H^+} C_{H^+}(f, \zeta),$$

where in each case H^+ ranges over all right horocyclic angles at ζ. The outer horocyclic angular cluster set of f at ζ is defined to be

$$C_{\mathfrak{A}}(f, \zeta) = C_{\mathfrak{A}^+}(f, \zeta) \cup C_{\mathfrak{A}^-}(f, \zeta),$$

and the inner horocyclic angular cluster set of f at ζ to be

$$C_{\mathfrak{B}}(f, \zeta) = C_{\mathfrak{B}^+}(f, \zeta) \cap C_{\mathfrak{B}^-}(f, \zeta).$$

Finally the right principal horocyclic cluster set of f at ζ is defined to be

$$\Pi_\omega^+(f, \zeta) = \bigcap_{0 < r < 1} C_{h_r^+}(f, \zeta),$$

while we define the principal horocyclic cluster set of f at ζ to be

$$\Pi_\omega(f, \zeta) = \Pi_\omega^+(f, \zeta) \cap \Pi_\omega^-(f, \zeta).$$

If $f: D \to W$, then a point $\zeta \in \Gamma$ is called a right horocyclic Fatou point of f with right horocyclic Fatou value $w \in W$ provided

$$C_{\mathfrak{A}^+}(f, \zeta) = \{w\};$$

ζ is called a right horocyclic Plessner point of f provided

$$C_{\mathfrak{B}^+}(f, \zeta) = W;$$

ζ is called a right horocyclic Meier point of f provided

$$\Pi_\omega^+(f, \zeta) = C(f, \zeta) \subset W,$$

where $\subset$ denotes proper inclusion. The sets of right horocyclic Fatou points, right horocyclic Plessner points and right horocyclic Meier points of f are denoted by $F_\omega^+(f)$, $I_\omega^+(f)$ and $M_\omega^+(f)$ respectively. One defines $F_\omega^-(f)$, $I_\omega^-(f)$ and $M_\omega^-(f)$ in an analogous manner.

The sets of horocyclic Fatou points, horocyclic Plessner points and horocyclic Meier points of $f: D \to W$ are denoted by $F_\omega(f)$, $I_\omega(f)$ and $M_\omega(f)$ respectively, and are defined as follows:

$$\zeta \in F_\omega(f) \text{ if } C_{\mathfrak{A}}(f, \zeta) \text{ is a singleton};$$

$$\zeta \in I_\omega(f) \text{ if } C_{\mathfrak{B}}(f, \zeta) = W; \text{ i.e. } I_\omega(f) = I_\omega^+(f) \cap I_\omega^-(f);$$

$$\zeta \in M_\omega(f) \text{ if } \Pi_\omega(f, \zeta) = C(f, \zeta) \subset W; \text{ i.e. } M_\omega(f) = M_\omega^+(f) \cap M_\omega^-(f).$$

By an arc at a point $\zeta \in \Gamma$ we mean a continuous curve $\Lambda: z = z(t)$ $(0 \leqslant t < 1)$ such that $|z(t)| < 1$ for $0 \leqslant t < 1$ and $\lim_{t \to 1} z(t) = \zeta$.

A point $\zeta \in \Gamma$ is said to be an ambiguous point of $f: D \to W$ if there exist two arcs Λ_1 and Λ_2 at ζ such that

$$C_{\Lambda_1}(f, \zeta) \cap C_{\Lambda_2}(f, \zeta) = \phi.$$

Bagemihl's ambiguous point theorem [1, Theorem 2] states that f has at most enumerably many ambiguous points. Thus,

$$[F_\omega^+(f) \cap F_\omega^-(f)] - F_\omega(f)$$

must be an enumerable set for $f: D \to W$.

If S_1 and S_2 are subsets of Γ such that $S_1 - S_2$ and $S_2 - S_1$ are of first Baire category (we sometimes say that nearly every point of S_1 is a point of S_2 and nearly every point of S_2 is a point of S_1), then S_1 and S_2 are said to be topologically equivalent. If meas$[S_1 - S_2] = $ meas$[S_2 - S_1] = 0$, then S_1 and S_2 are said to be metrically equivalent.

2.   Cluster sets of arbitrary functions.

Let $\mathscr{D}(1)$ be an open connected subset of D such that $\overline{\mathscr{D}(1)} \cap \Gamma = \{1\}$. By $\mathscr{D}(\zeta)$ we shall mean the transform of $\mathscr{D}(1)$ under the rotation about the origin that sends 1 into ζ. The following lemma is quite similar to that of Collingwood [8, Theorem 2].

LEMMA 1.   *Let* $f : D \to W$. *Then*

$$C_{\mathscr{D}(\zeta)}(f, \zeta) = C(f, \zeta)$$

for a residual G_δ *subset of* Γ.

Proof.   Let D be the set of points $\zeta \in \Gamma$ for which the condition of the lemma does not hold. It suffices to prove that E is an F_σ set of first category.

Considering W to be the Riemann sphere, let $\{Q_p : p = 1, 2, \cdots\}$ be the enumerable collection of open spherical discs on W such that the boundary of Q_p is a circle whose center has rational coordinates and whose radius has rational length. Let $\frac{1}{2} Q_p$ denote the open spherical disc on W with the same center as Q_p and area one-half that of Q_p.

Given $\zeta \in E$, there exists a disc Q_p such that

$$C(f, \zeta) \cap \frac{1}{2} Q_p \neq \phi \ \text{ and } \ C_{\mathscr{D}(\zeta)}(f, \zeta) \cap \overline{Q_p} = \phi.$$

Hence we can find a positive integer m such that

$$\overline{f(\mathscr{D}(\zeta) \cap \alpha_m)} \cap Q_p = \phi,$$

where α_m is the annulus $1 - 1/m < |z| < 1$. Thus we may write

$$E = \bigcup_{m, p} E_{m, p},$$

where

$$\overline{f(\mathscr{D}(\zeta) \cap \alpha_m)} \cap Q_p = \phi \ \text{ and } \ C(f, \zeta) \cap \frac{1}{2} Q_p \neq \phi, \ \zeta \in E_{m, p}.$$

Since $\mathscr{D}(1)$ is open, one can easily prove that

$$f(\mathscr{D}(\zeta) \cap \alpha_m) \cap Q_p = \phi, \ \zeta \in \overline{E_{m, p}}.$$

Also, it is readily seen that

$$C(f, \zeta) \cap \overline{\frac{1}{2} Q_p} \neq \phi, \quad \zeta \in \overline{\overline{E_{m,p}}}.$$

Thus, $\overline{\overline{E_{m,p}}} \subseteq E$ for all values of m and p. Hence we have

$$E = \underset{m,p}{\cup} E_{m,p} \subseteq \underset{m,p}{\cup} \overline{\overline{E_{m,p}}} \subseteq E.$$

Thus, E is an F_σ subset of Γ.

We now show that each set $\overline{\overline{E_{m,p}}}$ is nowhere dense, so that E is of first category. If $\overline{\overline{E_{m,p}}}$ is dense on any open arc Γ^* of Γ, then, setting

$$\alpha_m^* = \underset{\zeta \in \Gamma^*}{\cup} \mathscr{D}(\zeta) \cap \alpha_m,$$

we have

$$\overline{f(\alpha_m^*)} \cap Q_p = \phi.$$

Since $\mathscr{D}(1)$ is connected, we obtain α_m if we allow the points ζ to range over Γ in the previous union. Also $\overline{\mathscr{D}(1)} \cap \Gamma = \{1\}$, so that no point of Γ^* is a frontier point of $\alpha_m - \alpha_m^*$. Thus, given any $\zeta \in \Gamma^*$, there exists a positive integer $N = N(\zeta)$ such that

$$\{z \in D: |z - \zeta| < 1/n\} \subset \alpha_m^*, \quad n \geqslant N.$$

Since $\overline{f(\alpha_m^*)} \cap Q_p = \phi$,

$$C(f, \zeta) \cap Q_p = \phi, \quad \zeta \in \Gamma^*.$$

This contradicts the fact that

$$C(f, \zeta) \cap \frac{1}{2} Q_p \neq \phi, \quad \zeta \in E_{m,p} \cap \Gamma^* \neq \phi.$$

This completes the proof.

The following conventions will be used throughout the remainder of this paper.

Given a point $\zeta \in \Gamma$, $\Delta_{n,r}(\zeta)$, or more simply $\Delta_{n,r}$, represents the Stolz angle at ζ such that $\Delta_{n,r}$ has aperture $\pi/2^n$, n a positive integer; and the bisector of $\Delta_{n,r}$ at ζ makes a rational angle r $(-\pi/2 < r < \pi/2)$ with the radius at ζ. If α_m is the annulus $1 - 1/m < |z| < 1$ and $1 - 1/m > \sin\left(|r| + \frac{\pi}{2^{n+1}}\right)$, then we set

$$\Delta_{n,r,m} = \Delta_{n,r} \cap \alpha_m.$$

Then for each point $\zeta \in \Gamma$, we define $\Sigma(\zeta)$ to be the enumerable collection

of all such Stolz "triangles" $\Delta_{n,r,m}(\zeta)$ at ζ. When we wish to refer to this collection without specifying a point ζ, we write Σ.

Analogously, we define $\Sigma_\omega(\zeta)$ to be the enumerable collection of horocyclic angles $H_{r_1,r_2,r_3}(\zeta)$ at ζ with the radii r_1, r_2, r_3 rational.

Making use of the enumerability of Σ and Σ_ω we can prove

LEMMA 2. *Let* $f: D \to W$. *Then*

$$C_{\mathscr{B}}(f,\zeta) = C_{\mathfrak{B}}(f,\zeta) = C(f,\zeta)$$

for a residual G_δ *subset of* Γ.

Proof. For each $\Delta \in \Sigma$, we have $C_\Delta(f,\zeta) = C(f,\zeta)$ for a residual G_δ subset of Γ by Lemma 1. The intersection of these enumerably many residual G_δ sets is a residual G_δ subset E_1 of Γ such that

$$C(f,\zeta) = \bigcap_{\Delta \in \Sigma} C_\Delta(f,\zeta) = C_{\mathscr{B}}(f,\zeta), \ \ \zeta \in E_1.$$

Similarly, we can find a residual G_δ subset E_2 of Γ such that

$$C(f,\zeta) = \bigcap_{H \in \Sigma_\omega} C_H(f,\zeta) = C_{\mathfrak{B}}(f,\zeta), \ \ \zeta \in E_2.$$

Then $E_1 \cap E_2$ is the required subset of Γ.

THEOREM 1. (Bagemihl [3, Theorem 4]). *Let* $f: D \to W$. *Then the sets* $I(f)$, $I_\omega^+(f)$, $I_\omega^-(f)$ *and* $I_\omega(f)$ *are topologically equivalent.*

Proof. Since $C_{\mathfrak{B}}(f,\zeta) = C_{\mathfrak{B}^+}(f,\zeta) \cap C_{\mathfrak{B}^-}(f,\zeta)$ for each $\zeta \in \Gamma$, Lemma 2 implies that

$$C_{\mathscr{B}}(f,\zeta) = C_{\mathfrak{B}^+}(f,\zeta) = C_{\mathfrak{B}^-}(f,\zeta) = C_{\mathfrak{B}}(f,\zeta) = C(f,\zeta)$$

for a residual set of points $\zeta \in \Gamma$. This implies the desired result.

Remark 1. A further consequence of Lemma 2 is that if any one of the sets $I(f)$, $I_\omega^+(f)$, $I_\omega^-(f)$ or $I_\omega(f)$ is dense on an arc Γ^* of Γ (hence $C(f,\zeta) = W$ for each point $\zeta \in \Gamma^*$), then each of the four sets is residual on Γ^*.

Remark 2. (Bagemihl [3, Remark 3]). Let $f: D \to W$. Then the sets $F(f)$, $F_\omega^+(f)$, $F_\omega^-(f)$ and $F_\omega(f)$ are topologically equivalent. Since $C_{\mathscr{B}}(f,\zeta) \subseteq C_{\mathscr{A}}(f,\zeta)$, $C_{\mathfrak{B}^+}(f,\zeta) \subseteq C_{\mathfrak{A}^+}(f,\zeta)$, $C_{\mathfrak{B}^-}(f,\zeta) \subseteq C_{\mathfrak{A}^-}(f,\zeta)$ and $C_{\mathfrak{B}}(f,\zeta) \subseteq C_{\mathfrak{A}}(f,\zeta)$, Lemma 2 implies that

$$C_{\mathscr{A}}(f,\zeta) = C_{\mathfrak{A}^+}(f,\zeta) = C_{\mathfrak{A}^-}(f,\zeta) = C_{\mathfrak{A}}(f,\zeta)$$

for a residual set of points $\zeta \in \Gamma$. The result now follows.

Remark 3. It need not be true that the sets $M(f)$ and $M_\omega(f)$ are topologically equivalent for $f: D \to W$. Let S be an enumerable everywhere dense subset of Γ. Define $f(z)$ in D by $f(0) = 0$ and

$$f(z) = 1 \text{ for } z \in h_{\frac{1}{2}}^+(\zeta), \ \zeta \in S,$$
$$f(z) = 0 \text{ for } z \in h_{\frac{1}{2}}^+(\zeta), \ \zeta \in \Gamma - S.$$

Since both S and $\Gamma - S$ are everywhere dense on Γ,

$$\Pi_x(f,\zeta) = C(f,\zeta) = \{0,1\}, \ \zeta \in \Gamma.$$

However, $\Pi_\omega(f,\zeta) = \{0\}$ for $\zeta \in \Gamma - S$, and $\Pi_\omega(f,\zeta) = \{1\}$ for $\zeta \in S$. Thus $M(f) = \Gamma$, but $M_\omega(f) \cap \Gamma = \phi$. This example also shows that $M(f)$ and $M_\omega(f)$ need not be metrically equivalent for $f: D \to W$.

DEFINITION 1. If $f: D \to W$, then $K(f)$ consists of those points $\zeta \in \Gamma$ for which $C_{\varDelta_1}(f,\zeta) = C_{\varDelta_2}(f,\zeta)$ for any pair of Stolz angles $\varDelta_1$ and $\varDelta_2$ at ζ.

Collingwood [7, Theorem 4a] has shown that $K(f)$ is both residual and of measure 2π on Γ for f meromorphic in D. It is a consequence of the following lemma that the same result holds for an arbitrary function f in D.

LEMMA 3. *Let* $f: D \to W$. *Then at almost every and nearly every point* $\zeta \in \Gamma$,

$$C_{\mathscr{L}(\zeta)}(f,\zeta) \subseteq C_{\mathscr{B}}(f,\zeta)$$

where $\mathscr{L}(\zeta)$ *is any set for which there exists a Stolz angle at* ζ *containing* $\mathscr{L}(\zeta)$.

Proof. If E is the set of points $\zeta \in \Gamma$ for which the lemma fails to hold, then for each $\zeta \in E$ there exists a set $\mathscr{L}(\zeta)$ lying in the interior of a Stolz angle at ζ such that $C_{\mathscr{L}(\zeta)}(f,,\zeta) \nsubseteq C_{\varDelta(\zeta)}(f,\zeta)$ for some (not necessarily the same) Stolz angle $\varDelta(\zeta)$ at ζ. Then there exists a disc Q_p on the Riemann sphere W such that

$$C_{\mathscr{L}(\zeta)}(f,\zeta) \cap Q_p \neq \phi \text{ and } C_{\varDelta(\zeta)}(f,\zeta) \cap \overline{Q_p} = \phi.$$

It is then possible to find a Stolz triangle $\varDelta_{n,r,m}(\zeta) \in \Sigma(\zeta)$ such that $\overline{f(\varDelta_{n,r,m}(\zeta))} \cap Q_p = \phi$. Thus we may write

$$E = \bigcup_{n,r,m,p} E_{n,r,m,p},$$

where $\zeta \in E_{n,r,m,p}$ provided there exists at least one set $\mathscr{S}(\zeta)$ lying in a Stolz angle at ζ such that

$$C_{\mathscr{S}(\zeta)}(f,\zeta) \cap Q_p \neq \phi \text{ and } \overline{f(\varDelta_{n,r,m}(\zeta))} \cap Q_p = \phi.$$

Now suppose that some set $E_{n,r,m,p}$ has positive exterior measure. If $X \equiv E_{n,r,m,p}$, then

(1)
$$\overline{f(\varDelta_{n,r,m}(\zeta))} \cap Q_p = \phi, \quad \zeta \in \bar{X}.$$

Note that it is not necessarily true that $C_{\mathscr{S}(\zeta)}(f,\zeta) \cap Q_p \neq \phi$ for at least one set $\mathscr{S}(\zeta)$ lying in some Stolz angle at ζ for each $\zeta \in \bar{X}$.

If

(2)
$$G = \bigcup_{\zeta \in \bar{X}} \varDelta_{n,r,m}(\zeta),$$

then G is composed of finitely many open simply connected subregions $G_1, \cdots, G_N$ of D. There are only finitely many such subregions because $\Gamma - \bar{X}$ contains only finitely many arcs with length exceeding a fixed number between 0 and 2π. As in [23, p. 220], we conclude that each subregion G_k $(1 \leqslant k \leqslant N)$ has a rectifiable Jordan curve J_k $(1 \leqslant k \leqslant N)$ as boundary.

Now $X \cap J_k$ must be of positive exterior measure for at least one curve J_k. Also the tangent to J_k at almost every point of $\Gamma \cap J_k$ coincides with the tangent to Γ. Consequently, there exist points of X belonging to $\Gamma \cap J_k$ at which the tangent to J_k coincides with the tangent to Γ. At any such point each Stolz angle at the point has a terminal portion (i.e. a Stolz triangle at ζ) contained in G_k. Thus there exist points $\zeta \in X$, such that $C_{\mathscr{S}(\zeta)}(f,\zeta) \subseteq \overline{f(G_k)}$ for every set $\mathscr{S}(\zeta)$ at ζ which is contained in a Stolz angle at ζ. By (1) and (2),

$$\overline{f(G_k)} \cap Q_p = \phi.$$

However, according to the definition of X, we must have $C_{\mathscr{S}(\zeta)}(f,\zeta) \cap Q_p \neq \phi$ for at least one set $\mathscr{S}(\zeta)$ lying in some Stolz angle at ζ for every $\zeta \in X$ which is inconsistent with the previous statement. Hence each set $E_{n,r,m,p}$, and consequently E, has measure zero.

It is evident that our proof needs only minor modifications to establish that each set $E_{n,r,m,p}$, and consequently E, is of first category.

THEOREM 2. *Let $f: D \to W$. Then $K(f)$ is both residual and of measure 2π on Γ.*

Proof. At each point $\zeta \in \Gamma - K(f)$ there exists a Stolz angle $\Delta(\zeta)$ such that $C_{\Delta(\zeta)}(f, \zeta) \nsubseteq C_{\mathscr{B}}(f, \zeta)$. Lemma 3 implies that $\Gamma - K(f)$ is of measure zero and first category.

DEFINITION 2. If $f: D \to W$, then $K_\omega(f)$ consists of those points $\zeta \in \Gamma$ for which $C_{H_1}(f, \zeta) = C_{H_2}(f, \zeta)$ for any pair of horocyclic angles H_1 and H_2 at ζ.

Remark 4. A most crucial line of reasoning in the proof of Lemma 3 was that each Jordan curve J_k was rectifiable so that the tangent to J_k coincided with the tangent to Γ at almost every point $\zeta \in \Gamma \cap J_k$; and consequently, at almost every point $\zeta \in \Gamma \cap J_k$, each Stolz angle at ζ had a terminal portion interior to G_k.[†]

For a fixed horocyclic angle $H_{r_1, r_2, r_3}(\zeta)$ and a closed set $P \subset \Gamma$, define

$$G^\omega = \bigcup_{\zeta \in P} H_{r_1, r_2, r_3}(\zeta).$$

By [3, Lemma 1], G^ω is composed of finitely many simply connected subregions $G_1^\omega, \cdots, G_N^\omega$ having as their respective boundaries the rectifiable Jordan curves $J_1^\omega, \cdots, J_N^\omega$. Hence the tangent to J_k^ω ($1 \leqslant k \leqslant N$) at almost every point $\zeta \in \Gamma \cap J_k^\omega$ coincides with the tangent to Γ. However, this does not imply that at almost every point $\zeta \in \Gamma \cap J_k^\omega$, each horocyclic angle H has a terminal portion which lies in G_k^ω, because the tangent to H at ζ also coincides with the tangent to Γ at ζ. But if we can verify that this latter statement is true, then by virtually the same proof as of Lemma 3 we can obtain a horocyclic analogue of Lemma 3 (see Lemma 6).

LEMMA 4. *Let P be a perfect nowhere dense subset of $[0, 1]$. For almost every point $p \in P$, if $\{(a_n, b_n)\}$ is any sequence of open intervals in $[0, 1] - P$ converging to p, then*

$$|a_n - p|/(b_n - a_n) \text{ tends to } +\infty.$$

 [†] If $S \subset D$ such that $\bar{S} \cap \Gamma = \{\zeta\}$, then $S' \subseteq S$ is called a terminal portion of S if $S' \cap D - \alpha_m = \phi$ and $S' \cap \alpha_p = S \cap \alpha_p$, where $p \geqslant m > 0$.

Proof. According to Hobson [12, p. 194], the metric density exists and is unity at almost every point $p \in P$. Let $p \in P$ be such a point, and suppose the sequence $\{(a_n, b_n)\}$ converges to p from the right. Then by the definition of metric density

$$(3) \qquad \lim_{n \to +\infty} \frac{\text{meas}\,(P \cap (p, b_n))}{\text{meas}\,(p, b_n)} = 1$$

and

$$(4) \qquad \lim_{n \to +\infty} \frac{\text{meas}\,(P \cap (p, a_n))}{\text{meas}\,(p, a_n)} = 1.$$

Let $x_n = \text{meas}\,(P \cap (p, b_n))$, $y_n = a_n - p$ and $z_n = b_n - a_n$. Then (3) implies

$$\frac{x_n}{y_n + z_n} \to 1$$

and, since $P \cap (p, b_n) = P \cap (p, a_n)$, (4) implies

$$\frac{x_n}{y_n} \to 1.$$

Since $x_n > 0$, $y_n > 0$ and $z_n > 0$, these conditions imply that

$$\frac{z_n}{y_n} \to 0; \text{ i.e. } \frac{y_n}{z_n} \to +\infty.$$

Thus $(a_n - p)/(b_n - a_n) \to +\infty$ and in general, $|a_n - p|/(b_n - a_n) \to +\infty$.

LEMMA 5. *Let P be a perfect nowhere dense subset of Γ and set*

$$G^\omega = \bigcup_{\zeta \in P} H_{r_1, r_2, r_3}(\zeta),$$

where H_{r_1, r_2, r_3} is a fixed horocyclic angle. Then at almost every point $\zeta \in P$ each disc $\Omega_r(\zeta)$ $(0 < r < 1)$ has a terminal portion lying interior to G^ω.

Proof. Without explicitly going through all the details we note that it is possible, by means of a bilinear mapping $L(z)$, to transfer the setting of our lemma from D to the upper half-plane and arrive at an equivalent formulation. We now give this formulation in a somewhat extensive form.

Let P be a perfect nowhere dense set on the finite interval I of the real axis, and let the two circles (take $(a_n, b_n) \subset I - P$)

$$(5) \qquad C_1: (x - a_n)^2 + (y - R)^2 = R^2 \text{ and } C_2: (x - b_n)^2 + (y - r)^2 = r^2$$

have radii satisfying

(6) $$0 < R_1 \leqslant r \leqslant R_2 < R_3 \leqslant R \leqslant R_4.$$

We choose r and R in this fashion because the two horocycles $h_{r_1}(\zeta)$ and $h_{r_2}(\zeta)$ forming part of $H_{r_1, r_2, r_3}(\zeta)$, and hence part of the boundary of G^ω, would be mapped by $L(z)$, as ζ ranges over $P \subset \Gamma$, onto circles of the form (5) whose radii satisfy a condition of the form (6).
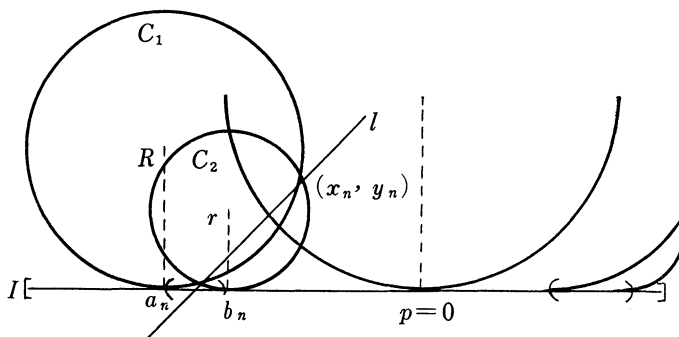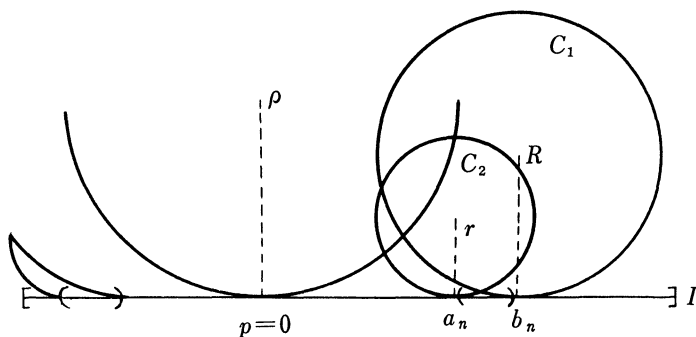
Figure 1

Figure 2

At the left and right endpoints of each interval in $I - P$ construct circles C_1 and C_2 respectively (see Figure 1). In the proof it shall become apparent that we could choose C_1 to be at the right endpoint and C_2 at the left endpoint of each interval in $I - P$ (see Figure 2). These two situations correspond to the choice of $H_{r_1, r_2, r_3}(\zeta)$ as a left and right horocyclic angle, respectively.

Our ultimate goal is to prove:

(7)   At almost every point $p \in P$, for any sequence $\{(a_n, b_n)\}$ of arcs in $I-P$

converging to p, the point $(x_n, y_n) \in C_1 \cap C_2$ (see Figure 1) lies interior to any given circle tangent to the x-axis at p for at most finitely many values of n.

Our method of proof will be to show that the condition on p in (7) is satisfied at each point $p \in P$ at which Lemma 4 holds. Since Lemma 4 holds for almost every point $p \in P$, (7), and hence our lemma, will be established.

Suppose to the contrary that there exists a point $p \in P$ at which Lemma 4 holds and the condition on p in (7) fails to be true. Without loss of generality we may assume that $p = 0$. Hence, we are assuming that there exists a circle C: $x^2 + (y - \rho)^2 = \rho^2 \, (0 < \rho < + \infty)$ and a sequence $\{(a_n, b_n)\}$ in $I - P$ converging to $p = 0$ for which $|a_n|/(b_n - a_n) \to + \infty$, but the point $(x_n, y_n) \in C_1 \cap C_2$ lies interior to C for infinitely many values of n; i.e.

$$(8) \qquad\qquad x_n^2 + (y_n - \rho)^2 < \rho^2 \text{ for infinitely many } n.$$

Also, since $|a_n|/(b_n - a_n) \to + \infty$ and $sgn(a_n) = sgn(b_n)$,

$$(9) \qquad\qquad |b_n + a_n|/(b_n - a_n) \to + \infty.$$

Consider the radical axis l of C_1 and C_2 passing through $C_1 \cap C_2$. The equation for l is given by

$$(x - a_n)^2 + (y - R)^2 - R^2 - [(x - b_n)^2 + (y - r)^2 - r^2] = 0,$$

or

$$x = \frac{R - r}{b_n - a_n} \, y + \frac{b_n + a_n}{2} \, .$$

Hence,

$$(10) \qquad\qquad x_n = \frac{R - r}{b_n - a_n} \, y_n + \frac{b_n + a_n}{2} \, .$$

Solving (10) simultaneously with the equation of C_1 in (5) for y_n, we have

$$\left(\frac{R - r}{b_n - a_n} \, y_n + \frac{b_n + a_n}{2} - a_n \right)^2 + (y_n - R)^2 = R^2.$$

This can be rewritten as

$$(R - r)^2 \, \frac{y_n}{(b_n - a_n)^2} + \frac{(b_n - a_n)^2}{y_n} = R + r - y_n.$$

Since $y_n \to 0^+$ we immediately have

$$y_n = 0((b_n - a_n)^2),$$

and hence,

(11) $y_n < K(b_n - a_n)^2, \quad K > 0, \quad$ for all sufficiently large n.

Now we show that (8) is impossible. Substituting (10) in (8) yields

(12) $\left(\dfrac{R - r}{b_n - a_n} \right)^2 y_n + (R - r) \left(\dfrac{b_n + a_n}{b_n - a_n} \right) + \left(\dfrac{b_n + a_n}{2} \right)^2 \dfrac{1}{y_n} + y_n < 2\rho.$

The left-hand side of (12) is greater than

$$(R - r) \left(\frac{b_n + a_n}{b_n - a_n} \right) + \left(\frac{b_n + a_n}{2} \right)^2 \frac{1}{y_n} ,$$

and by (6) and (11), this expression is greater than

$$(R_3 - R_2) \left(\frac{b_n + a_n}{b_n - a_n} \right) + \left(\frac{b_n + a_n}{2} \right)^2 \frac{1}{K(b_n - a_n)^2}$$

$$= \frac{b_n + a_n}{b_n - a_n} \left[R_3 - R_2 + \frac{1}{4K} \frac{b_n + a_n}{b_n - a_n} \right].$$

By (9) this latter expression tends to $+ \infty$ so that (12), and hence (8), can hold for at most finitely many values of n, which is a contradiction. Thus our lemma is proved.

LEMMA 6. *Let $f : D \to W$. Then at almost every and nearly every point $\zeta \in \Gamma$,*

$$C_{\mathscr{H}(\zeta)}(f, \zeta) \subseteq C_{\mathfrak{B}}(f, \zeta)$$

where $\mathscr{H}(\zeta)$ is any set for which there exists a disc $\Omega_r(\zeta)$ at ζ containing $\mathscr{H}(\zeta)$.

Proof. As stated in Remark 4, the proof of Lemma 3 with only minor modifications can be used here. We replace Stolz angles by horocyclic angles, the region G by a region G^{ω} and apply Lemma 5 where needed.

THEOREM 3. *Let $f : D \to W$. Then $K_{\omega}(f)$ is both residual and of measure 2π on Γ.*

Proof. At each point $\zeta \in \Gamma - K_{\omega}(f)$ there exists a horocyclic angle $H(\zeta)$ such that $C_{H(\zeta)}(f, \zeta) \nsubseteq C_{\mathfrak{B}}(f, \zeta)$. Lemma 6 implies that $\Gamma - K_{\omega}(f)$ is of measure zero and first category.

COROLLARY 1.   *Let $f: D \to W$.   Then the sets $F_\omega^+(f)$, $F_\omega^-(f)$ and $F_\omega(f)$ are metrically equivalent, and the sets $I_\omega^+(f)$, $I_\omega^-(f)$ and $I_\omega(f)$ are metrically equivalent.*

Proof. If ζ belongs to at least one of the sets $F_\omega^+(f)$, $F_\omega^-(f)$, $F_\omega(f)$, but not to all of them, then $C_{H_1}(f, \zeta) \neq C_{H_2}(f, \zeta)$ for some pair of horocyclic angles H_1 and H_2 at ζ. By Theorem 3, the set of such points $\zeta \in \Gamma$ is of measure zero. This proves the first part of Corollary 1, and the proof of the second part is identical.

Remark 5.   Lemma 2 affords some additional information concerning $K(f)$ and $K_\omega(f)$. The relation

$$C_\Delta(f, \zeta) = C_H(f, \zeta) = C(f, \zeta)$$

holds at nearly every point $\zeta \in K(f) \cap K_\omega(f)$ for any Stolz angle Δ at ζ and any horocyclic angle H at ζ.

THEOREM 4.   *Let $f: D \to W$.   Then at almost every and nearly every point $\zeta \in \Gamma$,*

$$C_\Delta(f, \zeta) \subseteq C_H(f, \zeta)$$

for each Stolz angle Δ at ζ and each horocyclic angle H at ζ.

Proof. If ζ is a point where the condition fails to hold, then $C_{\Delta(\zeta)}(f, \zeta) \nsubseteq C_{\mathfrak{B}}(f, \zeta)$ for some Stolz angle $\Delta(\zeta)$ at ζ. Lemma 6 implies the desired result.

We can now generalize two results of Bagemihl [3, Theorems 1 and 2].

COROLLARY 2.   *Let $f: D \to W$.   Then almost every horocyclic Fatou point of f is a Fatou point of f, and almost every Plessner point of f is a horocyclic Plessner point of f.*

Proof. If $\zeta \in F_\omega(f)$, then there exists a horocyclic angle $H(\zeta)$ at ζ and a point $w_\zeta \in W$ such that $C_{H(\zeta)}(f, \zeta) = \{w_\zeta\}$. From Theorem 4 we conclude that $C_{\mathscr{A}}(f, \zeta) = \{w_\zeta\}$ for almost every point $\zeta \in F_\omega(f)$; i.e. almost every point of $F_\omega(f)$ is a point of $F(f)$.

If $\zeta \in I(f)$, then $C_{\mathscr{B}}(f, \zeta) = W$. According to Theorem 4, $C_{\mathscr{B}}(f, \zeta) \subseteq C_{\mathfrak{B}}(f, \zeta)$ for almost every point $\zeta \in \Gamma$. Thus $C_{\mathfrak{B}}(f, \zeta) = W$ for almost every point $\zeta \in I(f)$, which is the desired conclusion.

3.   The set $F(f) \cap I_\omega(f)$.

The following example, a special case of an example of Lohwater and Piranian [17, Theorem 9], shows that $F(f)$ and $F_\omega(f)$ need not be metrically equivalent.

THEOREM 5.   *There exists a function $B(z)$ holomorphic and bounded in D such that the set of horocyclic Fatou points of $B(z)$ has measure zero.*

Proof.   The Blaschke product

$$B(z) = \prod_{n=1}^{\infty} \frac{(\rho_n)^{2^n} + (z)^{2^n}}{1 + (\rho_n z)^{2^n}}, \quad \rho_n = 1 - (n^2 2^n)^{-1}, \quad n = 1, 2, \cdots,$$

has zeros at the points

$$z_{n,k} = \rho_n e^{i(2k-1)2^{-n}\pi}, \quad n = 1, 2, \cdots; \ k = 1, 2, \cdots, 2^n.$$

In [10] it is shown that for each point $\zeta \in \Gamma$ and each horocycle $h_r(\zeta)$ $(0 < r < 1)$ at ζ, there exist sequences of these zeros lying interior to $\Omega_r^+(\zeta)$ and $\Omega_r^-(\zeta)$.   Thus, for each $\zeta \in \Gamma$,

(13) $0 \in C_{\Omega_r^+(\zeta)}(B, \zeta) \ (0 < r < 1)$ and $0 \in C_{\Omega_r^-(\zeta)}(B, \zeta) \ (0 < r < 1).$

It is well-known [24, p. 94] that a Blaschke product has a Fatou value of modulus one at almost every point $\zeta \in \Gamma$.   Take $\zeta \in F(B)$ such that B has Fatou value α, $|\alpha| = 1$, at ζ.   If ζ is a right horocyclic Fatou point of B, then the right horocyclic Fatou value must be 0 because a result of Lindelöf [6, p. 42] states that the right horocyclic Fatou value of B at ζ must equal

$$C_{\Omega_r^+(\zeta)}(B, \zeta) \qquad (0 < r < 1),$$

and, from (13), 0 belongs to each such cluster set.   Thus,

$$C_{\Omega_r^+(\zeta)}(B, \zeta) = \{0\} \qquad (0 < r < 1).$$

However, this contradicts the fact that $C_{\Delta(\zeta)}(B, \zeta) = \{\alpha\}$ for each Stolz angle $\Delta(\zeta)$ at ζ.   Thus the set of right horocyclic Fatou points of B is of measure zero.   By Corollary 1, $F_\omega(f)$ has measure zero, and the proof is complete.

To show that $I(f)$ and $I_\omega(f)$ need not be metrically equivalent, we cite the following theorem proven in [10].

THEOREM 6.   *There exists a function $f(z)$ holomorphic in D such that every point of Γ is a horocyclic Plessner point of f and almost every point of Γ is a Fatou point of f.*

The following corollary is interesting in view of Plessner's theorem [22] and Meier's theorem [18, Theorem 5].

COROLLARY 3.   *There exists a function $f(z)$ holomorphic in D such that almost every point of Γ is a Fatou point of f and nearly every point of Γ is a Plessner point of f.*

Proof. By Theorem 1, $I(f)$ and $I_\omega(f)$ are topologically equivalent. Since every point $\zeta \in \Gamma$ is a point of $I_\omega(f)$, the result follows.

Theorem 6 shows that $F(f) \cap I_\omega(f)$ may be large metrically even if f is holomorphic in D.   However, for $f : D \to W$, $F(f) \cap I_\omega(f)$ must be of first category by Theorem 1.

An arc Λ_ω at $\zeta \in \Gamma$ is said to be an admissible tangential arc at ζ if there exists a sequence $\{H r_1^{(n)}, r_2^{(n)}, r_3^{(n)}(\zeta)\}$ of nested right or of nested left horocyclic angles at ζ with $\lim_{n\to\infty}[r_2^{(n)} - r_1^{(n)}] = 0$, each term of which contains some terminal subarc of Λ_ω.

We now define

$$\Pi_{T_\omega}(f, \zeta) = \bigcap_{\Lambda_\omega} C_{\Lambda_\omega}(f, \zeta),$$

where the intersection is taken over all admissible tangential arcs Λ_ω at ζ.

THEOREM 7.   *If $f(z)$ is meromorphic in D, then*

$$\Pi_{T_\omega}(f, \zeta) \cup R(f, \zeta) = W$$

for each point $\zeta \in F(f) \cap I_\omega(f)$ with the possible exception of at most enumerably many such points.

Proof. If ζ is a point of $F(f) \cap I_\omega(f)$ such that $\Pi_{T_\omega}(f, \zeta) \cup R(f, \zeta) \subset W$, then either $W - [\Pi_{T_\omega}(f, \zeta) \cup R(f, \zeta)]$ is the Fatou value of f at ζ or there exists a value $w \notin \Pi_{T_\omega}(f, \zeta) \cup R(f, \zeta)$ different from the Fatou value of f at ζ.   We assert that in either case, ζ is an ambiguous point of f.   Bagemihl's ambiguous point theorem [1, Theorem 2] then implies the desired result.

In the first case $C_\chi(f, \zeta) \cap C_{\Lambda_\omega}(f, \zeta) = \phi$ for each chord χ at ζ and some admissible tangential arc Λ_ω at ζ, so that ζ is an ambiguous point of f.

In the second case there must be an admissible tangential arc Λ_ω at ζ such that $w \notin C_{\Lambda_\omega}(f, \zeta)$. Let χ be a chord at ζ disjoint from Λ_ω, and join the endpoints of χ and Λ_ω by means of a Jordan arc J^* so that $\{\zeta\} \cup \Lambda_\omega \cup J^* \cup \chi$ is a Jordan curve. Let G denote the interior of this Jordan curve and set $J = \Lambda_\omega \cup J^* \cup \chi$. Since Λ_ω is an admissible tangential arc at ζ, G must contain at least one right or left horocyclic angle at ζ. Thus $C_G(f, \zeta) = W$. Since w is not the Fatou value of f at ζ and $w \notin C_{\Lambda_\omega}(f, \zeta)$, $w \notin C_J(f, \zeta)$. Moreover, $w \notin R_G(f, \zeta)$, because $w \notin R(f, \zeta)$. Hence

$$w \in [C_G(f, \zeta) - C_J(f, \zeta)] \cap \operatorname{comp} R_G(f, \zeta),$$

so that by the Gross-Iversen theorem [9, p. 101], there exists an arc Λ at ζ such that $C_\Lambda(f, \zeta) = \{w\}$. Hence, ζ is an ambiguous point of f, and the theorem is proved.

COROLLARY 4. *If $f(z)$ is holomorphic in D, then*

$$\infty \in \Pi_{T_\omega}(f, \zeta)$$

for each point $\zeta \in F(f) \cap I_\omega(f)$ with the possible exception of at most enumerably many such points.

We now prove that Corollary 4 is no longer true if we replace $F(f) \cap I_\omega(f)$ by $I_\omega(f)$.

THEOREM 8. *Let P be a perfect nowhere dense subset of Γ. Then there exists a function $f(z)$ holomorphic in D such that almost every point of P is a point of $I_\omega(f)$, and $\Pi_{T_\omega}(f, \zeta) = \{0\}$ for each point $\zeta \in P$ with at most enumerably many exceptions.*

Proof. Set

$$T = \bigcup_{\zeta \in P} h_{\frac{1}{2}}^{+}(\zeta).$$

Then T is a tress in the sense of Bagemihl and Seidel [4, Definition 1], and there exists a function $f(z)$ holomorphic in D such that

(14) $$C_{h_{\frac{1}{2}}^{+}(\zeta)}(f, \zeta) = \{0\}$$

for each point $\zeta \in P$ [4, Corollary 1].

If $\operatorname{meas}[P \cap F(f)] > 0$, then, since $C_{h_{\frac{1}{2}}^{+}(\zeta)}(f, \zeta) = \{0\}$ for each point $\zeta \in P \cap F(f)$, f must have 0 as Fatou value at each point $\zeta \in P \cap F(f)$ with the possible exception of at most enumerably many ambiguous

points. But this is impossible by Priwalow's theorem [9, Theorem 8. 1]. Hence almost every point of P is a point of $I(f)$ by Plessner's theorem. By Corollary 2, almost every point of P is a point of $I_\omega(f)$. By (14), $\Pi_{T_\omega}(f, \zeta) = \{0\}$ at any point of P which is not an ambiguous point of f. This completes the proof of the theorem.

Remark 6. By [21, Remark, p. 74], it is not possible to construct the function $f(z)$ of Theorem 8 to have both a right and a left horocycle at almost every point $\zeta \in P$ on which f is bounded.

Remark 7. Theorem 4 states that $C_{\mathscr{G}}(f, \zeta) \subseteq C_{\mathfrak{B}}(f, \zeta)$ for almost every point $\zeta \in \Gamma$ for $f : D \to W$. It is a consequence of Theorem 8 that even if f is holomorphic in D, then it need not be true that $\Pi_x(f, \zeta) \subseteq \Pi_\omega(f, \zeta)$ for almost every point $\zeta \in \Gamma$.

If f is holomorphic in D, then, by applying the Gross-Iversen theorem, one sees that

$$\infty \in \Pi_x(f, \zeta) \cup \Pi_\omega(f, \zeta)$$

for each point $\zeta \in I(f) \cup I_\omega(f)$ with the possible exception of at most enumerably many ambiguous points. Thus, for the function $f(z)$ in Theorem 8, $\infty \in \Pi_x(f, \zeta)$ and $\infty \notin \Pi_\omega(f, \zeta)$ for almost every point $\zeta \in P$ since almost every point of P is a point of $I_\omega(f)$.

It is an open question whether $\Pi_x(f, \zeta) \subseteq \Pi_\omega(f, \zeta)$ for nearly every point $\zeta \in \Gamma$ if $f(z)$ is meromorphic in D.

4.   Horocyclic cluster sets of meromorphic functions.

THEOREM 9.   *There exists a function $f(z)$ holomorphic in D such that almost every point of Γ is a Fatou point of f, but*

$$\text{meas } [F_\omega(f) \cup M_\omega(f) \cup I_\omega(f)] = 0.$$

Proof. For the Blaschke product $B(z)$ of Theorem 5, almost every point $\zeta \in \Gamma$ is a Fatou point of B with Fatou value of modulus one. By a theorem of Lusin [12, p. 192], this set of Fatou points of B contains a set S of measure 2π such that $S = \bigcup_n S_n$, where $S_1 \subset S_2 \subset \cdots \subset S_n \subset S_{n+1} \subset \cdots \subset \Gamma$ and each S_n is a perfect nowhere dense set.

By essentially the same method as used in [10], it is possible to construct a function $g(z)$ holomorphic in D such that $g(z)$ is bounded on the

disc $\Omega_{\frac{1}{2}}(\zeta)$ for every point $\zeta \in S$; and for each point $\zeta \in \Gamma$, there exists a sequence $\{z_n\} \subset D$ converging to ζ for which $\mathscr{R}g(z_n) \to +\infty$ and $|B(z_n)| \geq \dfrac{1}{2}$. If we set $f(z) = B(z)\,e^{g(z)}$, then the latter property of $g(z)$ implies that $\infty \in C(f,\zeta)$ for each point $\zeta \in \Gamma$. The former property of $g(z)$ implies that $f(z)$ is bounded on $\Omega_{\frac{1}{2}}(\zeta)$ for each point $\zeta \in S$. Hence the set $M_\omega(f) \cup I_\omega(f)$ is of measure zero, while the set of Fatou points of f has measure 2π by Plessner's theorem.

Let $\zeta \in \Gamma$ be a point at which $f(z)$ has a non-zero Fatou value and $f(z)$ is bounded on $\Omega_{\frac{1}{2}}(\zeta)$. The set of such points has measure 2π since it contains all points of S. Since the zeros of $B(z)$ are zeros of $f(z)$,

$$0 \in C_{\Omega_r^+(\zeta)}(f,\zeta)\,(0 < r < 1) \text{ and } 0 \in C_{\Omega_r^-(\zeta)}(f,\zeta)\,(0 < r < 1).$$

By the same argument as in Theorem 5, the point ζ cannot be a right horocyclic Fatou point of f. Thus $F_\omega(f)$ has measure zero.

We now indicate how to modify the method in [10] in order to construct the function $g(z)$. For each $n = 1, 2, \cdots$, define

$$G_n = \left(\bigcup_{\zeta \in S_n} \Omega_{\frac{1}{2}}(\zeta)\right) \cup \{z : |z| < \rho_n\},$$

where $\dfrac{1}{2} < \rho_1 < \rho_2 < \cdots < \rho_n < \rho_{n+1} < \cdots < 1$ and $\rho_n \to 1$. Also, for each $n = 1, 2, \cdots$, let Z_n be a finite subset of $D - \overline{G_n}$ chosen as follows:

(1) in each component of $D - \overline{G_1}$ having area in the range $[\pi/2^n, \pi/2^{n-1})$, choose a point z in $D - \overline{G_n}$ at which $|B(z)| \geq \dfrac{1}{2}$ (recall that $B(z)$ has radial limit of modulus one on a dense set of radii);

(2) in each component of $D - \overline{G_2}$ having area in the range $[\pi/2^{n+1}, \pi/2^n)$ choose a point z in $D - \overline{G_n}$ at which $|B(z)| \geq \dfrac{1}{2}$;

$\cdot$
$\cdot$
$\cdot$

(n) in each component of $D - \overline{G_n}$ having area in the range $[\pi/2^{2n-1}, \pi/2^{2n-2})$ choose a point z at which $|B(z)| \geq \dfrac{1}{2}$.

It is easily proven that the collection $\bigcup_n Z_n$ has Γ as its derived set, so that for each $\zeta \in \Gamma$ there exists a sequence $\{z_{n_k}\}$ converging to ζ where $z_{n_k} \in Z_{n_k}$.

For the function $t(z)$ defined on the sets T_n we substitute the function $\tau(z)$ defined on the sets Z_n by $\tau(z) = n$, $z \in Z_n$, $n = 1, 2, \cdots$. Also, we define

$$F_n = \overline{G}_n \cup \left(\bigcup_{1 \leqslant j \leqslant n} Z_j \right), \quad n = 1, 2, \cdots,$$

so that each F_n is a compact set with connected complement. We obtain by induction a sequence of polynomials $\{p_n(z)\}$ converging (uniformly on compact subsets of D) to a function $g(z)$ holomorphic in D such that $g(z)$ is bounded in G_n, $n = 1, 2, \cdots$. Since $\Omega_{\frac{1}{3}}(\zeta)$ is a subset of G_n for each $\zeta \in S_n$ ($n = 1, 2, \cdots$), $g(z)$ is bounded on $\Omega_{\frac{1}{3}}(\zeta)$ for each $\zeta \in S_n$ ($n = 1, 2, \cdots$) as required.

The sequence $\{p_n(z)\}$ also satisfies

$$|p_n(z) - \tau(z)| < 2^{-n}, \quad z \in \bigcup_{1 \leqslant j \leqslant n} Z_j,$$

$$|g(z) - p_n(z)| < 2^{-n}, \quad z \in D_{\rho_n}.$$

Thus,

$$\lim_{\substack{z \to \zeta \in \Gamma \\ z \in \bigcup_n Z_n}} |g(z) - \tau(z)| = 0.$$

Hence for each point $\zeta \in \Gamma$ there exists a sequence $\{z_{n_k}\}$ converging to ζ, $z_{n_k} \in Z_{n_k}$, such that

$$\lim_{k \to \infty} |\mathscr{R}g(z_{n_k}) - \tau(z_{n_k})| = \lim_{k \to \infty} |\mathscr{R}g(z_{n_k}) - n_k| = 0.$$

The function $g(z)$ has the required properties, and the theorem is proved.

To determine the horocyclic behavior of the function $f(z)$ of Theorem 9, we begin with the definition of a normal meromorphic function in the unit disc D due to Noshiro [20].

DEFINITION 3. Let $f(z)$ be a meromorphic function in D. Denote by $z' = L(z)$ an arbitrary one-to-one conformal mapping of D onto itself. The function $f(z)$ is called normal in D if the family of functions $\{f(L(z))\}$ is normal in the sense of Montel, where convergence is defined in terms of the spherical metric.

LEMMA (Bagemihl [3, Lemma 4]). *If $f(z)$ is a normal meromorphic function in D and $\zeta \in K_\omega(f)$, then*

$$\Pi_{T_\omega}(f,\zeta) = C_{\mathfrak{A}}(f,\zeta).$$

Remark 8.　A meromorphic function assuming each of three values only finitely often in D is normal in D (see [19, pp. 125-126] or [15, p. 54]). If f is meromorphic in D and ζ is a horocyclic Meier point of f, then $C(f,\zeta) \subset W$. Thus f is normal on each disc $\Omega_r(\zeta)$ $(0 < r < 1)$. From this and the lemma of Bagemihl just cited, one can prove that

$$\Pi_{T_\omega}(f,\zeta) = C(f,\zeta) \subset W$$

at each horocyclic Meier point of a meromorphic function f.

DEFINITION 4.　The primary-tangential cluster set of f at ζ is defined to be

$$C_{\mathcal{Q}}(f,\zeta) = \overline{\underset{0<r<1}{\cup} C_{\Omega_r(\zeta)}(f,\zeta)}.$$

The term "primary-tangential" is used to differentiate this cluster set from similar cluster sets wherein tangential approach of higher order is used.

Remark 9.　It is evident that

$$C_{\mathfrak{B}}(f,\zeta) \subseteq C_{\mathfrak{A}}(f,\zeta) \subseteq C_{\mathcal{Q}}(f,\zeta)$$

for every point $\zeta \in \Gamma$. By Lemma 6,

$$C_{\mathcal{Q}}(f,\zeta) \subseteq C_{\mathfrak{B}}(f,\zeta)$$

at almost every point $\zeta \in \Gamma$. Thus, at almost every point $\zeta \in \Gamma$,

$$C_{\mathfrak{A}}(f,\zeta) = C_{\mathfrak{B}}(f,\zeta) = C_{\mathcal{Q}}(f,\zeta).$$

DEFINITION 5.　A point $\zeta \in \Gamma$ is said to be a primary-tangential pre-Meier point of $f: D \to W$ provided

$$\Pi_{T_\omega}(f,\zeta) = C_{\mathcal{Q}}(f,\zeta) \subset W.$$

The term "pre-Meier" is used because the condition

$$C_{h_r^-}(f,\zeta) = C_{h_{r'}^+}(f,\zeta) \subset W \ (0 < r < 1;\ 0 < r' < 1)$$

is fulfilled at each primary-tangential pre-Meier point of f, and this is a necessary condition that a point $\zeta \in \Gamma$ be a horocyclic Meier point of f. If it is also true that $C_{\mathcal{Q}}(f,\zeta) = C(f,\zeta) \subset W$, then the point ζ is in fact a horocyclic Meier point of f.

Each horocyclic Meier point of a function f meromorphic in D is a primary-tangential pre-Meier point of f because of Remark 8. An example can be easily constructed to show that the word "meromorphic" cannot be omitted.

Although a horocyclic analogue of Fatou's theorem does not exist, we can prove

THEOREM 10. *If $f(z)$ is a normal meromorphic function in D, then almost every point $\zeta \in \Gamma$ is either a primary-tangential pre-Meier point of f or a point at which $\Pi_{T_\omega}(f, \zeta) = W$.*

Proof. By Remark 9, $C_\mathfrak{A}(f, \zeta) = C_\Omega(f, \zeta)$ almost everywhere on Γ. Since $K_\omega(f)$ is of measure 2π, Bagemihl's lemma implies that

$$\Pi_{T_\omega}(f, \zeta) = C_\Omega(f, \zeta)$$

for almost every point $\zeta \in \Gamma$. The theorem now follows from the fact that at every point $\zeta \in \Gamma$, either $C_\Omega(f, \zeta) \subset W$ or $C_\Omega(f, \zeta) = W$.

Applying Theorem 10 to the holomorphic bounded function $B(z)$ in Theorem 5, we see that the set of primary-tangential pre-Meier points of B has measure 2π and the set of horocyclic Fatou points of B has measure zero.

Although a horocyclic analogue of Plessner's theorem does not exist, we can prove

THEOREM 11. *If $f(z)$ is meromorphic in D, then almost every point $\zeta \in \Gamma$ is either a primary-tangential pre-Meier point of f or a horocyclic Plessner point of f.*

Proof. At a point $\zeta \in \Gamma - I_\omega(f)$, $C_\mathfrak{B}(f, \zeta) \subset W$. By Theorem 3 and Remark 9, for almost every point $\zeta \in \Gamma - I_\omega(f)$,

(15) $\zeta \in K_\omega(f)$ and $C_\mathfrak{B}(f, \zeta) = C_\mathfrak{B}(f, \zeta) = C_\Omega(f, \zeta) \subset W$.

Let the point $\zeta \in \Gamma - I_\omega(f)$ satisfy (15), and let Λ_ω be an admissible tangential arc at ζ. Then there exists a disc $\Omega_{r_0}(\zeta)$ at ζ containing Λ_ω. Since $C_\Omega(f, \zeta) \subset W$, $f^*(z)$, the restriction of $f(z)$ to $\Omega_{r_0}(\zeta)$, is a normal meromorphic function in $\Omega_{r_0}(\zeta)$ by Remark 8. Furthermore, $\zeta \in K_\omega(f)$ implies that $\zeta \in K_\omega(f^*)$, where the meaning of $K_\omega(f^*)$ is the natural one. Bagemihl's lemma applied to the function $f^*(z)$ implies that

$$C_{\Lambda_\omega}(f,\zeta) = C_{\Lambda_\omega}(f^*,\zeta) = C_{\Omega_{r_0}(\zeta)}(f^*,\zeta) = C_{\Omega_{r_0}(\zeta)}(f,\zeta) = C_\Omega(f,\zeta),$$

where the last equality follows because $C_\mathfrak{B}(f,\zeta) = C_\Omega(f,\zeta)$. Since Λ_ω was an arbitrary admissible tangential arc, $\Pi_{T_\omega}(f,\zeta) = C_\Omega(f,\zeta)$. Thus almost every point $\zeta \in \Gamma - I_\omega(f)$ is a primary-tangential pre-Meier point of f, and the theorem is proved.

Theorem 11 implies that for the function $f(z)$ in Theorem 9 almost every point $\zeta \in \Gamma$ is a primary-tangential pre-Meier point of f, but meas $[F_\omega(f) \cup M_\omega(f) \cup I_\omega(f)] = 0$.

Since no primary-tangential pre-Meier point of a function is a Plessner point of the function, Plessner's theorem implies that almost every primary-tangential pre-Meier point of a meromorphic function $f(z)$ is a Fatou point of $f(z)$. Since meas $[F(f) \cap I_\omega(f)] = 2\pi$ for the function $f(z)$ of Theorem 6, the converse is not true.

Finally we point out that for a meromorphic function $f(z)$ almost every point of $F_\omega(f) \cup M_\omega(f)$ is a primary-tangential pre-Meier point of f. This follows from Theorem 11 and the fact that no point of $F_\omega(f) \cup M_\omega(f)$ is a point of $I_\omega(f)$. The function $f(z)$ in Theorem 9 shows that the converse need not be true.

5. The set $F(f) \cap M_\omega(f)$.

In the proof of our final theorem, we shall need

Remark 10. Let $c \subset D$ be the arc of a circle C orthogonal to Γ (i.e. $c = D \cap C$), and let $\zeta \in \Gamma$ be interior to C. Then, under inversion in c, the image of that part of each disc $\Omega_r(\zeta)$ $(0 < r < 1)$ which lies exterior to C again lies in $\Omega_r(\zeta)$.

Proof. Let $L(z) = i\dfrac{\zeta + z}{\zeta - z}$. Then $L(z)$ maps $h_r(\zeta)$ onto a straight line parallel to the real axis and c onto a semi-circle $L(c)$ with diameter on the real axis. The inversion in c corresponds to inversion in $L(c)$, and the assertion is evident.

THEOREM 12. *There exists a function $f(z)$ holomorphic and bounded in D such that almost every point $\zeta \in \Gamma$ is a horocyclic Meier point of f, while the set of Meier points of f has measure zero.*

Proof. We shall prove that the function $f(z)$ constructed by Jenkins in [13] has the required properties.

Let d be the domain obtained from the unit disc $|w| < 1$ by inserting at each point $e^{i(m/n)\pi}$ a radial slit of length $1/\sqrt{n}$ where m, n are integers, $n > 0$, $|m| \leqslant n$, and the fraction m/n is in its lowest terms.

We obtain from the domain d a Riemann surface R by the following construction. For each slit s_j $(j = 1, 2, \cdots)$ let d_j be a domain obtained from d by reflection in the diameter bearing s_j. Then we cross-join d_j to d along s_j and the corresponding slit on d_j. For each d_j, let the remaining boundary slits of d_j be denoted by s_{jk} $(k = 1, 2, \cdots; k \neq j)$, where s_{jk} corresponds to s_k. For each d_j and each slit s_{jk} on d_j, let the domain d_{jk} be obtained from d_j by reflection in the diameter bearing s_{jk}. We cross-join d_{jk} to d_j along s_{jk} and the corresponding slit on d_{jk} for each admissible value of k. For each d_{jk}, let the remaining boundary slits of d_{jk} be denoted by s_{jkl} $(l = 1, 2, \cdots; l \neq k, l \neq j)$, where s_{jkl} corresponds to s_{jk}.

Continuing this process, we obtain a Riemann surface R which has no relative boundary over $|w| < 1$. Evidently R is simply connected and of hyperbolic type so that there exists a function $w = f(z)$ which maps D in a one-to-one conformal manner onto the surface R. We assume that f carries the origin $z = 0$ onto the point of d covering the origin $w = 0$.

The surface is invariant under the following transformations. Let d' and d'' be two sheets of R cross-joined along the slit s. Select any point p' in d', and let p'_w denote the point in $|w| < 1$ covered by p'. Let p''_w denote the point in $|w| < 1$ obtained from p'_w by reflection in the diameter which contains the radial segment covered by s. With p' we associate the point p'' in d'' which covers p''_w. Under such an association d' is transformed into d'' and conversely, while the slit s is fixed. Any sheet d^* attached to d' is transformed into a sense-reversed (with respect to the diameter bearing the slit along which it is cross-joined to d') replica of itself attached to d'', and any sheet d^{**} attached to d'' is transformed into a sense-reversed (with respect to the diameter bearing the slit along which it is cross-joined to d'') replica of itself attached to d', etc. We may extend such a mapping to the points on the cross-joins by continuity to obtain, for each choice of d', d'' and s, a transformation which leaves R invariant. Note that the slit s is the only pointwise fixed subset of R.

Each corresponding transformation in D is an anti-conformal transformation of D onto itself, and thus must be the conjugate of a linear transformation. Since each transformation on R fixes pointwise a slit s, the

transformation in D fixes pointwise an arc in D with its endpoints on Γ. The conjugate of a linear transformation carrying D onto itself can leave such an arc pointwise fixed only if the arc lies on a circle orthogonal to Γ and the mapping in question is inversion in that circle.

We can now give a geometric description of $f(z)$. In the mapping $f(z)$ of D onto R, the subset of D mapped onto the initial sheet d of R is a subdomain δ of D bounded by a countable set of open arcs c_j $(j = 1, 2, \cdots)$ on circles orthogonal to Γ (one for each slit $s_j; j = 1, 2, \cdots$) together with a set H on Γ. Since the length of an arc (in D) of a circle orthogonal to Γ is for a suitable constant, say K^*, less than K^* times the length of the arc on Γ which the circle intercepts, the boundary of δ is a rectifiable Jordan curve. If Φ denotes a one-to-one conformal mapping of the disc $|Z| < 1$ onto d, then $f^{-1}(\Phi(z))$ maps $|Z| < 1$ in a one-to-one conformal manner onto δ. The boundary of d consists of Γ_w: $|w| = 1$ and the enumerable collection of slits $s_1, s_2, \cdots$. Due to the choice of the lengths of the slits $s_1, s_2, \cdots$, no Stolz triangle with a vertex on Γ_w can be completely contained in d. According to a theorem of Lavrentieff [14, Theorem 1], the set of points on $|Z| = 1$ mapped onto Γ_w by Φ, say E, must be of measure zero. Since the domain δ has a rectifiable boundary and H is the image under $f^{-1}(\Phi(z))$ of the set E of measure zero, H is of measure zero by the Riesz theorem [24, p. 49].

The function $f(z)$ defined on D can be thought of as the continuation of $f(z)$ defined on δ. If we reflect δ in each of the arcs c_j $(j = 1, 2, \cdots)$ and continue this process, we sweep out the domain D while the corresponding transformations on R completely cover R as the image of d. The images of H under these successive inversions have measure zero. Thus, their enumerable union K has measure zero.

We shall show that $C_\Omega(f, \zeta) = \{w: |w| \leqslant 1\}$ for each point $\zeta \in \Gamma - K$. Then, since $|f(z)| \leqslant 1$, $C(f, \zeta) = \{w: |w| \leqslant 1\}$ for each point $\zeta \in \Gamma - K$ (and hence for each point $\zeta \in \Gamma$). Since f has a radial limit almost everywhere, the set of Meier points of f is of measure zero. By Theorem 10, $\Pi_{T_\omega}(f, \zeta) = C_\Omega(f, \zeta)$ for almost every point $\zeta \in \Gamma$, so that

$$C(f, \zeta) = C_\Omega(f, \zeta) = \Pi_{T_\omega}(f, \zeta) \subseteq \Pi_\omega(f, \zeta)$$

for almost every point $\zeta \in \Gamma - K$. Thus $\Pi_\omega(f, \zeta) = C(f, \zeta)$ for almost every

point $\zeta \in \Gamma$, and the set of horocyclic Meier points of f is of measure 2π as asserted.

If $\zeta \in \Gamma - K$, then ζ is not an endpoint of any arc c_j $(j = 1, 2, \cdots)$ nor is ζ an endpoint of the reflection of any such arc. So there exists a sequence c_j, c_{jk}, c_{jkl}, $\cdots$ of arcs on circles orthogonal to Γ such that ζ lies interior to each such circle. These arcs correspond under f to cross-joins s_j, s_{jk}, s_{jkl}, $\cdots$ on R, where d and d_j are cross-joined along s_j, etc. Also, if $\delta_j \subset D$ is the domain obtained from δ by reflection in c_j, then f carries δ_j onto d_j, etc.

Now if $C_\Omega(f, \zeta) \neq \{w: |w| \leqslant 1\}$, then there exists a point w_0, $|w_0| < 1$, and a closed neighborhood $N(w_0)$ of w_0 contained in $\{w: |w| \leqslant 1\}$ such that $N(w_0)$ has area $\eta > 0$ and

$$N(w_0) \cap C_\Omega(f, \zeta) = \phi.$$

Since $f(\delta) = d$, we can choose the disc $\Omega_r(\zeta)$ so large that

$$\text{area}\,[f(\delta \cap \Omega_r(\zeta))] > \pi - \eta/2.$$

Hence, we must have

$$f(\delta \cap \Omega_r(\zeta)) \cap N(w_0) \neq \phi.$$

Now let $\delta_j^* \subset \delta_j$ be the reflection of $\delta \cap \Omega_r(\zeta)$ in c_j. Then $f(\delta_j^*) \subset f(\delta_j) = d_j$. As previously stated, f in δ_j is the continuation of f in δ by reflection in the arc c_j. The corresponding transformation on R between d and d_j preserves area so that, since $f(\delta_j^*)$ is the image of $f(\delta \cap \Omega_r(\zeta))$ under this transformation on R,

$$\text{area}\, f(\delta_j^*) = \text{area}\, f(\delta \cap \Omega_r(\zeta)).$$

Now $\delta_j^* \subset \delta_j$ and by Remark 10, $\delta_j^* \subset \Omega_r(\zeta)$. Thus, $\delta_j^* \subset \delta_j \cap \Omega_r(\zeta)$, so that

$$\text{area}\, f(\delta_j \cap \Omega_r(\zeta)) > \text{area}\, f(\delta_j^*) = \text{area}\, f(\delta \cap \Omega_r(\zeta)) > \pi - \eta/2.$$

Thus

$$f(\delta_j \cap \Omega_r(\zeta)) \cap N(w_0) \neq \phi.$$

Proceeding in this fashion we obtain the sequence of domains

$$\delta \cap \Omega_r(\zeta), \quad \delta_j \cap \Omega_r(\zeta), \quad \delta_{jk} \cap \Omega_r(\zeta), \quad \cdots$$

which converges to ζ, while the image under f of each such domain inter-

sects $N(w_0)$. Since $N(w_0)$ is closed and bounded, there exists a point in $N(w_0)$ which belongs to $C_{\Omega_r(\zeta)}(f,\zeta)$. Thus,

$$C_{\Omega}(f,\zeta) \cap N(w_0) \neq \phi,$$

which contradicts our assumption that this intersection is empty. This completes the proof of the theorem.

REEFRENCES

[1] F. Bagemihl, Curvilinear cluster sets of arbitrary functions, *Proc. Nat. Acad. Sci.*, **41** (1955), 379–382.

[2] ————————, Some approximation theorems for normal functions, *Ann. Acad. Sci. Fennicae* AI **335** (1963), 1–5.

[3] ——————, Horocyclic boundary properties of meromorphic functions, *Ann. Acad. Sci. Fennicae* AI **385** (1966), 1–18.

[4] —————— and W. Seidel, Some boundary properties of analytic functions, *Math. Z.* **61** (1954), 186–199.

[5] —————— and ————, Sequential and continuous limits of meromorphic functions, *Ann. Acad. Sci. Fennicae* AI **280** (1960), 1–17.

[6] C. Carathéodory, Theory of Functions, Vol. II, 2nd ed., *Chelsea Publishing Co.*, New York, 1960.

[7] E.F. Collingwood, On sets of maximum indetermination of analytic functions, *Math. Z.* **67** (1957), 377–396.

[8] ————————————, Cluster sets of arbitrary functions, *Proc. Nat. Acad. Sci.* **46** (1960), 1236–1242.

[9] ———————— and A.J. Lohwater, The Theory of Cluster Sets, Cambridge Tracts in Mathematics and Mathematical Physics, No. 56, *Cambridge Univ. Press*, New York, 1966.

[10] S. Dragosh, Horocyclic boundary properties of meromorphic functions, *J. d'Analyse Math.* (to appear).

[11] P. Fatou, Séries trigonométriques et séries de Taylor, *Acta Math.* **30** (1906), 335–400.

[12] E.W. Hobson, The Theory of Functions of a Real Variable, Vol. I, 3rd ed., *Harren Press*, Wash., D.C., 1950.

[13] J.A. Jenkins, On a problem of Lusin, *Mich. Math. J.* **3** (1955–1956), 187–189.

[14] M. Lavrentieff, On certain properties of univalent functions (in Russian), *Mat. Sbornik* **43** (1936), 815–844, (French Summary) 845–846.

[15] O. Lehto and K.I. Virtanen, Boundary behaviour and normal meromorphic functions, *Acta Math.* **97** (1957), 47–65.

[16] J.E. Littlewood, Mathematical Notes (4): On a theorem of Fatou, *J. London Math. Soc.* **2** (1927), 172–176.

[17] A.J. Lohwater and G. Piranian, The boundary behavior of functions analytic in a disk, *Ann. Acad. Sci. Fennicae* AI **239** (1957), 1–17.

[18] K. Meier, Über die Randwerte der meromorphen Funktionen, *Math. Annalen* **142** (1961), 328–344.

[19] P. Montel, Leçons sur les familles normales de fonctions analytiques et leurs applications, *Paris*, 1927.

[20] K. Noshiro, Contributions to the theory of meromorphic functions in the unit circle, *J. Fac. Sci. Hokkaido Imperial Univ.* **7** (1939), 149–159.

[21] ————, Cluster Sets, *Berlin*, 1960.
[22] A.I. Plessner, Über das Verhalten analytischer Funktionen am Rande ihres Definitionsbereiches, *J. reine angew. Math.* **158** (1927), 219–227.
[23] I.I. Priwalow, Randeigenschaften analytischer Funktionen, *Berlin*, 1956.
[24] F. Riesz, Über die Randwerte einer analytischen Funktion, *Math. Z.* **18** (1923), 87–95.

Michigan State University