K. Hirata
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SEPARABLE EXTENSIONS AND CENTRALIZERS OF RINGS

KAZUHIKO HIRATA

We have introduced in [9] a type of separable extensions of a ring as a generalization of the notion of central separable algebras. Unfortunately it was unsuitable to call such extensions 'central' as Sugano pointed out in [15] (Example below Theorem 1.1). Some additional properties of such extensions were given in [15]. Especially Propositions 1.3 and 1.4 in [15] are interesting and suggested us to consider the commutor theory of separable extensions. Let Λ be a ring and Γ a subring of Λ. When $\Lambda \otimes_\Gamma \Lambda$ is a direct summand of a finite direct sum of Λ as a two-sided Λ-module we shall denote it by ${}_\Lambda\Lambda \otimes_\Gamma \Lambda_\Lambda < \oplus\, {}_\Lambda(\Lambda \oplus \cdots \oplus \Lambda)_\Lambda$ and call Λ an H-separable extension of Γ (cf. [9] and [15]). Let Δ be a subring of Λ containing the center C of Λ and let Γ be the centralizer of Δ in Λ, $\Gamma = V_\Lambda(\Delta) = \Lambda^\Delta = \{\lambda \in \Lambda \mid \delta\lambda = \lambda\delta,\ \delta \in \Delta\}$. If ${}_\Lambda\Lambda \otimes_C \Delta_\Delta < \oplus\, {}_\Lambda(\Lambda \oplus \cdots \oplus \Lambda)_\Delta$ and Δ is C-finitely generated and projective then Λ is an H-separable extension of Γ and Λ is right Γ-finitely generated and projective. Conversely for such an H-separable extension Λ over Γ, if we set $\Delta' = V_\Lambda(\Gamma)$, then ${}_\Lambda\Lambda \otimes_C \Delta'_{\Delta'} < \oplus\, {}_\Lambda(\Lambda \oplus \cdots \oplus \Lambda)_{\Delta'}$ and Δ' is C-finitely generated and projective. In this way we can give a one to one correspondence between Γ's and Δ's. A more general situation than H-separable extensions is possible and is symmetric to each other. Let B and Γ be subrings of Λ such that $B \supset \Gamma$. Let $\Delta = V_\Lambda(\Gamma)$ and $D = V_\Lambda(B)$. If ${}_B B \otimes_\Gamma \Lambda_\Lambda < \oplus\, {}_B(\Lambda \oplus \cdots \oplus \Lambda)_\Lambda$ and B is right Γ-finitely generated and projective then ${}_\Lambda\Lambda \otimes_D \Delta_\Delta < \oplus {}_\Lambda(\Lambda \oplus \cdots \oplus \Lambda)_\Delta$ and Δ is left D-finitly generated and projective. Same considerations are possible for H-separable subextensions. These are treated in §2, 3 and 4. §1 is a continuation of §1 in [9] and the results are applied to the following sections. In §5 we give some notes on two-sided modules. It is well known that any finitly generated projective module over a commutative ring is a generator (completely faithful) if it is faithful. Let M be a two-sided module over a

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ring R and assume that ${}_R M_R < \oplus_R (R \oplus \cdots \oplus R)_R$. (It is natural to say such a module 'centrally projective'.) Set $M^R = \{m \in M \mid rm = mr,\ r \in R\}$. Then if M^R is C-faithful, where C is the center of R, then ${}_R R_R < \oplus_R (M \oplus \cdots \oplus M)_R$.

Throughout this paper we assume that all rings have a unit element, subrings contain this element and modules are unitary.

§1. Continuation of §1 in [9]

Let R be a ring and let A and B be left R-modules respectively. Put $S = \mathrm{End}_R(A)$ and $T = \mathrm{End}_R(B)$. Following to [9] we note that S and T-operate on the right of A and B respectively. Then $\mathrm{Hom}_R(A, B)$ is a left S- and right T-module, and $\mathrm{Hom}_R(B, A)$ is a left T- and right S-module.

THEOREM 1. 1. *For R-modules A and B the following conditions are equivalent.*

(1) ${}_R B < \oplus_R (A \oplus \cdots \oplus A)$.

(2) $\mathrm{Hom}_R(B, A)$ *is S-finitely generated projective and B is isomorphic to* $\mathrm{Hom}_S(\mathrm{Hom}_R(B, A), A)$ *as an R-module.*

(3) $\mathrm{Hom}_R(B, A) \otimes_S \mathrm{Hom}_R(A, M) \cong \mathrm{Hom}_R(B, M)$ *for any left R-module M.*

Proof. By (1. 2) in [9], (1) implies (2). Assume (2). Then since $\mathrm{Hom}_R(B, A)$ is S-finitely generated and projective $\mathrm{Hom}_R(B, A) \otimes_S \mathrm{Hom}_R(A, M) \cong \mathrm{Hom}_R(\mathrm{Hom}_S(\mathrm{Hom}_R(B, A), A), M)$ and by the second condition of (2) the last is isomorphic to $\mathrm{Hom}_R(B, M)$. If we put $M = B$ then (3) implies (1) by (1. 1) in [9].

PROPOSITION 1. 2. *Assume that ${}_R B < \oplus_R (A \oplus \cdots \oplus A)$. If A is an S-generator so is B as a T-module.*

Proof. By (1. 2) in [9] B is isomorphic to $A \otimes_S \mathrm{Hom}_R(A, B)$ as a right T-module. Since $S_S < \oplus (A \oplus \cdots \oplus A)_S$ tensoring with $\mathrm{Hom}_R(A, B)$ over S we have $\mathrm{Hom}_R(A, B)_T < \oplus (A \otimes_S \mathrm{Hom}_R(A, B) \oplus \cdots \oplus A \otimes_S \mathrm{Hom}_R(A, B))_T \cong (B \oplus \cdots \oplus B)_T$. As $\mathrm{Hom}_R(A, B)$ is a T-generator so is B.

PROPOSITION 1. 3. *Assume that both ${}_R B < \oplus_R (A \oplus \cdots \oplus A)$ and ${}_R A < \oplus_R (B \oplus \cdots \oplus B)$. Then*

(1) $\mathrm{End}_T(B) \cong \mathrm{End}_S(A)$ *as rings.*

(2) *A is S-finitely generated projective if and only if B is so as a T-module.*

(3) *A is an S-generator if and only if B is so as a T-module.*

Proof. (1) By (1. 2) in [9] we have both $B_T \cong A \otimes_S \mathrm{Hom}_R(A, B)_T$ and $A_S \cong \mathrm{Hom}_T(\mathrm{Hom}_R(A, B), B)_S$. Then we have $\mathrm{Hom}_T(B, B) \cong \mathrm{Hom}_T(A \otimes_S \mathrm{Hom}_R(A, B), B) \cong \mathrm{Hom}_S(A, \mathrm{Hom}_T(\mathrm{Hom}_R(A, B), B)) \cong \mathrm{Hom}_S(A, A)$.

(2) Assume that A is S-finitely generated and projective. So $A_S <\oplus (S \oplus \cdots \oplus S)_S$. Tensoring with $\mathrm{Hom}_R(A, B)$ over S we have $B_T \cong A \otimes_S \mathrm{Hom}_R(A, B)_T <\oplus (\mathrm{Hom}_R(A, B) \oplus \cdots \oplus \mathrm{Hom}_R(A, B))_T$. Since $\mathrm{Hom}_R(A, B)$ is T-finitely generated and projective by (1. 5) in [9] so is B. The converse is similar. (3) was proved in (1. 2) already.

Remark 1. When the assumptions in (1. 3) are fulfiled the category of left (right) S-modules is equivalent to the category of left (right) T-modules ((1. 5) in [9]). Therefore Proposition 1. 3 is an obvious fact. Furthermore the property 'direct summand' is preserved in the above equivalences. We shall use this fact in §2.

Remark 2. The isomorphism $\mathrm{End}_T(B) \cong \mathrm{End}_S(A)$ is given as follows. Let $v \in \mathrm{End}_S(A)$. Then corresponding $u \in \mathrm{End}_T(B)$ is given by the composition $B \cong A \otimes_S \mathrm{Hom}_R(A, B) \xrightarrow{v \otimes 1} A \otimes_S \mathrm{Hom}_R(A, B) \cong B$, and so, the isomorphisms stated in (1. 2) in [9] are all $\mathrm{End}_T(B) \cong \mathrm{End}_S(A)$-admissible.

§2. Pairs of subrings and their centralizers

Let Λ be a ring and let B and Γ be subrings of Λ such that $B \supset \Gamma$. We consider the case that ${}_B B \otimes_\Gamma \Lambda_\Lambda <\oplus {}_B(\Lambda \oplus \cdots \oplus \Lambda)_\Lambda$. Then $\mathrm{End}_{(B, \Lambda)}(\Lambda, \Lambda)$, left B- and right Λ-endomorphisms of Λ, is isomorphic to the left multiplication of $D = V_\Lambda(B) = \Lambda^B$, the centralizer of B in Λ, and $\mathrm{Hom}_{(B, \Lambda)}(B \otimes_\Gamma \Lambda, \Lambda)$ is isomorphic to $\Delta = V_\Lambda(\Gamma) = \Lambda^\Gamma$, the centralizer of Γ in Λ. We have, by (1. 2) in [9], $B \otimes_\Gamma \Lambda \cong \mathrm{Hom}_D({}_D\Delta, {}_D\Lambda)$, $b \otimes \lambda \longrightarrow (\delta \longrightarrow b\delta\lambda)$, as left B- and right Λ-modules and Δ is left D-finitely generated and projective. Furthermore we have following isomorphisms.

$\Lambda \otimes_D \Delta \cong \mathrm{Hom}_\Lambda(\Lambda_\Lambda, \Lambda_\Lambda) \otimes_D \Delta \cong \mathrm{Hom}_\Lambda(\mathrm{Hom}_D({}_D\Delta, {}_D\Lambda)_\Lambda, \Lambda_\Lambda) \cong \mathrm{Hom}_\Lambda(B \otimes_\Gamma \Lambda_\Lambda, \Lambda_\Lambda) \cong \mathrm{Hom}_\Gamma(B_\Gamma, \mathrm{Hom}_\Lambda(\Lambda_\Lambda, \Lambda_\Lambda)) \cong \mathrm{Hom}_\Gamma(B_\Gamma, \Lambda_\Gamma)$. The isomorphism of $\Lambda \otimes_D \Delta$ to $\mathrm{Hom}_\Gamma(B_\Gamma, \Lambda_\Gamma)$ is given by $\lambda \otimes \delta \longrightarrow (b \longrightarrow \lambda b \delta)$. Therefore this is left Λ- and right Δ-admissible. If B is right Γ-finitely generated and projective, then ${}_\Lambda\mathrm{Hom}_\Gamma(B_\Gamma, \Lambda_\Gamma)_\Delta <\oplus {}_\Lambda\mathrm{Hom}_\Gamma((\Gamma \oplus \cdots \oplus \Gamma)_\Gamma, \Lambda_\Gamma)_\Delta \cong {}_\Lambda(\mathrm{Hom}_\Gamma(\Gamma_\Gamma, \Lambda_\Gamma) \oplus \cdots \oplus \mathrm{Hom}_\Gamma(\Gamma_\Gamma, \Lambda_\Gamma))_\Delta \cong {}_\Lambda(\Lambda \oplus \cdots \oplus \Lambda)_\Delta$. We have

PROPOSITION 2. 1. *Let Λ be a ring and let B and Γ be subrings of Λ such* *that* $B \supset \Gamma$. *If* ${}_B B \otimes_\Gamma \Lambda_\Lambda < \oplus\, {}_B(\Lambda \oplus \cdots \oplus \Lambda)_\Lambda$ *then* ${}_B B \otimes_\Gamma \Lambda_\Lambda \cong {}_B\mathrm{Hom}_D({}_D\Delta, {}_D\Lambda)_\Lambda$, ${}_\Lambda \Lambda \otimes_D \Delta_\Delta \cong {}_\Lambda\mathrm{Hom}_\Gamma(B_\Gamma, \Lambda_\Gamma)_\Delta$ *and* Δ *is left D-finitely generated and projective. If, further,* B *is right* Γ*-finitely generated and projective then* ${}_\Lambda \Lambda \otimes_D \Delta_\Delta < \oplus\, {}_\Lambda(\Lambda \oplus \cdots \oplus \Lambda)_\Delta$.

We shall call a subring of a ring Λ be closed if it coincides with its second centralizer in Λ. From the above proposition we have

THEOREM 2. 2. *There is a one to one correspondence between the set of pairs* (B, Γ) *of closed subrings of a ring* Λ *such that* $B \supset \Gamma$, ${}_B B \otimes_\Gamma \Lambda_\Lambda < \oplus\, {}_B(\Lambda \oplus \cdots \oplus \Lambda)_\Lambda$ *and* B *is right* Γ*-finitely generated projective and the set of pairs* (Δ, D) *of closed subrings of* Λ *such that* $\Delta \supset D$, ${}_\Lambda \Lambda \otimes_D \Delta_\Delta < \oplus\, {}_\Lambda(\Lambda \oplus \cdots \oplus \Lambda)_\Delta$ *and* Δ *is left D-finitely generated projective.*

Now the endomorphism ring of $B \otimes_\Gamma \Lambda$ as a (B, Λ)-module is isomorphic to $(B \otimes_\Gamma \Lambda)^\Gamma = \{\xi \in B \otimes_\Gamma \Lambda \mid \gamma\xi = \xi\gamma,\ \gamma \in \Gamma\}$ and, as is easily seen, it is also isomorphic to $\mathrm{Hom}_D({}_D\Delta, {}_D\Delta)$ if ${}_B B \otimes_\Gamma \Lambda_\Lambda < \oplus\, {}_B(\Lambda \oplus \cdots \oplus \Lambda)_\Lambda$, where $\Delta = V_\Lambda(\Gamma)$ and $D = V_\Lambda(B)$. Contrary to §1 we consider $B \otimes_\Gamma \Lambda$ as a left $(B \otimes_\Gamma \Lambda)^\Gamma$-module.

PROPOSITION 2. 3. *Let* $B \supset \Gamma$ *be subrings of a ring* Λ *such that* ${}_B B \otimes_\Gamma \Lambda_\Lambda < \oplus\, {}_B(\Lambda \oplus \cdots \oplus \Lambda)_\Lambda$ *and let* $\Delta = V_\Lambda(\Gamma)$ *and* $D = V_\Lambda(B)$. *Then the following hold.*

(1) *If* $\Gamma_\Gamma < \oplus B_\Gamma$ *then the contraction map* φ_Δ: $\Lambda \otimes_D \Delta \longrightarrow \Lambda$, $\varphi_\Delta(\lambda \otimes \delta) = \lambda\delta$, *splits as a* (Λ, Δ)*-homomorphism.*

(2) *If the contraction map* φ_B: $B \otimes_\Gamma \Lambda \longrightarrow \Lambda$, $\varphi_B(b \otimes \lambda) = b\lambda$, *splits as a* (B, Λ)*-homomorphism then* ${}_D D < \oplus\, {}_D\Delta$.

(3) *Let* C *be the center of* Λ *and define the map* η: $\Lambda \otimes_\Gamma \Lambda \longrightarrow \mathrm{Hom}_c(\Delta, \Lambda)$ *by* $\eta(x \otimes y)(\delta) = x\delta y$. *If* $B_\Gamma < \oplus \Lambda_\Gamma$ *and* η *is a monomorphism, or if* B *is right* Γ*-finitely generated projective,* $V_\Lambda(V_\Lambda(\Gamma)) = \Gamma$ *and* ${}_\Gamma\Gamma < \oplus\, {}_\Gamma\Lambda$, *then* $V_\Lambda(V_\Lambda(B)) = B$.

(4) *Assume that* ${}_B\Lambda_\Lambda < \oplus\, {}_B B \otimes_\Gamma \Lambda_\Lambda$. *Then* $(B \otimes_\Gamma \Lambda)^\Gamma < \oplus B \otimes_\Gamma \Lambda$ *as left* $(B \otimes_\Gamma \Lambda)^\Gamma$*-modules if and only if* ${}_D\Delta < \oplus\, {}_D\Lambda$.

(5) *Assume that* $V_\Lambda(V_\Lambda(\Gamma)) \subset B$. (*This is the case when* $V_\Lambda(V_\Lambda(B)) = B$.) *If* $\Gamma_\Gamma < \oplus B_\Gamma$ *or* ${}_\Gamma\Gamma < \oplus\, {}_\Gamma\Lambda$ *then* $V_\Lambda(V_\Lambda(\Gamma)) = \Gamma$.

Proof. (1) Let ψ_B: $\mathrm{Hom}_\Gamma(B_\Gamma, \Lambda_\Gamma) \longrightarrow \Lambda$ be the map defined by $\psi_B(f) = f(1)$, $f \in \mathrm{Hom}_\Gamma(B_\Gamma, \Lambda_\Gamma)$. Then the following diagram

$$\begin{tikzcd} \Lambda \otimes_D \Delta \arrow[rr, "\cong"] \arrow[dr, "\varphi_\Delta"'] & & \mathrm{Hom}_\Gamma(B_\Gamma, \Lambda_\Gamma) \arrow[dl, "\psi_B"] \\ & \Lambda & \end{tikzcd}$$

is commutative. If $\Gamma_\Gamma <\oplus B_\Gamma$, let $\pi\colon B \longrightarrow \Gamma$ be the projection and define $\psi'_B\colon \Lambda \longrightarrow \mathrm{Hom}_\Gamma(B_\Gamma, \Lambda_\Gamma)$ by $\psi'_B(\lambda) = \lambda_l \circ \pi$ where λ_l is the left multiplication of λ on B. Then ψ'_B is a (Λ, Δ)-homomorphism such that $\psi_B \circ \psi'_B = 1_\Lambda$. Therefore φ_B splits.

(2) By (2. 1) $B \otimes_\Gamma \Lambda \cong \mathrm{Hom}_D({}_D\Delta, {}_D\Lambda)$ and the diagram

$$\begin{tikzcd} B \otimes_\Gamma \Lambda \arrow[r] \arrow[dr, "\varphi_B"'] & \mathrm{Hom}_D({}_D\Delta, {}_D\Lambda) \arrow[d, "\psi_\Delta"] \\ & \Lambda \end{tikzcd}$$

is commutative, where $\psi_\Delta(g) = g(1)$, $g \in \mathrm{Hom}_D({}_D\Delta, {}_D\Lambda)$. If $\varphi_B\colon B \otimes_\Gamma \Lambda \longrightarrow \Lambda$ splits as a (B, Λ)-homomorphism, then there exists $\psi'_\Delta\colon \Lambda \longrightarrow \mathrm{Hom}_D({}_D\Delta, {}_D\Lambda)$ such that $\psi_\Delta \circ \psi'_\Delta = 1_\Lambda$. If we let $\psi'_\Delta(1) = \rho$, then $b \circ \rho = \rho \circ b$, $b \in B$ and $\rho(1) = 1$. From this D is a left D-direct summand of Δ. We note that $\varphi_B\colon B \otimes_\Gamma \Lambda \longrightarrow \Lambda$ splits if and only if there exists an element $\sum b_i \otimes \lambda_i \in B \otimes_\Gamma \Lambda$ such that $\sum bb_i \otimes \lambda_i = \sum b_i \otimes \lambda_i b$ for $b \in B$ and $\sum b_i \lambda_i = 1$. Then the projection from Δ to D is given by $\delta \longrightarrow \sum b_i \delta \lambda_i$, $\delta \in \Delta$.

(3) Assume that $B_\Gamma <\oplus \Lambda_\Gamma$ and $\eta\colon \Lambda \otimes_\Gamma \Lambda \longrightarrow \mathrm{Hom}_C(\Delta, \Lambda)$ is monomorphic. Let x be in $V_\Lambda(V_\Lambda(B)) = V_\Lambda(D)$ and consider the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \Lambda \otimes_\Gamma \Lambda \arrow[r, "\eta"] & \mathrm{Hom}_C(\Delta, \Lambda) \\ & B \otimes_\Gamma \Lambda \arrow[r, "\cong"] \arrow[u] & \mathrm{Hom}_D({}_D\Delta, {}_D\Lambda) \arrow[u] \\ & 0 \arrow[u] & 0 \arrow[u] \end{tikzcd}$$

Then since $\eta(x \otimes 1)$ may consider as is in $\mathrm{Hom}_D({}_D\Delta, {}_D\Lambda)$ we have $x \otimes 1 \in B \otimes_\Gamma \Lambda$. Therefore $x \in B$, as $B_\Gamma <\oplus \Lambda_\Gamma$. Next we assume that B is right Γ-projective, $V_\Lambda(V_\Lambda(\Gamma)) = \Gamma$ and ${}_\Gamma\Gamma <\oplus {}_\Gamma\Lambda$. Since B is right Γ-finitely generated and projective, ${}_\Lambda\Lambda \otimes_D \Delta_\Delta <\oplus_\Lambda (\Lambda \oplus \cdots \oplus \Lambda)_\Delta$ by (2. 1). Therefore if we put $V_\Lambda(\Delta) = B'$ then $B' \otimes_\Gamma \Lambda \cong \mathrm{Hom}_D({}_D\Delta, {}_D\Lambda)$. Since $B \otimes_\Gamma \Lambda \cong \mathrm{Hom}_D({}_D\Delta, {}_D\Lambda)$, from the sequence

$$0 \longrightarrow B \longrightarrow B' \longrightarrow B'/B \longrightarrow 0$$

we have $B'/B \otimes_\Gamma \Lambda = 0$. As ${}_\Gamma\Gamma <\oplus {}_\Gamma\Lambda$, $B'/B = 0$ and $B = B'$.

(4) Since ${}_B\Lambda_\Lambda <\oplus {}_B B \otimes_\Gamma \Lambda_\Lambda <\oplus_B (\Lambda \oplus \cdots \oplus \Lambda)_\Lambda$ we can use Remark 1 in §1. By (1. 1) in [9] we have $(B \otimes_\Gamma \Lambda)^\Gamma \cong \mathrm{Hom}_{(B,\Lambda)}(B \otimes_\Gamma \Lambda, B \otimes_\Gamma \Lambda) \cong \mathrm{Hom}_{(B,\Lambda)}$

$(\Lambda, B\otimes_\Gamma\Lambda)\otimes_D \mathrm{Hom}_{(B,\Lambda)}(B\otimes_\Gamma\Lambda, \Lambda)$. On the other hand by (1. 2) in **[9]** $B\otimes_\Gamma\Lambda\cong \mathrm{Hom}_{(B,\Lambda)}(\Lambda, B\otimes_\Gamma\Lambda)\otimes_D\Lambda$. Here we are considering Λ and $B\otimes_\Gamma\Lambda$ as left D- and left $(B\otimes_\Gamma\Lambda)^\Gamma$-modules respectively. Then ${}_{(B\otimes_\Gamma\Lambda)^\Gamma}(B\otimes_\Gamma\Lambda)^\Gamma < \oplus_{(B\otimes_\Gamma\Lambda)^\Gamma} B\otimes_\Gamma\Lambda$ means that $\mathrm{Hom}_{(B,\Lambda)}(\Lambda, B\otimes_\Gamma\Lambda)\otimes_D \mathrm{Hom}_{(B,\Lambda)}(B\otimes_\Gamma\Lambda, \Lambda) <\oplus \mathrm{Hom}_{(B,\Lambda)}(\Lambda, B\otimes_\Gamma\Lambda)\otimes_D\Lambda$. By Remark 1 in §1, this implies that ${}_D\Delta\cong\mathrm{Hom}_{(B,\Lambda)}(B\otimes_\Gamma\Lambda, \Lambda) <\oplus {}_D\Lambda$. The converse is obtained by tensoring with $\mathrm{Hom}_{(B,\Lambda)}(\Lambda, B\otimes_\Gamma\Lambda)$ over D.

(5) Let x be in $V_\Lambda(V_\Lambda(\Gamma)) = V_\Lambda(\Delta)$. Since $B\otimes_\Gamma\Lambda\cong \mathrm{Hom}_D({}_D\Delta, {}_D\Lambda)$ we have $x\otimes 1 = 1\otimes x$ in $B\otimes_\Gamma\Lambda$. Assume $B_\Gamma = (\Gamma\oplus\Gamma')_\Gamma$ and write $x = y+z$, $y\in\Gamma$, $z\in\Gamma'$. Then $B\otimes_\Gamma\Lambda = \Gamma\otimes_\Gamma\Lambda\oplus\Gamma'\otimes_\Gamma\Lambda$ and $y\otimes 1 + z\otimes 1 = x\otimes 1 = 1\otimes x\in\Gamma\otimes\Lambda$. Therefore $x\otimes 1 = y\otimes 1$ and $x = y\in\Gamma$. The case of ${}_\Gamma\Gamma <\oplus {}_\Gamma\Lambda$ is similar.

Remark 1. η in (3) of (2. 3) is a monomorphism (isomorphism) if Λ is H-separable over B. For, then we have $\Lambda\otimes_\Gamma\Lambda\cong\Lambda\otimes_B B\otimes_\Gamma\Lambda<\oplus\Lambda\otimes_B\Lambda\oplus\cdots\oplus\Lambda\otimes_B\Lambda<\oplus\Lambda\oplus\cdots\oplus\Lambda$ and Λ is H-separable over Γ, and so $\Lambda\otimes_\Gamma\Lambda\cong\mathrm{Hom}_c(\Delta,\Lambda)$ (cf. §2 in **[9]**).

Remark 2. If ${}_B\Lambda_\Lambda<\oplus {}_B(B\otimes_\Gamma\Lambda\oplus\cdots\oplus B\otimes_\Gamma\Lambda)_\Lambda$ then ${}_B\Lambda_\Lambda <\oplus {}_B B\otimes_\Gamma\Lambda_\Lambda$ and the contraction map $B\otimes_\Gamma\Lambda\longrightarrow\Lambda$ splits as a (B,Λ)-homomorphism.

PROPOSITION 2. 4. *Assume that* ${}_B\Lambda_\Lambda<\oplus_B B\otimes_\Gamma\Lambda_\Lambda<\oplus_B(\Lambda\oplus\cdots\oplus\Lambda)_\Lambda$ *and let* $\Delta = V_\Lambda(\Gamma)$ *and* $D = V_\Lambda(B)$. *Then* ${}_D D <\oplus {}_D\Lambda$ *if and only if* ${}_D\Delta <\oplus {}_D\Lambda$.

Proof. By (1. 3) $V = \mathrm{End}_D(\Lambda)\cong\mathrm{End}_T(B\otimes_\Gamma\Lambda) = U$ where $T = \mathrm{End}_{(B,\Lambda)}(B\otimes_\Gamma\Lambda)\cong(B\otimes_\Gamma\Lambda)^\Gamma$. If ${}_D D <\oplus {}_D\Lambda$ then Λ is V-finitely generated and projective. Since $\mathrm{Hom}_{(B,\Lambda)}(\Lambda, B\otimes_\Gamma\Lambda)$ is D-finitely generated and projective by (1. 2) in **[9]**, $\mathrm{Hom}_{(B,\Lambda)}(\Lambda, B\otimes_\Gamma\Lambda)\otimes_D\Lambda$ is V-finitely generated and projective. Since the isomorphism of U to V is given through the isomorphism $B\otimes_\Gamma\Lambda\cong \mathrm{Hom}_{(B,\Lambda)}(\Lambda, B\otimes_\Gamma\Lambda)\otimes_D\Lambda$ (Remark 2 in §1) $B\otimes_\Gamma\Lambda$ is U-finitely generated and projective. On the other hand $U\longrightarrow B\otimes_\Gamma\Lambda$ defined by $f\longrightarrow f(1\otimes 1)$, $f\in U$, is epimorphic since B_l and Λ_r are in U, and so splits as a U-homomorphism. Therefore $\mathrm{End}_U(B\otimes_\Gamma\Lambda) = \mathrm{End}_{(B,\Lambda)}(B\otimes_\Gamma\Lambda)\cong(B\otimes_\Gamma\Lambda)^\Gamma$ is a direct summand of $B\otimes_\Gamma\Lambda$ as a $(B\otimes_\Gamma\Lambda)^\Gamma$-module. So ${}_D\Delta <\oplus {}_D\Lambda$ by (4) in (2. 3). The converse is a similar argument. Or, by (2) in (2. 3) ${}_D D <\oplus {}_D\Delta$ and so ${}_D D <\oplus {}_D\Lambda$.

PROPOSITION 2. 5. *Assume that* ${}_B B\otimes_\Gamma\Lambda_\Lambda<\oplus_B(\Lambda\oplus\cdots\oplus\Lambda)_\Lambda$ *and let* $\Delta = V_\Lambda(\Gamma)$ *and* $D = V_\Lambda(B)$. *Then for every right* Λ*-module* M, $\mathrm{Hom}_\Gamma(B_\Gamma, M_\Gamma)\cong M\otimes_D\Delta$.

If further B is right Γ-finitely generated and projective then $B\otimes_\Gamma N\cong \mathrm{Hom}_D({}_D\Delta, {}_DN)$ *for any left Λ-module N.*

Proof. Since $B\otimes_\Gamma\Lambda\cong\mathrm{Hom}_D({}_D\Delta, {}_D\Lambda)$ and Δ is D-finitely generated and projective, we have $\mathrm{Hom}_\Gamma(B_\Gamma, M_\Gamma)\cong\mathrm{Hom}_\Gamma(B_\Gamma, \mathrm{Hom}_\Lambda(\Lambda, M)_\Gamma)\cong\mathrm{Hom}_\Lambda(B\otimes_\Gamma\Lambda, M)\cong\mathrm{Hom}_\Lambda(\mathrm{Hom}_D({}_D\Delta, {}_D\Lambda), M)\cong\mathrm{Hom}_\Lambda(\Lambda, M)\otimes_D\Delta = M\otimes_D\Delta$. Similarly we have $\mathrm{Hom}_D({}_D\Delta, {}_DN)\cong\mathrm{Hom}_D(\Delta, \mathrm{Hom}_\Lambda(\Lambda, N))\cong\mathrm{Hom}_\Lambda(\Lambda\otimes_D\Delta, N)\cong\mathrm{Hom}_\Lambda(\mathrm{Hom}_\Gamma(B_\Gamma, \Lambda_\Gamma), N)\cong B\otimes_\Gamma\mathrm{Hom}_\Lambda(\Lambda, N)\cong B\otimes_\Gamma N$ since B is right Γ-finitely generated and projective.

§3. Separable extensions

In §2 if we take $B=\Lambda$ then we have the condition ${}_\Lambda\Lambda\otimes_\Gamma\Lambda_\Lambda<\oplus\,{}_\Lambda(\Lambda\oplus\cdots\oplus\Lambda)_\Lambda$ for a ring Λ and its subring Γ. When this condition holds we have proved that Λ is a separable extension of Γ, that is, the contraction map φ: $\Lambda\otimes_\Gamma\Lambda\longrightarrow\Lambda$, $\varphi(x\otimes y)=xy$, splits as a (Λ, Λ)-homomorphism ((2. 2) in [9]). We shall call such an extension an H-separable extension. Let $\Delta=V_\Lambda(\Gamma)$ and $C=$ the center of Λ. Then by (2. 1)

Proposition 3. 1. *If Λ is an H-separable extension of Γ, then* $\Lambda\otimes_\Gamma\Lambda\cong\mathrm{Hom}_c(\Delta, \Lambda)$, $\Lambda\otimes_c\Delta\cong\mathrm{Hom}_\Gamma(\Lambda_\Gamma, \Lambda_\Gamma)$, $\Delta\otimes_c\Lambda\cong\mathrm{Hom}_\Gamma({}_\Gamma\Lambda, {}_\Gamma\Lambda)$ *and Δ is C-finitely generated and projective. Furthermore, if Λ is right Γ-finitely generated and projective then* ${}_\Lambda\Lambda\otimes_c\Delta_\Delta<\oplus_\Lambda(\Lambda\oplus\cdots\oplus\Lambda)_\Delta$, *and, if Λ is left Γ-finitely generated and projective then* ${}_\Delta\Delta\otimes_c\Lambda_\Lambda<\oplus\,{}_\Delta(\Lambda\oplus\cdots\oplus\Lambda)_\Lambda$.

Remark. We shall show further $\Delta\otimes_c\Delta\cong\mathrm{Hom}_{(\Gamma,\Gamma)}(\Lambda, \Lambda)$ in §4.

Proposition 3. 2. *Let Λ be an H-separable extension of Γ and let $\Delta=V_\Lambda(\Gamma)$ and $C=$ the center of Λ. Then $\Gamma_\Gamma<\oplus\,\Lambda_\Gamma$ if and only if the contraction map $\Lambda\otimes_C\Delta\longrightarrow\Lambda$ splits as a (Λ, Δ)-homomorphism and $V_\Lambda(\Delta)=\Gamma$. Similarly ${}_\Gamma\Gamma<\oplus{}_\Gamma\Lambda$ if and only if $\Delta\otimes_C\Lambda\longrightarrow\Lambda$ splits as a (Δ, Λ)-homomorphism and $V_\Lambda(\Delta)=\Gamma$.*

Proof. The following diagram

$$\begin{tikzcd} \Lambda\otimes_C\Delta \arrow[rr, "i"] \arrow[dr, "\varphi"'] & & \mathrm{Hom}_\Gamma(\Lambda_\Gamma, \Lambda_\Gamma) \arrow[dl, "\psi"] \\ & \Lambda & \end{tikzcd}$$

is commutative where i, φ and ψ are defined as follows: $i(\lambda\otimes\delta)(x)=\lambda x\delta$, $\varphi(\lambda\otimes\delta)=\lambda\delta$ and $\psi(f)=f(1)$ respectively. If $\Gamma_\Gamma<\oplus\,\Lambda_\Gamma$ then letting π be the projection from Λ to Γ, the map ψ': $\Lambda\longrightarrow\mathrm{Hom}_\Gamma(\Lambda_\Gamma, \Lambda_\Gamma)$, $\psi'(\lambda)=\lambda_l\circ\pi$, is a (Λ, Δ)-homomorphism and $\psi\circ\psi'=1_\Lambda$. Therefore φ: $\Lambda\otimes\Delta\longrightarrow\Lambda$ splits as a

(Λ, Δ)-homomorphism. That $V_\Lambda(\Delta) = \Gamma$ is Proposition 1. 2 in **[15]**. Conversely if there exists $\varphi' : \Lambda \longrightarrow \Lambda \otimes_C \Delta$ such that $\varphi \circ \varphi' = 1_\Lambda$, let $\pi = i \circ \varphi'(1)$. Then $\delta \circ \pi = \pi \circ \delta$ for any $\delta \in \Delta$ and $\pi(1) = 1$. Therefore $\pi(\lambda) \in V_\Lambda(\Delta) = \Gamma$ for $\lambda \in \Lambda$ and $\pi(\gamma) = \gamma$ for $\gamma \in \Gamma$, and so $\Gamma_\Gamma <\oplus \Lambda_\Gamma$. Another statement is similar.

PROPOSITION 3. 3. *Let Λ be a ring C the center of Λ, Δ a subring of Λ containing C and let $\Gamma = V_\Lambda(\Delta)$. If ${}_\Lambda\Lambda \otimes_C \Delta_\Delta <\oplus {}_\Lambda(\Lambda \oplus \cdots \oplus \Lambda)_\Delta$ then $\Lambda \otimes_C \Delta \cong$ $\mathrm{Hom}_\Gamma(\Lambda_\Gamma, \Lambda_\Gamma)$, $\Lambda \otimes_\Gamma \Lambda \cong \mathrm{Hom}_C(\Delta, \Lambda)$ and Λ is right Γ-finitely generated projective. If ${}_\Delta\Delta \otimes_C \Lambda_\Lambda <\oplus {}_\Delta(\Lambda \oplus \cdots \oplus \Lambda)_\Lambda$ then $\Delta \otimes_C \Lambda \cong \mathrm{Hom}_\Gamma({}_\Gamma\Lambda, {}_\Gamma\Lambda)$, $\Lambda \otimes_\Gamma \Lambda \cong \mathrm{Hom}_C$ (Δ, Λ) and Λ is left Γ-finitely generated projective.*

Proof. This is a special case of (2. 1).

From (3. 3) and (2. 3) we can easily prove the following proposition by the same argument.

PROPOSITION 3. 4. *Let Λ be a ring with the center C, Δ a subring of Λ containing C and let $\Gamma = V_\Lambda(\Delta)$. Assume that ${}_\Lambda\Lambda \otimes_C \Delta_\Delta <\oplus {}_\Lambda(\Lambda \oplus \cdots \oplus \Lambda)_\Delta$. Then*

(1) *${}_C C <\oplus {}_C\Delta$ if and only if Λ is a separable extension of Γ.*

(2) *If Δ is C-finitely generated and projective then Λ is an H-separable extension of Γ.*

(3) *If the contraction map $\Lambda \otimes_C \Delta \longrightarrow \Lambda$ splits as a (Λ, Δ)-homomorphism then $\Gamma_\Gamma <\oplus \Lambda_\Gamma$.*

(4) *If ${}_C\Delta <\oplus {}_C\Lambda$ and $\eta : \Lambda \otimes_C \Lambda \longrightarrow \mathrm{Hom}_C(\Lambda, \Lambda)$ is a monomorphism or if ${}_C C <\oplus {}_C\Lambda$ and Δ is C-finitely generated projective then $V_\Lambda(V_\Lambda(\Delta)) = \Delta$.*

There is a similar statement for Λ, Δ and C such that ${}_\Delta\Delta \otimes_C \Lambda_\Lambda <\oplus {}_\Delta(\Lambda \oplus \cdots \oplus \Lambda)_\Lambda$.

From (3. 1), (3. 3) and (3. 4) we have the following theorem.

THEOREM 3. 5. *There is a one to one correspondence between the set of closed subrings Γ's of a ring Λ such that Λ is H-separable over Γ and Λ is right (left) Γ-finitely generated projective, and the set of closed subrings Δ's of Λ containing the center C of Λ such that ${}_\Lambda\Lambda \otimes_C \Delta_\Delta <\oplus {}_\Lambda(\Lambda \oplus \cdots \oplus \Lambda)_\Delta$ $({}_\Delta\Delta \otimes_c \Lambda_\Lambda <\oplus {}_\Delta(\Lambda \oplus \cdots \oplus \Lambda)_\Lambda)$ and Δ is C-finitely generated projective.*

From (2. 3) and (2. 4) letting $B = \Lambda$ we have

PROPOSITION 3. 6. *Let Λ be a ring with the center C, Γ a subring of Λ.*

Assume that ${}_{\Lambda}\Lambda\otimes_{\Gamma}\Lambda_{\Lambda}<\oplus\ {}_{\Lambda}(\Lambda\oplus\cdots\oplus\Lambda)_{\Lambda}$ *and let* $T=\mathrm{End}_{(\Lambda,\Lambda)}(\Lambda\otimes_{\Gamma}\Lambda)\cong(\Lambda\otimes_{\Gamma}\Lambda)^{\Gamma}$. *Then the following are equivalent.*

(1) ${}_{C}C<\oplus\ {}_{C}\Lambda$.

(2) ${}_{T}(\Lambda\otimes_{\Gamma}\Lambda)^{\Gamma}<\oplus\ {}_{T}\Lambda\otimes_{\Gamma}\Lambda$.

(3) ${}_{C}\Delta<\oplus\ {}_{C}\Lambda$.

THEOREM 3.7. *Let Λ be a ring with the center C, Γ a subring of Λ. Assume that C is a C-direct summand of Λ. Then there is a one to one correspondence between the set of subrings Γ's of Λ such that Λ is H-separable over Γ, Λ is right (left) Γ-finitely generated projective and $\Gamma_{\Gamma}<\oplus\ \Lambda_{\Gamma}({}_{\Gamma}\Gamma<\oplus\ {}_{\Gamma}\Lambda)$, and the set of subrings Δ's of Λ containing C such that ${}_{\Lambda}\Lambda_{\Delta}<\oplus\ {}_{\Lambda}\Lambda\otimes_{C}\Delta_{\Delta}<\oplus\ {}_{\Lambda}(\Lambda\oplus\cdots\oplus\Lambda)_{\Delta}$ $({}_{\Delta}\Lambda_{\Lambda}<\oplus\ {}_{\Delta}\Delta\otimes_{C}\Lambda_{\Lambda}<\oplus\ {}_{\Delta}(\Lambda\oplus\cdots\oplus\Lambda)_{\Lambda})$, and Δ is C-finitely generated projective.*

Proof. If $\Gamma_{\Gamma}<\oplus\Lambda_{\Gamma}$ then Γ is closed by (3.2). If Δ satisfies the assumptions of the theorem then Δ is closed by (4) of (3.4). Therefore the theorem follows from (3.5).

Note that ${}_{\Lambda}\Lambda\otimes_{C}\Delta_{\Delta}<\oplus\ {}_{\Lambda}(\Lambda\oplus\cdots\oplus\Lambda)_{\Delta}$ means that left $\Lambda\otimes_{C}\Delta^{0}$-module Λ is a generator where Δ^{0} is the opposite ring of Δ.

PROPOSITION 3.8. *Let Λ be a ring with the center C and Γ a subring of Λ. Assume that Λ is an H-separable extension of Γ and let $T=\mathrm{End}_{(\Lambda,\Lambda)}(\Lambda\otimes_{\Gamma}\Lambda)$. Then $\mathrm{End}_{T}(\Lambda\otimes_{\Gamma}\Lambda)\cong\mathrm{Hom}_{C}(\Lambda,\Lambda)$, and Λ is C-finitely generated projective if and only if $\Lambda\otimes_{\Gamma}\Lambda$ is T-finitely generated projective.*

Proof. Since ${}_{\Lambda}\Lambda_{\Lambda}<\oplus\ {}_{\Lambda}\Lambda\otimes_{\Gamma}\Lambda_{\Lambda}$ we can apply (1.3).
From (2.5) we have

PROPOSITION 3.9. *Let Λ be an H-separable extension of Γ and let $\Delta=V_{\Lambda}(\Gamma)$ and C the center of Λ. Then for any right (left) Λ-module M (N) $\mathrm{Hom}_{\Gamma}(\Lambda_{\Gamma},M_{\Gamma})\cong M\otimes_{C}\Delta$ $(\mathrm{Hom}_{\Gamma}({}_{\Gamma}\Lambda,{}_{\Gamma}N)\cong\Delta\otimes_{C}N)$. If further Λ is right (left) Γ-finitely generated projective then $\Lambda\otimes_{\Gamma}N\cong\mathrm{Hom}_{C}(\Delta,N)$ $(M\otimes_{\Gamma}\Lambda\cong\mathrm{Hom}_{C}(\Delta,M))$.*

§4. Separable subextensions

In this section we shall deal with a ring Λ and its subrings $B\supset\Gamma$ such that B is H-separable over Γ. Since ${}_{B}B\otimes_{\Gamma}B_{B}<\oplus\ {}_{B}(B\oplus\cdots\oplus B)_{B}$, tensoring with Λ over B there yields ${}_{\Lambda}\Lambda\otimes_{\Gamma}B_{B}$ ${}_{\Lambda}(\Lambda\oplus\cdots\oplus\Lambda)_{B}$ or ${}_{B}B\otimes_{\Gamma}\Lambda_{\Lambda}<\oplus_{B}(\Lambda\oplus\cdots\oplus\Lambda)_{\Lambda}$. Therefore all propositions in §2 hold for the da'ta Λ, B and Γ such that B is H-separable over Γ. We shall study about further properties of them.

Let B^Γ be the centralizer of Γ in B and B^B the center of B. Then, since B is H-separable over Γ, for any two-sided B-module $M, M^\Gamma \cong B^\Gamma \otimes_{BB} M^B$ by Theorem 1. 2 in [15] where $M^\Gamma = \{m \in M \mid \gamma m = m\gamma, \gamma \in \Gamma\}$ and $M^B = \{m \in M \mid bm = mb, b \in B\}$. Therefore if we put $\Lambda^\Gamma = \Delta$ and $\Lambda^B = D$ then $\Delta \cong B^\Gamma \otimes_{BB} D$.

Proposition 4. 1. *Let Λ be a ring, B and Γ subrings of Λ such that $B \supset \Gamma$. Let Δ and D be the centralizers of Γ and B in Λ respectively. If B is H-separable over Γ then $\Delta \otimes_D \Delta \cong \mathrm{Hom}_{(\Gamma,\Gamma)}(B, \Lambda)$ and ${}_D D_D <\oplus {}_D \Delta_D <\oplus {}_D(D \oplus \cdots \oplus D)_D$. If further B is closed in Λ ($V_\Lambda(V_\Lambda(B)) = B$) then $B \otimes_\Gamma B \cong \mathrm{Hom}_{(D,D)}(\Delta, \Lambda)$.*

Proof. Since B is H-separable over Γ, $B \otimes_\Gamma B \cong \mathrm{Hom}_{B^B}(B^\Gamma, B)$ and B^Γ is B^B-finitely generated and projective. And so B^B is B^B-direct summand of B^Γ. We have $B^B_{BB} <\oplus B^\Gamma_{BB} <\oplus (B^B \oplus \cdots \oplus B^B)_{BB}$. Tensoring with D over B^B this yields $D <\oplus \Delta <\oplus D \oplus \cdots \oplus D$ as two-sided D-modules.

Next, we have $\Delta \otimes_D \Delta \cong B^\Gamma \otimes_{BB} D \otimes_D \Delta \cong B^\Gamma \otimes_{BB} \Delta \cong B^\Gamma \otimes_{BB} \mathrm{Hom}_{(B,\Gamma)}(B, \Lambda)$ $\cong \mathrm{Hom}_{(B,\Gamma)}(\mathrm{Hom}_{BB}(B^\Gamma, B), \Lambda)$ (B^Γ is B^B-finitely generated and projective) $\cong \mathrm{Hom}_{(B,\Gamma)}(B \otimes_\Gamma B, \Lambda) \cong \mathrm{Hom}_{(\Gamma,\Gamma)}(B, \mathrm{Hom}_B({}_B B, {}_B \Lambda)) \cong \mathrm{Hom}_{(\Gamma,\Gamma)}(B, \Lambda)$.

Last, we assume that B is closed. We have $\mathrm{Hom}_{(D,D)}(\Delta, \Lambda) \cong \mathrm{Hom}_{(D,D)}$ $(B^\Gamma \otimes_{BB} D, \Lambda) \cong \mathrm{Hom}_{BB}(B^\Gamma, \mathrm{Hom}_{(D,D)}(D, \Lambda)) \cong \mathrm{Hom}_{BB}(B^\Gamma, B)$ as $\mathrm{Hom}_{(D,D)}(D, \Lambda)$ $\cong V_\Lambda(D) = B$. Since $B \otimes_\Gamma B \cong \mathrm{Hom}_{BB}(B^\Gamma, B)$ we have $\mathrm{Hom}_{(D,D)}(\Delta, \Lambda) = B \otimes_\Gamma B$.

Corollary 4. 2. *Let Λ be a ring, B and Γ subrings of Λ such that B is H-separable over Γ. If ${}_\Gamma \Gamma_\Gamma <\oplus {}_\Gamma B_\Gamma$ then Δ is separable over D, and if ${}_\Gamma B_\Gamma <\oplus {}_\Gamma(\Gamma \oplus \cdots \oplus \Gamma)_\Gamma$ then Δ is H-separable over D.*

Proof. We have following commutative diagram

$$\begin{array}{ccc} \Delta \otimes_D \Delta & \xrightarrow{\cong} & \mathrm{Hom}_{(\Gamma,\Gamma)}(B, \Lambda) \\ & {\scriptstyle\varphi}\searrow \quad \Delta \quad \swarrow{\scriptstyle\psi} & \end{array}$$

where φ is the contraction map and $\psi(f) = f(1)$, $f \in \mathrm{Hom}_{(\Gamma,\Gamma)}(B, \Lambda)$. If ${}_\Gamma \Gamma_\Gamma <\oplus {}_\Gamma B_\Gamma$ then, letting π be the projection of B to Γ, ψ': $\Delta \longrightarrow \mathrm{Hom}_{(\Gamma,\Gamma)}$ (B, Λ) defined by $\psi'(\delta) = \delta_l \circ \pi = \delta_r \circ \pi$ is a two-sided Δ-homomorphism. Therefore Δ is separable over D.

If ${}_\Gamma B_\Gamma <\oplus {}_\Gamma(\Gamma \oplus \cdots \oplus \Gamma)_\Gamma$ then $\Delta \otimes_D \Delta \cong \mathrm{Hom}_{(\Gamma,\Gamma)}(B, \Lambda) <\oplus \mathrm{Hom}_{(\Gamma,\Gamma)}$

$(\Gamma\oplus\cdots\oplus\Gamma, \Lambda)\cong \mathrm{Hom}_{(\Gamma,\Gamma)}(\Gamma,\Lambda)\oplus\cdots\oplus \mathrm{Hom}_{(\Gamma,\Gamma)}(\Gamma,\Lambda)\cong\Delta\oplus\cdots\oplus\Delta$. Therefore Δ is H-separable over D.

Proposition 1. 4 in [15] asserts that for a separable subextension B of Γ in an H-separable extension Λ of Γ, Λ is an H-separable extension of B if Λ, Γ and B satisfy the assumption in Proposition 1. 3 in [15]. But the last assumption is not necessary. That is

PROPOSITION 4. 3. *Let Λ be an H-separable extension of Γ and B a separable subextension of Γ in Λ. Then Λ is H-separable over B and ${}_DD_D<\oplus\, {}_D\Delta_D$ where $\Delta=V_\Lambda(\Gamma)$ and $D=V_\Lambda(B)$.*

Proof. Since B is separable over Γ, ${}_BB_B<\oplus\, {}_BB\otimes_\Gamma B_B$. Tensoring with Λ over B on both sides, we have ${}_\Lambda\Lambda\otimes_B\Lambda_\Lambda<\oplus\, {}_\Lambda\Lambda\otimes_\Gamma\Lambda_\Lambda$ and since Λ is H-separable over Γ we have ${}_\Lambda\Lambda\otimes_\Gamma\Lambda_\Lambda<\oplus\, {}_\Lambda(\Lambda\oplus\cdots\oplus\Lambda)_\Lambda$. Therefore ${}_\Lambda\Lambda\otimes_B\Lambda_\Lambda$ $<\oplus_\Lambda(\Lambda\oplus\cdots\oplus\Lambda)_\Lambda$ and Λ is H-separable over B. That ${}_DD_D<\oplus\, {}_D\Delta_D$ has been proved in [15] without further assumptions.

Instead of the assumption ${}_BB_\Gamma<\oplus\, {}_B\Lambda_\Gamma$ in Proposition 1. 3 in [15] we can assume that B is H-separable over Γ or more weakly ${}_BB\otimes_\Gamma\Lambda_\Lambda<\oplus\, {}_B(\Lambda\oplus\cdots\oplus\Lambda)_\Lambda$.

LEMMA 4. 4. *Let Λ be a ring, $B\supset\Gamma$ subrings of Λ. If B is H-separable over Γ and $\Gamma_\Gamma<\oplus\,\Lambda_\Gamma$ (${}_\Gamma\Gamma<\oplus\, {}_\Gamma\Lambda$) then $B_B<\oplus\,\Lambda_B$ (${}_BB<\oplus\, {}_B\Lambda$).*

Proof. Since ${}_BB\otimes_\Gamma B_B<\oplus\, {}_B(B\oplus\cdots\oplus B)_B$ tensoring with Λ over B we have ${}_\Lambda\Lambda\otimes_\Gamma B_B<\oplus\, {}_\Lambda(\Lambda\oplus\cdots\oplus\Lambda)_B$. If $\Gamma_\Gamma<\oplus\Lambda_\Gamma$ then $B_B\cong\Gamma\otimes_\Gamma B<\oplus\,\Lambda\otimes_\Gamma B$. Therefore $B_B<\oplus\,(\Lambda\oplus\cdots\oplus\Lambda)_B$ and $B_B<\oplus\,\Lambda_B$ since Λ is a ring.

LEMMA 4. 5. *Assume that Λ is H-separable over Γ and that B is an H-separable subextension of Γ in Λ. If $\Gamma_\Gamma<\oplus\,\Lambda_\Gamma$ or ${}_\Gamma\Gamma<\oplus\, {}_\Gamma\Lambda$ then $V_\Lambda(V_\Lambda(B))=B$.*

Proof. By (4. 3) Λ is H-separable over B, and by (4. 4) $B_B<\oplus\,\Lambda_B$ or ${}_BB<\oplus\, {}_B\Lambda$. Therefore by Proposition 1. 2 in [15] $V_\Lambda(V_\Lambda(B))=B$.

Let R be a ring, M a two-sided R-module. If ${}_RM_R<\oplus\, {}_R(R\oplus\cdots\oplus R)_R$ we shall call M a centrally projective module. We shall prove in §5 the following fact in more general form. Let S be an overring of a ring R. If S is R-centrally projective then ${}_RR_R<\oplus\, {}_RS_R$.

LEMMA 4. 6. *Let Λ be a ring, $B\supset\Gamma$ subrings of Λ. If B is H-separable over Γ and Λ is Γ-centrally projective then Λ is B-centrally projective and B is Γ-centrally projective.*

Proof. Since ${}_{\Gamma}\Lambda_{\Gamma} < \oplus_{\Gamma}(\Gamma \oplus \cdots \oplus \Gamma)_{\Gamma}$ tensoring with B over Γ we have ${}_{B}B \otimes_{\Gamma}\Lambda_{\Gamma} < \oplus_{B}(B \oplus \cdots \oplus B)_{\Gamma}$. On the other hand since ${}_{B}B_{B} < \oplus_{B}B \otimes_{\Gamma} B_{B}$ we have ${}_{B}\Lambda_{\Lambda} \cong {}_{B}B \otimes_{B}\Lambda_{\Lambda} < \oplus_{B}B \otimes_{\Gamma}\Lambda_{\Lambda}$. Therefore ${}_{B}\Lambda_{\Gamma} < \oplus_{B}(B \oplus \cdots \oplus B)_{\Gamma}$. Furthermore tensoring with B over Γ we have ${}_{B}\Lambda \otimes_{\Gamma} B_{B} < \oplus_{B}(B \otimes_{\Gamma} B \oplus \cdots \oplus B \otimes_{\Gamma} B)_{B}$. Since ${}_{\Lambda}\Lambda_{B} < \oplus\, {}_{\Lambda}\Lambda \otimes_{\Gamma} B_{B}$ and ${}_{B}B \otimes_{\Gamma} B_{B} < \oplus\, {}_{B}(B \oplus \cdots \oplus B)_{B}$ we have ${}_{B}\Lambda_{B} < \oplus\, {}_{B}(B \oplus \cdots \oplus B)_{B}$. As we noted above we have also ${}_{B}B_{B} < \oplus\, {}_{B}\Lambda_{B}$ and of course ${}_{\Gamma}B_{\Gamma} < \oplus_{\Gamma}\Lambda_{\Gamma}$. Since ${}_{\Gamma}\Lambda_{\Gamma} < \oplus_{\Gamma}(\Gamma \oplus \cdots \oplus \Gamma)_{\Gamma}$ we have ${}_{\Gamma}B_{\Gamma} < \oplus_{\Gamma}(\Gamma \oplus \cdots \oplus \Gamma)_{\Gamma}$.

Letting $B = \Lambda$ in (4. 1) and (4. 2) we have

PROPOSITION 4. 7. *Let Λ be an H-separable extension of Γ and let $\Delta = V_{\Lambda}(\Gamma)$, C the center of Λ. Then $\Delta \otimes_{C} \Delta \cong \mathrm{Hom}_{(\Gamma,\Gamma)}(\Lambda, \Lambda)$ and Δ is C-finitely generated projective. If further ${}_{\Gamma}\Gamma_{\Gamma} < \oplus\, {}_{\Gamma}\Lambda_{\Gamma}$ then Δ is a separable C-algebra, and if ${}_{\Gamma}\Lambda_{\Gamma} < \oplus\, {}_{\Gamma}(\Gamma \oplus \cdots \oplus \Gamma)_{\Gamma}$ then Δ is an H-separable C-algebra.*

Combining these lemmas and propositions we have

THEOREM 4. 8. *Let Λ be a ring, $B \supset \Gamma$ subrings of Λ. Assume that Λ is a Γ-centrally projective H-separable extension of Γ and B is an H-separable subextension of Γ in Λ. Let $\Delta = V_{\Lambda}(\Gamma)$, $D = V_{\Lambda}(B)$ and $C =$ the center of Λ. Then* (1) *Δ is a finitely generated projective, H-separable C-algebra and closed in Λ.* (2) *D is a C-finitely generated projective H-separable C-subalgebra of Δ.* (3) *$V_{\Lambda}(V_{\Lambda}(B)) = B$ and $V_{\Lambda}(V_{\Lambda}(\Gamma)) = \Gamma$. Conversely assume that Δ is a subring of Λ containing C, that Δ is a finitely generated projective, H-separable C-algebra and that D is an H-separable C-subalgebra of Δ. Then* (4) *Λ is $V_{\Lambda}(\Delta)$-centrally projective and H-separable over $V_{\Lambda}(\Delta)$.* (5) *$V_{\Lambda}(D)$ is H-separable over $V_{\Lambda}(\Delta)$.* (6) *$V_{\Lambda}(V_{\Lambda}(D)) = D$. In this way there is a one to one correspondence between the set of H-separable subextensions of Γ in Λ and the set of H-separable C-subalgebras of Δ.*

Proof. If Λ is a centrally projective H-separable extension of Γ then, by (4. 7), Δ is C-finitely generated projective and H-separable over C. Closedness of Δ is clear. If B is an H-separable subextension of Γ then, by (4. 3), Λ is H-separable over B and B-centrally projective by (4. 6). Therefore D is C-finitely generated projective and H-separable over C. As we have noted above, ${}_{\Gamma}\Gamma_{\Gamma} < \oplus\, {}_{\Gamma}\Lambda_{\Gamma}$ and ${}_{B}B_{B} < \oplus\, {}_{B}\Lambda_{B}$ since Λ is both Γ- and B-centrally projective. Therefore $V_{\Lambda}(V_{\Lambda}(\Gamma)) = \Gamma$ and $V_{\Lambda}(V_{\Lambda}(B)) = B$ by Proposition 1. 2 in **[15]**, since Λ is H-separable over Γ and over B. The converse is similar. We note that under these assumptions for Δ, D and C, Δ is D-

centrally projective and H-separable over D, and so (5) follows form (4. 1) and (4. 2). That $V_\Lambda(V_\Lambda(D)) = D$ follows from (5) in (2. 3).

Finally we give the converse of Proposition 3. 4 in [9]. Let Λ be an H-separable extension of its subring Γ and assume that ${}_\Gamma\Gamma_\Gamma <\oplus {}_\Gamma\Lambda_\Gamma$. Let $\Delta = V_\Lambda(\Gamma)$ and C the center of Λ. Then $V_\Lambda(\Delta) = \Gamma$ by Proposition 1. 2 [15]. So center $\Gamma = \Gamma \cap \Delta = V_\Lambda(\Delta) \cap \Delta =$ center $\Delta \supset C$. Let $C' =$ center $\Gamma =$ center Δ and $\Lambda' = V_\Lambda(C')$. Since Δ is separable over C by (4. 7), Δ is central separable over C' and so H-separable over C'. By Theorem 1. 2 in [15] $\Lambda' = \Gamma \otimes_C \Delta$. If $C' = C$ then $\Lambda = \Gamma \otimes_C \Delta$.

PROPOSITION 4. 10. *Let Λ be a ring with the center C, Γ a subring of Λ with the center equal to Λ. If Λ is H-separable over Γ and ${}_\Gamma\Gamma_\Gamma <\oplus {}_\Gamma\Lambda_\Gamma$ then $V_\Lambda(\Gamma)$ is central separable over C, $\Lambda \cong \Gamma \otimes_C V_\Lambda(\Gamma)$ and Λ is Γ-centrally projective.*

§5. Centrally projective modules

As we have seen in the last section there is a type of two-sided modules which we have called 'centrally projective'. In this section we shall study some properties of these modules. Let R be a ring with the center C, M a two-sided R-module. If ${}_RM_R <\oplus {}_R(R \oplus \cdots \oplus R)_R$ we shall call M a centrally projective module. Note that $\mathrm{Hom}_{(R,R)}(R, M)$ is isomorphic to $M^R = \{m \in M \mid rm = mr, r \in R\}$. Let $\Omega = \mathrm{End}_{(R,R)}(M)$. By (1. 1) in [9] we have

PROPOSITION 5. 1. *M is centrally projective if and only if $\mathrm{Hom}_{(R,R)}(M, R) \otimes_C M^R \cong \Omega$.*

The isomorphism is given by $g \otimes m \longrightarrow (x \longrightarrow g(x)m)$, where $g \otimes m \in \mathrm{Hom}_{(R,R)}(M, R) \otimes_C M^R$ and $x \in M$.

From (1. 2) in [9] we have

PROPOSITION 5. 2. *If M is centrally projective then M^R is C-finitely generated projective as well as an Ω-generator, $M \cong R \otimes_C M^R$ and $\mathrm{End}_C(M^R) = \Omega$.*

The isomorphism $M \cong R \otimes_C M^R$ is given by $r \otimes m \longrightarrow rm$ for $r \otimes m \in R \otimes_C M^R$.

PROPOSITION 5. 3. *If M is centrally projective and M^R is C-faithful then ${}_RR_R <\oplus {}_R(M \oplus \cdots \oplus M)_R$.*

Proof. Since M^R is C-finitely generated projective, if it is C-faithful then ${}_CC <\oplus_C (M^R \oplus \cdots \oplus M^R)$. Therefore tensoring with R over C we have $R <\oplus R \otimes_C M^R \oplus \cdots \oplus R \otimes_C M^R \cong M \oplus \cdots \oplus M$ as two-sided R-modules.

Let $\mathrm{Tr}_{(R,R)}(M)$ be the two-sided ideal in R generated by $g(m)$, $g \in \mathrm{Hom}_{(R,R)}(M, R)$ and $m \in M$. Then by (1. 2) in [9]

Proposition 5. 4. *${}_R R_R < \oplus_R (M \oplus \cdots \oplus M)_R$ if and only if $\mathrm{Tr}_{(R,R)}(M) = R$. When this is the case M^R is Ω-finitely generated projective as well as a C-generator and $\mathrm{Hom}_\Omega(M^R, M^R) \cong C$.*

Let $\mathrm{Tr}_C(M^R)$ be the ideal in C generated by $f(m)$, $f \in \mathrm{Hom}_C(M^R, C)$ and $m \in M^R$. If $M \cong R \otimes_C M^R$ then since $\mathrm{Hom}_{(R,R)}(M, R) \cong \mathrm{Hom}_C(M^R, C)$ it is easily seen that $R \cdot \mathrm{Tr}_C(M^R) = \mathrm{Tr}_{(R,R)}(M)$. Let $\mathfrak{A} = \{x \in R \mid xM = 0, Mx = 0\}$ and $\mathfrak{a} = \{x \in C \mid xM^R = 0\}$. If $M \cong R \otimes_C M^R$ then it is clear that $R \cdot \mathfrak{a} \subset \mathfrak{A}$.

Proposition 5. 5. *If M is centrally projective then $\mathfrak{A} + \mathrm{Tr}_{(R,R)}(M) = R$.*

Proof. Since M^R is C-finitely generated and projective, by Proposition A. 3 [1], $\mathfrak{a} + \mathrm{Tr}_C(M^R) = C$. From the above remarks we have the conclusion.

Next we consider an overring of R which is centrally projective.

Proposition 5. 6. *Let S be an overring of a ring R, C the center of R. If S is R-centrally projective then $S \cong R \otimes_C S^R$, S^R is C-finitely generated projective and ${}_R R_R < \oplus\, {}_R S_R$.*

Proof. The first two assertions follow from (5. 2). Since S^R is C-finitely generated projective and $S^R \supset C$, ${}_C C < \oplus\, {}_C S^R$ and $R < \oplus\, R \otimes_C S^R$ as two-sided R-modules.

We also note that if ${}_R R_R < \oplus\, {}_R (S \oplus \cdots \oplus S)_R$ then ${}_R R_R < \oplus\, {}_R S_R$.

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Chiba University