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A NOTE ON TANGENTIAL EQUIVALENCES

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The main objective of this paper is to prove the following theorem, which generalizes some results of [1], [2], [6]. Our theorem is also suggested by the work of Novikov [5].

THEOREM. *Let M, M' be closed smooth $2n$-manifolds of the same homotopy type. Let $\tau(M)$ and $\tau(M')$ be the tangent bundles of M and M'. Suppose we are given a homotopy equivalence $f: M \to M'$ such that the induced bundle $f^*\tau(M')$ is stably equivalent to $\tau(M)$. (cf.* [4]). *Then $f^*\tau(M')$ is actually equivalent to $\tau(M)$.*

COROLLARY. *Under the same assumption, M and M' have the same span, that is the maximal numbers of linearly independent vector fields on M and M' are equal. (cf.* [1]).

Proof of the theorem. Let M^{2n-1} be the $(2n-1)$-skeleton of M. Set $\tau = \tau(M)$ and $\tau' = f^*\tau(M')$. Let $\tau | M^{2n-1}$ and $\tau' | M^{2n-1}$ be the restrictions of τ and τ' on M^{2n-1}. Let $O(k)$ be the orthogonal group of the k-dimensional euclidean space R^k. Then $(O(2n+1), O(2n))$ is $(2n-1)$-connected. By our assumption $\tau | M^{2n-1}$ is equivalent to $\tau' | M^{2n-1}$, and using the obstruction theory we have an equivalence $\alpha: \tau | M^{2n-1} \approxeq \tau' | M^{2n-1}$ which can be extended to a stable equivalence of $\tau \oplus 1 \approxeq \tau' \oplus 1$ over M, where 1 is the trivial line bundle over M.

Let $i: O(2n) \to O(2n+1)$ be the canonical inclusion. Then we have the following exact sequence

$$O \to \text{Ker } i_* \overset{j}{\to} \pi_{2n-1}(O(2n)) \overset{i*}{\to} \pi_{2n-1}(O(2n+1)) \to O,$$

where Ker $i_* \approx Z$ (the additive group of integers) (cf. [3]). Let c be the obstruction cocycle for extending α to an equivalence $\tau \approxeq \tau'$ over the whole M. (The coefficients group $\pi_{2n-1}(O(2n))$ of this cocycle is twisted if M is non-orientable, and the operation of $\pi_1(M)$ is given in [7] § 23). Then, by our previous remark on α, the value $c(\sigma_i^{2n})$ of c on each simplex σ_i^{2n} of M

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belongs to Ker i_*. We shall show the cohomology class $\{c\} \in H^{2n}(M, \pi_{2n-1}(O(2n))$ is zero. Then we are done. However, c may be considered a cocycle with coefficients in Ker i_*. Thus it is enough to show $\{c\} \in H^{2n}(M, \text{Ker } i_*)$ is zero.

Take a closed disc D_i^{2n} in the interior of σ_i^{2n}. Set $N = M - \text{Int}\, D_i^{2n}$. Then M^{2n-1} is a deformation retract of N. Thus $\alpha : \tau | M^{2n-1} \approxeq \tau' | M^{2n-1}$ is extended to an equivalence $\alpha : \tau | N \approxeq \tau' | N$. Since D_i^{2n} is contractible, $\tau | D_i^{2n}$ and $\tau' | D_i^{2n}$ are trivial. Let S_i^{2n-1} be the boundary of D_i^{2n}. Using some fixed trivialization $\tau | D_i^{2n} \approx D_i^{2n} \times R^{2n}$ and $\tau' | D_i^{2n} \approx D_i^{2n} \times R^{2n}$, we can express

$$\alpha | S_i^{2n-1} : \underset{\tau | S_i^{2n-1}}{\underset{\wr\wr}{S_i^{2n-1} \times R^{2n}}} \to \underset{\tau' | S_i^{2n-1}}{\underset{\wr\wr}{S_i^{2n-1} \times R^{2n}}}$$

in the following form;

$$\alpha(x, y) = (x, f_i(x)y), \text{ where } f_i(x) \in O(2n).$$

By definition $c(\sigma_i^{2n}) = \{f_i\}$, the homotopy class of f_i in $\pi_{2n-1}(O(2n))$. And by our assumption on α, $\{f_i\} \in \text{Ker } i_*$.

Let $\pi : O(2n) \to S^{2n-1}$ be the projection given by $\pi(r) = re$, where e is a base point of S^{2n-1}. Then the following composition of homomorphisms

$$k : \text{Ker } i_* = Z \to \pi_{2n-1}(O(2n)) \to \pi_{2n-1}(S^{2n-1}) = Z$$

is the multiplication by two. (cf. [7]). Let $k_* : H^{2n}(M, \text{Ker } i_*) \to H^{2n}(M, \pi_{2n-1}(S^{2n-1}))$ be the induced homomorphism. Then the both groups are isomorphic to Z since the coefficients are twisted in case M is non-orientable, and k_* is also the multiplication by two. Therefore, if $k_*\{c\} = 0$, then $\{c\} = 0$ and we are through.

Let $[M]$ and $[M']$ be the fundamental homology classes of M and M'. Let $\langle\ ,\ \rangle$ be the Kronecker product which gives the duality of $H^{2n}(M, \pi_{2n-1}(S^{2n-1}))$ and $H_{2n}(M, \pi_{2n-1}(S^{2n-1}))$. Let $X(\tau)$, $X(\tau')$ be the Euler classes of τ and τ'. Then

$$\langle X(\tau), [M] \rangle = \langle X(\tau(M)), [M] \rangle = \text{the Euler number of } M.$$

$$\begin{aligned} \langle X(\tau), [M] \rangle &= \langle f^* X(\tau(M')), [M] \rangle \\ &= \langle X(\tau(M')), [M'] \rangle \\ &= \text{the Euler number of } M'. \end{aligned}$$

Since M and M' are of the same homotopy type, the above shows that $X(\tau)=X(\tau')$. Thus our theorem will follow from

$$(*)\quad X(\tau)-X(\tau')=-k_*\{c\}\in H^{2n}(M, \pi_{2n-1}(S^{2n-1})).$$

The proof of $(*)$ proceeds as follows. First, we shall construct a $2n$-plane bundle δ over M from the disjoint union $N\times R^{2n}+\bigcup_i D_i^{2n}\times R^{2n}$ by identifying a point $(x, y)\in S_i^{2n-1}\times R^{2n}\subset N\times R^{2n}$ with $(x, f_i(x)y)\in S_i^{2n-1}\times R^{2n}\subset D_i^{2n}\times R^{2n}$. Define $s: N\to N\times R^{2n}$ by $s(x)=(x, e)$, $e\in S^{2n-1}\subset R^{2n}$. Then s is a non-zero section of δ over N, and the obstruction cohomology class for extending s to a non-zero section over M is the Euler class $X(\delta)$ of δ. But the construction of δ shows that $X(\delta)$ is represented by a cocycle d such that $d(\sigma_i^n)=\{\bar{f}_i\}$, where $\bar{f}_i: S_i^{2n-1}\to S^{2n-1}$ is given by $\bar{f}_i(x)=f_i(x)e$. Therefore, $d(\sigma_i^n)=k_\# \, c(\sigma_i^n)$, where $k_\#$ is the induced cochain map by $k:\mathrm{Ker}\, i_*\to\pi_{2n-1}(S^{2n-1})$.

Let $E(\tau)$ and $E(\tau')$ be the total spaces of τ and τ' respectively. τ has a non-zero section $t: N\to E(\tau)$, and the obstruction cohomology class for extending t over M is the Euler class $X(\tau)$. Since D_i^{2n} is contractible, $E(\tau)|D_i^{2n}$ can be identified with $D_i^{2n}\times R^{2n}$. Then $t|S_i^{2n-1}: S_i^{2n-1}\to E(\tau)|S_i^{2n-1}\subset D_i^{2n}\times R^{2n}$ may be given by $t(x)=(x, \bar{t}_i(x))$, where $\bar{t}_i(x)\in S^{2n-1}\subset R^{2n}$. And $X(\tau)$ is represented by the cocycle z_1, defined by

$$z_1(\sigma_i^{2n})=\{\bar{t}_i\}\in\pi_{2n-1}(S^{2n-1}).$$

On the other hand, using $\alpha: E(\tau)|N\approx E(\tau')|N$ we have a non-zero section $t': N\to E(\tau')$ defined by $t'(x)=\alpha(t(x))$. $X(\tau')$ is the obstruction for extending t' over M. Since α is given by $\alpha(x, y)=(x, f_i(x)y)$ on S_i^{2n-1}, $X(\tau')$ is represented by the cocycle z_2 defined by $z_2(\sigma_i^{2n})=\{\bar{t}'_i\}\in\pi_{2n-1}(S^{2n-1})$, where $\bar{t}'_i(x)=f_i(x)\bar{t}_i(x)$ for $x\in S_i^{2n-1}$. Thus, $(*)$ is proved if we show $z_1(\sigma_i^{2n})-z_2(\sigma_i^{2n})=-d(\sigma_1^{2n})$. And this follows from $\{\bar{t}_i\}-\{\bar{t}'_i\}=-\{\bar{f}_i\}$ in $\pi_{2n-1}(S^{2n-1})$.

Define $g_i, g'_i: S_i^{2n-1}\to O(2n)\times S^{2n-1}$ by $g_i(x)=(1, \bar{t}_i(x))$, and $g'_i(x)=(f_i(x), \bar{t}_i(x))$, where $1\in O(2n)$ is the unit. Let $\phi: O(2n)\times S^{2n-1}\to S^{2n-1}$ be the canonical operation of $O(2n)$ on S^{2n-1}. Then $\bar{t}_i=\phi\circ g_i$ and $\bar{t}'_i=\phi\circ g'_i$.

Consider the following sequence of homomorphisms.

$$\begin{array}{ccccc}\pi_{2n-1}(S_i^{2n-1}) & \underset{g_i'{}_*}{\overset{g_{i*}}{\rightrightarrows}} & \pi_{2n-1}(O(2n)\times S^{2n-1}) & \overset{\phi_*}{\rightarrow} & \pi_{2n-1}(S^{2n-1}) \\ & & \| & & \\ & & \pi_{2n-1}(O(2n))\oplus\pi_{2n-1}(S^{2n1}) & & \end{array}$$

Let $\iota \in \pi_{2n-1}(S_i^{2n-1})$ be the canonical generator. Then $\phi * \circ g_{i*}(\iota) = \{ \bar{t}_i \}$ and $\phi * \circ g'_{i*}(\iota) = \{ \bar{t}'_i \}$. Therefore, $\{ \bar{t}_i \} - \{ \bar{t}'_i \} = \phi*(g_{i*} - g'_{i*})(\iota)$. Since $g_{i*}(a) = O + \bar{t}_{i*}(a)$ and $g'_i(a) = f_{i*}(a) \oplus \bar{t}_{i*}(a)$, we have $\{ \bar{t}_i \} - \{ \bar{t}'_i \} = - \phi * \circ (f_{i*}(\iota) \oplus O) = - \phi*(f_{i*}(\iota) \oplus C_*(\iota))$, where $C : S_i^{2n-1} \to S^{2n-1}$ is the constant map given by $C(x) = e$.

On the other hand, put $h_i : S_i^{2n-1} \to O(2n) \times S^{2n-1}$ by $h_i(x) = (f_i(x), e) = (f_i(x), C(x))$. Then $\bar{f}_i = \phi \circ h_i$ and it is clear that

$$\{ \bar{f}_i \} = \phi * \circ h_{i*}(\iota) = \phi*(f_{i*}(\iota) \oplus C_*(\iota)). \quad \text{Thus, } \{ \bar{t}_i \} - \{ \bar{t}'_i \} = - \{ \bar{f}_i \}.$$

This completes the proof.

ADDENDUM. Let M, M' be oriented closed smooth $2n$-manifolds of the same homotopy type. Suppose we are given a homotopy equivalence such that $f^*\tau(M')$ is stably equivalent to $\tau(M)$ as an oriented bundle. Then $f^*\tau(M')$ is equivalent to $\tau(M)$ as an oriented bundle.

This follows completely analogously to our proof.

CONJECTURE. Can our theorem be generalized to the odd dimensional case?

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