CIRCULAR SLIT DISK WITH INFINITE RADIUS

KÔTARO OIKAWA and NOBUYUKI SUITA

To Professor Kiyoshi Noshiro on the occasion of his 60th birthday

Introduction

1. Let W be a plane domain such that $\infty \notin W$. Given a point $a \in W$ and a boundary component C of W, consider the family $\mathfrak{F}_{aC}=\mathfrak{F}_{aC}(W)$ consisting of all the functions f satisfying the following conditions: f is regular and univalent in W, $f(a)=0$, $f'(a)=1$, and the image $f(C)$ of C under f is the outer boundary component of the image domain $f(W)$. Set

$$M[f]=\sup_{z\in W}|f(z)|$$

and

$$r_{aC}=r_{aC}(W)=\inf_{f\in\mathfrak{F}_{aC}} M[f].$$

In the present paper we shall call r_{aC} the *mapping radius* of W with respect to a and C.

If r_{aC} is finite, it is now classical that there exists a function minimizing $M[f]$ within $\mathfrak{F}_{aC}$ and that it maps W onto a circular slit disk. If r_{aC} is infinite, however, to the best knowledge of the authors, no one has studied this kind of conformal mappings.

The purpose of the present paper is to show that a considerable part of the results for finite r_{aC} is extended to the case of infinite r_{aC}.

2. *Standard known results for* $r_{aC}<\infty$.

(I) *If* $r_{aC}<\infty$, *there exists a function* $\varphi\in\mathfrak{F}_{aC}$ *with* $M[\varphi]=r_{aC}$. *It is determined uniquely.*

This function φ will be denoted by φ_{aC} or φ^W_{aC}.

We mean by a *circular slit disk with radius* $Q(\leqq\infty)$ a domain $\varDelta$ such that $0\in\varDelta\subset\{w \mid |w|<Q\}$, and $\{w \mid |w|=Q\}$ is a boundary component, and further every other boundary component is a single point or a circular arc on $|w|=\text{const}$.

Received July 12, 1966.

(II) *The image of W under* φ *is a circular slit disk with radius* r_{aC}. *The total area* ($=2$ *dimensional measure*) *of slits vanishes.*

This property does not in general characterize φ_{aC}. For this reason the following is introduced: A circular slit disk Δ with radius $Q(<\infty)$ is said to be *minimal* if $\varphi_{0\Gamma}^{\Delta}(w)=w$ with respect to $\Gamma=\{w|\ |w|=Q\}$.

(III) *The image of W under* φ_{aC} *is a minimal circular slit disk with radius* r_{aC}. *Conversely, an* $f\in\mathfrak{F}_{aC}$ *such that* $f(W)$ *is a minimal circular slit disk* (*with a certain radius*) *is necessarily* φ_{aC}.

Consider an exhaustion $a\in W_n\uparrow W$ in the ordinary sense. Let C_n be the boundary contour of W_n which separates C from a. Put $r_n=r_{aC_n}(W_n)$ and $\varphi_n=\varphi_{aC_n}^{W_n}$.

(IV) r_n *increases with n and* $r_{aC}=\lim r_n$. *Further* $\varphi_{aC}=\lim\varphi_n$ *uniformly on every compact set in W.*

Proofs of the above are found in, e.g., Reich-Warschawski [7] , which contains also a list of literatures.

3. LEMMA 1. *A domain* Δ *with* $0\in\Delta\subset\{w|\ |w|<Q\}$, $Q<\infty$, *is a minimal circular slit disk with radius Q if and only if*

$$r_{0\Gamma}(\Delta)=Q,$$

where Γ *is the outer boundary of* Δ.

It is a direct consequence of (I) and the proof may be omitted.

4. *Main result.* If $r_{aC}=\infty$, the counterpart of (I) is meaningless. Indeed, $M[f]=\infty$ for all $f\in\mathfrak{F}_{aC}$, that is, all the f are extremal functions. It is known also that the first half of (IV) is valid for $r_{aC}=\infty$. Our main result is that (II), (III), and the latter half of (IV) are true for $r_{aC}=\infty$. It may be summarized as follows:

THEOREM 1. *Under the assumption of* $r_{aC}=\infty$, *there exists a uniquely determined function* $\varphi\in\mathfrak{F}_{aC}$ *such that, for every exhaustion* $a\in W_n\uparrow W$,

$$\varphi=\lim_{n\to\infty}\varphi_n$$

uniformly on every compact set in W. *The image* $\varphi(W)$ *is a circular slit disk with infinite radius, and the total area of the slits vanishes.*

The function φ will be denoted by φ_{ac} or φ_{ac}^{W}. A circular slit disk $\varDelta$ with infinite radius will be called *minimal* if $\varphi_{0\Gamma}^{\varDelta}(w)=w$ with respect to $\Gamma=\{\infty\}$. Then it would be clear that the counterparts of (II)-(IV) are derived from Theorem 1.

Let us sketch how the proof of Theorem 1 will be carried out. For any exhaustion, the sequence $\{\varphi_n\}$ is normal and, therefore, it contains a subsequence which converges to a $\psi\in\mathfrak{F}_{ac}$ uniformly on every compact set in W. We shall show, first, that $\psi(W)$ is a circular slit disk with infinite radius, which satisfies a certain condition enjoyed by minimal ones with finite radii (Theorem 2). This latter property implies that the total area of slits vanishes. Besides, it is satisfied by at most one function in $\mathfrak{F}_{ac}$ (Theorem 3). Thus the proof of Theorem 1 will be complete.

§§ 5–11 are devoted to the preparation, a part of which is contained in [6, 11]. In §§ 16–20 is discussed the corresponding case of mappings onto circular slit annuli.

Circular slit disk with finite radius

5. *The linear operator method.* We shall present several particular results for the case of finite r_{ac} which are needed later. To this end the linear operator method developed by Sario [9] will be used. Let us review the definition and basic properties of the operator L_1 in Ahlfors-Sario [1].

Let W be an open Riemann surface, let V be the union of a finite number of regularly imbedded non-compact subdomains with compact relative boundary. For any real analytic function f on α, the relative boundary of V, consider the problem of constructing a harmonic function u on $V\cup\alpha$ such that $u=f$ on α.

If V is the interior of the union of a finite number of compact bordered surfaces, we require u to satisfy the following conditions so that it may be determined uniquely:

$$u=\text{const} \quad \text{and} \quad \int du^{*}=0$$

on every contour of (border of V)$-\alpha$. The correspondence $f\rightarrow u$ is denoted by L_1. Note that it is the $(P)L_1$ in Ahlfors-Sario's book with respect to the canonical partition P (See [1, p. 160]).

If V is arbitrary we define $u=L_1 f$ by means of exhaustion $\alpha\subset W_n\uparrow W$. Let L_{1n} be the above defined operator L_1 acting from α into $V\cap W_n$. The sequence

$L_{1n}f$ converges to a harmonic function uniformly on every compact set in $V \cup \alpha$. The limiting function is independent of the exhaustion, which will be denoted by $L_1 f$:

$$L_1 f = \lim_{n\to\infty} L_{1n} f.$$

If V is the interior of the union of a finite number of compact bordered surfaces, this definition coincides with the previous.

We shall need the following properties:

Linearity, $\quad L_1(c_1 f_1 + c_2 f_2) = c_1 L_1 f_1 + c_2 L_1 f_2$

Maximum-principle, $\quad \min_\alpha f \leqq L_1 f \leqq \max_\alpha f \qquad \text{in } V.$

6. It is not difficult to derive the following *Consistency of* L_1:

(a) If $V' \subset V$, then

$$L_{1V'}(L_{1V} f) = L_{1V} f \qquad \text{on } V'$$

for every f on α. Here subscripts V' and V represent the domains where L_1 is considered.

(b) Conversely, let $V_1, \dots, V_k \subset V$ be mutually disjoint and such that $V - \cup_{i=1}^k V_i$ is relatively compact. Given f on α, suppose u is harmonic on $V \cup \alpha$, coincides with f on α, and satisfies

$$u = L_{1V_i} u \qquad \text{on } V_i \quad (i=1, \dots, k).$$

Then

$$u = L_1 f \qquad \text{on } V.$$

7. *Properties of* φ_{aC} *in terms of* L_1. Let W, a, and C be as in §1. Suppose $V \subset W$ is as in §5 and is such that

$$a \notin \bar{V}, \ \bar{V} \cap C = \phi;$$

by the closure we do *not* mean the one in the relative topology on W.

LEMMA 2. *If* $r_{aC} < \infty$, *then*

$$(1) \qquad L_1(\log|\varphi_{aC}|) = \log|\varphi_{aC}|$$

on every V. *If, further,* C *is a simple closed analytic curve isolated from* $\partial W - C$, *then* $\varphi = \varphi_{aC}$ *is conversely characterized by* (1) *and the following:* $\varphi \in \mathfrak{F}_{aC}$, *regular on* $W \cup C$, *and* $|\varphi| = const$ *on* C.

Proof. Consider an exhaustion $\{a\}\cup\alpha\subset W_n\uparrow W$. Put $u=\log|\varphi_{aC}|$ and $u_n=\log|\varphi_{aC_n}^{W_n}|$. Clearly

$$u_n=L_{1n}u_n \qquad \text{on } V\cap W_n,$$

$n=1, 2, \ldots$. Then

$$|u_n-L_1u|\leqq|L_{1n}u_n-L_{1n}u|+|L_{1n}u-L_1u|\leqq\max_\alpha|u_n-u|+|L_{1n}u-L_1u|\to 0$$

as $n\to\infty$. Accordingly $L_1u=u$.

To prove the latter half, let φ and $\tilde{\varphi}$ satisfy the conditions. Then $v=\log|\varphi|-\log|\tilde{\varphi}|$ is harmonic on $W\cup C$ and satisfies $L_{1V}v=v$ on every V. By the Consistency of L_1 we have $L_1v=v$ on W, where L_1 acts from C into W. Since $v=\text{const}$ on C, we obtain $v=\text{const}$ in W and, therefore, $\varphi=\tilde{\varphi}$. q.e.d.

From the latter half of Lemma 2 we can easily derive the following proposition:

Suppose Δ is a circular slit disk with radius $Q<\infty$ and $\Gamma=\{w|\ |w|=Q\}$ is isolated from $\partial\Delta-\Gamma$. Then it is minimal if and only if

$$L_1(\log|w|)=\log|w|$$

holds on every subdomain V such that $a\notin\bar{V}$, $W\cap\partial V$ consists of a simple closed analytic curve α, and $V=\Delta\cap(\operatorname{Int}\alpha)$. Here $\operatorname{Int}\alpha$ means the interior of α.

8. Definition. A set E in the w-plane with $0, \infty\notin E$ is called a *minimal set of circular slits* if, for every simple closed analytic curve α such that $0, \infty\notin\alpha$ and $E\cap(\operatorname{Int}\alpha)$ is compact, $E^c\cap(\operatorname{Int}\alpha)=V$ is a domain and

$$L_1(\log|w|)=\log|w| \qquad \text{on } V,$$

where L_1 acts from α into V.

In view of the last paragraph of the previous section, it may be characterized as follows: Given a set E with $0, \infty\notin E$, let α be, as is mentioned in the Definition, a simple closed analytic curve such that $0, \infty\notin\alpha$ and $E\cap(\operatorname{Int}\alpha)=E_1$ is compact; if E is a minimal set of circular slits, then, for every (or equivalently some) $Q<\infty$ with $E_1\subset\{w|\ |w|<Q\}$, the domain $\{w|\ |w|<Q\}-E_1$ is a minimal circular slit disk with radius Q; conversely, this property is sufficient for E to be a minimal set of circular slits.

Every component of a minimal set of circular slits is either a single point or an arc on $|w|=\text{const}$. It has vanishing area.

9. In the last paragraph of §7, the assumption for Γ can be removed as follows:

LEMMA 3. *A domain Δ with $0\in\Delta\subset\{w|\ |w|<Q\}, Q<\infty$, is a minimal circular slit disk with radius Q if and only if*

$$\Delta^c\cap\{w|\ |w|<Q\}$$

is a minimal set of circular slits.

Proof. The necessity is a direct consequence of the first half of Lemma 2.

To prove the sufficiency, let Γ be the outer boundary of Δ. Because of Lemma 1 and (IV) in §2, it suffices to find an exhaustion $0\in\Delta_n\uparrow\Delta$ such that

$$\lim_{n\to\infty} r_n=Q,$$

where $r_n=r_{0\Gamma_n}(\Delta_n)$.

It is clear that Γ coincides with the circle $|w|=Q$. Therefore it is possible to find a simple closed analytic curves Γ_n, $n=1,2,\ldots$ in $\Delta\cap\{w|Q-1/n<|w|<Q\}$ separating Γ from 0 and such that $\overline{(\text{Int }\Gamma_n)}\subset(\text{Int }\Gamma_{n+1})$. By the assumption, $E_n=(\text{Int }\Gamma_n)\cap\Delta^c$ has the property that

$$\tilde{\Delta}_n=\{w|\ |w|<Q\}-E_n$$

is a minimal circular slit disk with radius Q. As a consequence $\tilde{r}_n=r_{0\Gamma}(\tilde{\Delta}_n)$ coincides with Q.

Exhaust $\tilde{\Delta}_n$ by $0\in\Delta_{m,n}\uparrow\tilde{\Delta}_n$ $(m\uparrow\infty)$. The mapping radius $r_{m,n}=r_{0\Gamma_{m,n}}(\Delta_{m,n})$ satisfies

$$r_{m,n}\uparrow\tilde{r}_n \qquad (m\uparrow\infty).$$

We may assume $\Gamma_n\subset\Delta_{m,n}$, $m=1,2,\ldots$. Since $\tilde{\Delta}_n$ is a minimal circular slit disk, the sequence $\varphi_{m,n}(w)=\varphi_{\Gamma_{m,n}}^{\Delta_{m,n}}(w)$ converges to w uniformly on Γ_n as $m\to\infty$. Thus it is possible to find $m=m(n)$ such that $\varphi_{m,n}(\Gamma_n)$ is contained in $Q-1/n<|w|<Q$.

Take $m=m(n)$ sufficiently large so that

$$\Delta_n=(\text{Int }\Gamma_n)\cap\Delta_{m(n),n}, \qquad n=1,2,\ldots$$

satisfy $\bar{\Delta}_n\subset\Delta_{n+1}$ and $\cup_{n=1}^{\infty}\Delta_n=\Delta$. Then they are what we wish to get. In fact, first

$$r_n\leqq r_{m(n),n}\leqq\tilde{r}_n=Q$$

holds. Second, $\varphi_{m(n),n}(\Delta_n)$ has only a finite number of slits and contains $\varphi_{m(n),n}(\Delta_n)\cap\{w \mid |w|<Q-1/n\}$, which is a minimal circular slit disk with radius $Q-1/n$. Hence, by (I) in §2,

$$Q-\frac{1}{n}\leqq r_n.$$

As a consequence $\lim r_n=Q$.

10. *Extension of One-Quarter-Theorem.* The following has essentially been obtained by Grötzsch [3]:

LEMMA 4. *If* $r_{aC}<\infty$, *then*

$$\min_{f\in\mathfrak{F}_{aC}} m_C[f]=\frac{r_{aC}}{4},$$

where $m_C[f]=\min_{w\in f(C)}|w|$.

Proof. Consider the Koebe function

$$F_r(w)=\frac{r^2w}{(r+w)^2} \qquad |w|<r.$$

Put $F_{r\Theta}(w)=e^{i\Theta}F_r(e^{-i\Theta}w)$, $0\leqq\Theta<2\pi$. With respect to $r=r_{aC}$, the function $F_r\circ\varphi_{aC}$ belongs to $\mathfrak{F}_{aC}$ and has m_C equal to $r/4$. Thus $\inf m_C[f]\leqq r_{aC}/4$. Suppose, next, there exists an $f_0\in\mathfrak{F}_{aC}$ with $m_C[f_0]<r_{aC}/4$. If $f_0(C)$ is not a half-line on the ray $\arg w=\text{const}$, a simple application of One-Quarter-Theorem shows the existence of $f_1\in\mathfrak{F}_{aC}$ with $m_C[f_1]<m_C[f_0]$. Accordingly, we may assume from the beginning that $f_0(C)$ is a half-line on the ray $\arg w=\Theta$. With respect to $r=4m_C[f_0]$ the function $f_2=F_{r\Theta}^{-1}\circ f_0$ belongs to $\mathfrak{F}_{aC}$ and has $M[f_2]=r<r_{aC}$. This contradiction denies the existence of f_0, showing $\inf m_C[f]=r_{aC}/4$.

Weak boundary components

11. The mapping radius r_{aC} is infinite if and only if $f(C)=\{\infty\}$ for every $f\in\mathfrak{F}_{aC}$. This is equivalent to the fact that, for every univalent function F on W, the image $F(C)$ consists of a single point; note that the validity of this is independent of the reference point a. In this case the boundary component C is called vollkommenpunktförmig by Grötzsch [4], and *weak* by Sario [10].

Sario and others have generalized the concept of weakness for boundary components of open Riemann surfaces. A number of properties of parabolic Riemann surfaces are generalized for surfaces having weak boundary components. We need the maximum-principle due to Constantinescu [2, p. 55]. In the statement of the result he assumed that the function is positive. But it is seen easily from his proof that this assumption is unnecessary as long as the present operator L_1 concerns. We shall state it for plane domains.

LEMMA 5. *Suppose $r_{aC}=\infty$. Let V be a subdomain such that $W-V$ is compact and $\alpha=W\cap\partial V$ consists of a finite number of simple closed analytic curves. Let u be a harmonic function on $V\cup\alpha$. If u is bounded above in V and satisfies*

$$L_1 u=u$$

in every $V'\subset V$ with $\bar{V}'\cap C=\phi$, then

$$u\leqq\max_{\alpha} u \qquad \text{in } V.$$

Circular slit disk with infinite radius

12. Let W, a, and C be as in §1, and consider an exhaustion $a\in W_n\uparrow W$ as in §2. The functions $\varphi_n=\varphi_{aC_n}^{W_n}$ $(1, 2, \ldots)$ form a normal family, so that it contains a subsequence $\{\varphi_{n_j}\}$ for which

$$\lim \varphi_{n_j}=\psi\in\mathfrak{F}_{aC}$$

holds uniformly on every compact set in W.

By exactly the same argument as in the Proof of Lemma 2, we see that

$$L_1(\log|\psi|)=\log|\psi| \tag{2}$$

is satisfied on every $V\subset W$ with $a\notin\bar{V}$ and $\bar{V}\cap C=\phi$. If $r_{aC}=\infty$, then $\psi(C)=\{\infty\}$. Thus we immediately obtain the following:

THEOREM 2. *If $r_{aC}=\infty$, every limiting function ψ maps W onto a circular slit disk with infinite radius. $\partial\psi(W)-\{\infty\}$ is a minimal set of circular slits and, therefore, has vanishing area.*

13. THEOREM 3. *If $r_{aC}=\infty$, then $\mathfrak{F}_{aC}$ contains at most one function satisfying* (2).

Proof. Suppose there are two, say ψ and ψ_1. It suffices to show that the function $u=\log|\psi/\psi_1|$, harmonic in W, reduces to constant.

Let us first prove that it is bounded above. For $n=1, 2, \ldots$, $\Delta_n=\psi(W)\cap\{w|\ |w|<n\}$ is a minimal circular slit disk with radius n (Lemma

3). On applying Lemma 4 for the function $\psi_1 \circ \psi^{-1}$ on Δ_n, we observe that the image of $|w|=n$, a boundary component of Δ_n under this, lies outside the disk $|w| \leqq n/4$. It is not difficult to find a simple closed curve in $\psi(W) \cap \{w \mid n < |w| < n+1\}$ separating $|w|=n$ from $|w|=n+1$. It may be assumed to be an analytic curve. On denoting its counterimage under ψ by C_n, we obtain $|\psi| \leqq n+1$ and $|\psi_1| \geqq n/4$ on C_n, so that $u \leqq \log 4(n+1)n^{-1}$ on C_n.

C_n divides W into two subdomains. Let W_n be the one containing a. By the assumption (2) and by the Consistency of L_1, we have $L_1 u = u$ on W_n, where the operator acts from C_n into W_n. The maximum-principle implies $u \leqq \log 4(n+1)n^{-1}$ on W_n. On letting $n \to \infty$, we conclude that

$$u \leqq \log 4 \qquad \text{in } W.$$

Now let N be a relatively compact parametric disk about a. Apply Lemma 5 to u on $W-N$. The function u is dominated by $\max_{\partial N} u$ in $W-N$, so is in W. We conclude $u=\text{const}$.

14. As a trivial consequence of Theorems 2 and 3, we obtain the following extension of Lemma 3:

THEOREM 4. *A circular slit disk Δ with infinite radius is minimal if and only if $\Delta^c - \{\infty\}$ is a minimal set of circular slits.*

15. If $r_{aC} = \infty$ and C is isolated from $\partial W - C$, then C coincides with $\{\infty\}$. In this case $\tilde{W} = W \cup \{\infty\}$ is also a domain. The function on W obtained by Theorem 1 coincides with the well-known extremal function which maps $\tilde{W}$ onto a circular slit plane.

Circular slit annulus

16. *Introduction.* Let W be a plane domain having more than one boundary components. Assign two of them, C' and C. Consider the family $\mathfrak{F}_{C'C} = \mathfrak{F}_{C'C}(W)$ consisting of all the functions f satisfying the following conditions: f is regular and univalent, $f(z) \neq 0$ in W, $f(C')$ is the inner and $f(C)$ is the outer boundary component of $f(W)$. It would be convenient not to give further restrictions.

Set

$$M[f] = \sup_{z \in W} |f(z)|, \quad m[f] = \inf_{z \in W} |f(z)|,$$

and

$$r_{C'C}=r_{C'C}(W)=\inf_{f\in\mathfrak{F}_{C'C}}\frac{M[f]}{m[f]}.$$

We shall call $r_{C'C}$ the *modulus* of W with respect to C' and C.

If the modulus is finite, then the function minimizing $M[f]/m[f]$ within $\mathfrak{F}_{C'C}$ is known to exist. It is determined uniquely up to a constant factor. It maps W onto a circular slit annulus the ratio of whose outer and inner radii is equal to $r_{C'C}$. Inspite of the ambiguity of the constant factor, we shall denote this function by $\varphi_{C'C}$ or $\varphi^{W}_{C'C}$. Further results analogous to (II)−(IV) in §2 are also well known (see, e.g., Reich-Warschawski [8]).

In the following we shall prove the analogue of Theorems 2 and 3.

17. Let us begin with the following remark:

LEMMA 6. $r_{C'C}=\infty$ *if and only if* C' *or* C *is weak.*

Proof. Take a point $a\in W$ and an exhaustion $a\in W_n\uparrow W$. Let C_n' and C_n be contours of ∂W_n separating a from C' and C, respectively; they do not coincide if n is sufficiently large. Let $u_n=\log|\varphi_{aC_n}|, u_n'=\log|\varphi_{aC'_n}|$, and $v_n=\log|\varphi_{C_n'C_n}|$ on W_n, where the last one is normalized so that $|\varphi_{C_n'C_n}|=1$ on C_n'. Compute the Dirichlet integral in two ways:

$$D_{W_n}[u_n-u_n', v_n]=\int_{\partial W_n}(u_n-u_n')dv_n^*=\int_{\partial W_n}v_n d(u_n-u_n')^*.$$

Use also the relation

$$\int_{\partial W_n}u_n du_n'^*=\int_{\partial W_n}u_n' du_n.^*$$

We obtain

$$\log r_{aC_n}+\log r_{aC_n'}=\log r_{C_n'C_n},$$

from which we get the conclusion immediately.

18. With respect to the exhaustion $W_n\uparrow W$ consider

$$\varphi_n=\varphi_{C_n'C_n}, \qquad n=1,2,\dots.$$

If the constant factor is chosen suitably, this is a normal sequence. For example if $\varphi_n(a)=1, n=1,2,\dots$, at a fixed point $a\in W$, then every φ_n omits three values 0, 1, and ∞ on $W-\{a\}$. Take a subsequence which converges on every compact set in W:

$$\lim_{j\to\infty}\varphi_{nj}=\psi\in\mathfrak{F}_{C'C}\,.$$

THEOREM 5. *If C is weak and C' is not weak then $\psi(W)$ is a circular slit annulus with positive inner radius and infinite outer radius. If C and C' are weak, then $\psi(W)$ is a circular slit annulus with zero inner radius and infinite outer radius. In both cases $\partial\psi(W)-\psi(C)-\psi(C')$ is a minimal set of circular slits and, therefore, has vanishing area.*

Proof. The proof is completely analogous to that of Theorem 2 except the following fact: In the first case $\psi(C')$ is a circle with positive radius. We only give its proof.

Take $Q_0>0$ so that $\psi(C')$ is contained in the disk $|w|<Q_0$. Let

$$\tilde{\Delta}=\psi(W)\cap\{w|\ |w|<Q_0\}$$

and let $\tilde{r}$ be the modulus of $\tilde{\Delta}$ with respect to $\psi(C')$ and $|w|=Q_0$. For every $\varepsilon>0$ take a simple closed analytic curve C_ε in $\psi(W)\cap\{w|Q_0-\varepsilon<|w|<Q_0\}$ separating $\psi(C')$ from $|w|=Q_0$. Let

$$\tilde{\Delta}_\varepsilon=\psi(W)\cap(\mathrm{Int}\ C_\varepsilon)$$

and let $\tilde{r}_\varepsilon$ be the modulus of $\tilde{\Delta}_\varepsilon$ with respect to $\psi(C')$ and C_ε. Since $\psi(C')$ is not weak, $\tilde{r}$ is finite and, by the analogue of (I) in §2,

$$\tilde{r}_\varepsilon\leqq\tilde{r}.$$

Denote by Q_j' and Q_j the inner and outer radii, respectively, of the slit annulus $\varphi_{nj}(W_{nj})$. If j is sufficiently large, then $Q_j'<Q_0<Q_j$. Since $\varphi_{nj}\to\psi$ is uniform on $\psi^{-1}(C_\varepsilon)$, the curve $\varphi_{nj}(\psi^{-1}(C_\varepsilon))$ is contained in $Q_0-\varepsilon<|w|<Q_0$ provided j is sufficiently large. It is not difficult to see

$$\frac{Q_0-\varepsilon}{Q_j'}\leqq\tilde{r}_\varepsilon,$$

so that $Q_j'(j=1,2,\ldots)$ are bounded away from zero. For a suitable subsequence of $\{n_j\}$, being expressed by the same notation, the following limit exists:

$$\lim Q_j'=Q'>0.$$

Then, by a standard argument, we see that $\psi(C')$ lies exterior to the disk $|w|<Q'$. Thus

$$\frac{Q_0-\varepsilon}{Q'}\leqq\tilde{r}_\varepsilon \text{ and } \tilde{r}\leqq\frac{Q_0}{Q'}\,.$$

On letting $\varepsilon \rightarrow 0$ we get

$$\tilde{r}=\frac{Q_0}{Q'}.$$

From this relation and the fact $\tilde{\varDelta}\subset\{w\,|\,Q'<|w|<Q_0\}$ we see, by the analogue of Lemma 1, that $\tilde{\varDelta}$ is a minimal circular slit annulus with inner and outer radii Q' and Q_0, respectively. In particular we conclude that $\psi(C')$ is the circle $|w|=Q'>0$.

19. To obtain the analogue of Theorem 3 we need lemmas corresponding to Lemma 4. We shall use, in place of One-Quarter-Theorem, extremal domains of Grötzsch and Teichmüller (Teichmüller [12]). Since everything is completely analogous to Lemma 4, we shall omit the proofs of Lemmas 7 and 8 below.

For $f\in\mathfrak{F}_{C'C}$, put

$$m_C[f]=\min_{W\in f(C)}|w| \text{ and } M_{C'}[f]=\max_{W\in f(C')}|w|.$$

LEMMA 7. *If* $r_{C'C}<\infty$, *then*

$$m_C[f]\geqq\Phi^{-1}(r_{C'C})$$

for every $f\in\mathfrak{F}_{C'C}$ *such that* $f(C')$ *coincides with the circle* $|w|=1$.

LEMMA 8. *If* $r_{C'C}<\infty$, *then*

$$\frac{m_C[f]}{M_{C'}[f]}\geqq\Psi^{-1}(r_{C'C})$$

for every $f\in\mathfrak{F}_{C'C}$.

For the definitions of the functions Φ and Ψ the reader is referred to [12]. We shall also need the following: They are increasing functions and satisfy

$$\lim_{P\to\infty}\frac{P}{\Phi^{-1}(P)}=4, \qquad \lim_{P\to\infty}\frac{P}{\Psi^{-1}(P)}=16.$$

20. THEOREM 6. *If C is weak and C' is not weak, then $\mathfrak{F}_{C'C}$ contains at most one ψ, up to a constant factor, such that $\psi(C')$ is a circle with positive radius and $\partial\psi(W)-\psi(C)-\psi(C')$ is a minimal set of circular slits. If C and C' are weak then $\mathfrak{F}_{C'C}$ contains at most one ψ, up to a constant factor, such that $\partial\psi(W)-\psi(C)-\psi(C')$ is a minimal set of circular slits.*

Proof. In the first case we normalize ψ so that the radius of $\psi(C')$ is equal to one. The proof is similar to that of Theorem 3 if we substitute Lemma 4 by Lemma 7. It may be left to the reader.

To prove the second half, let ϕ and ϕ_1 be functions in $\mathfrak{F}_{C'C}$ satisfying the conditions. It suffices to show that $u=\log|\phi/\phi_1|$ is constant.

Take a simple closed analytic curve $C_0 \subset W$ separating C from C'. $W-C_0$ consists of two components; let W_0 be the one between C and C_0, and let W_0' be the other. Suppose the following are satisfied on C_0:

$$A \leqq |\phi(z)| \leqq B, \quad A_1 \leqq |\phi_1(z)| \leqq B_1.$$

By the analogue of Lemma 3, the domain $\Delta_n=\phi(W)\cap\{w \mid B<|w|<n\}$ with an integer $n>B$ is a minimal circular slit annulus. Its modulus with respect to $|w|=B$ and $|w|=n$ is equal to n/B. Therefore, the modulus $\tilde{r}_n$ of

$$\tilde{\Delta}_n=\phi(W_0)\cap\{w \mid \ |w|<n\}\supset\Delta_n$$

with respect to $\phi(C_0)$ and $|w|=n$ satisfies

$$\tilde{r}_n \geqq \frac{n}{B}.$$

By Lemma 8 for $\tilde{\Delta}_n$, the image of $|w|=n$ under $\phi_1\circ\phi^{-1}$ lies exterior to the disk $|w|<B_1\Psi^{-1}(\tilde{r}_n)$. On a simple closed analytic curve $C_n \subset W_0$ which separates C from C_0 and is such that $\phi(C_n)$ is in $n<|w|<n+1$,

$$|\phi(z)| \leqq n+1 \qquad \text{and} \qquad |\phi_1(z)| \geqq B_1\Psi^{-1}(n/B)$$

are satisfied. Further $|\phi(z)| \leqq B$ and $|\phi_1(z)| \geqq A_1$ hold on C_0. Then the maximum-principle for the operator L_1 yields

$$u(z) \leqq \max\left(\log\frac{B}{A_1}, \quad \log\frac{n+1}{B_1\Psi^{-1}(n/B)}\right)$$

for every z in W_0 lying between C_0 and C_n. On letting $n\to\infty$, we obtain

$$u(z) \leqq \max\left(\log\frac{B}{A_1}, \quad \log\frac{16B}{B_1}\right).$$

in W_0. Thus u is bounded above in W_0 and, by Lemma 5, $u \leqq \max_{C_0} u$ in W_0.

On the domain W_0', we consider $\Delta_n'=\phi_1(W)\cap\{w \mid 1/n<|w|<A_1\}$ and the curve C_n' such that $\phi_1(C_n')$ is in $1/(n+1)<|w|<1/n$. We see similarly, on applying Lemma 8, that

$$u(z) \leqq \max\left(\log\frac{B}{A_1}, \quad \log\frac{(n+1)A}{\Psi^{-1}(nA_1)}\right)$$

at z in W_0' lying between C_0 and C_n'. Thus

$$u(z) \leqq \max\left(\log \frac{B}{A_1}, \quad \log \frac{16A}{A_1}\right)$$

in W_0' and, by Lemma 5, $u \leqq \max_{C_0} u$ in W_0'.

Consequently $u \leqq \max_{C_0} u$ in W, showing that $u = \text{const}$.

References

[1] Ahlfors, L.V. and Sario, L. Riemann Surfaces. Princeton Univ. Press, 1960.

[2] Constantinescu, C. Ideale Randkomponenten einer Riemannschen Fläche. Revue Math. Pur. et Appl. 4 (1959), 43–76.

[3] Grötzsch, H. Über die Verzerrung bei schlichter konformer Abbildung mehrfach zusammenhängender schlichter Bereiche. Sächsiche Akad. Wiss., Berichte, 81 (1929), 38–47.

[4] ———————— ,Eine Bemerkung zum Koebeschen Kreisnormierungsprinzip. Ibid. 87 (1935), 319–324.

[5] Jenkins, J.A. Univalent Functions and Conformal Mapping. Springer Verlag, 1958.

[6] Oikawa, K. Minimal slit regions and linear operator method. Kōdai Math. Sem. Rep. 17 (1965), 187–190.

[7] Reich, E. and Warschawski, S.E. On canonical conformal maps of regions of arbitrary connectivity. Pacific J. 10 (1960), 965–985.

[8] ———— ,Canonical conformal maps onto a circular slit annulus. Scripta Math. 25 (1960), 137–146.

[9] Sario, L. A linear operator method on arbitrary Riemann surfaces. Trans. Amer. Math. Soc. 72 (1952), 281–295.

[10] ———— ,Strong and weak boundary components. J. Anal. Math. 5 (1956/57), 389–398.

[11] Suita, N. Minimal slit domains and minimal sets. Kōdai Math. Sem. Rep. 17 (1965), 166–186.

[12] Teichmüller, O. Untersuchungen über konforme und quasikonforme Abbildung. Deutsche Math., 3 (1938), 621–678.

College of General Education, University of Tokyo
and
Department of Mathematics, Tokyo Institute of Technology