T. Mabuchi
Nagoya Math. J.
Vol. 69 (1978), 33-64

C^3-ACTIONS AND ALGEBRAIC THREEFOLDS WITH AMPLE TANGENT BUNDLE

TOSHIKI MABUCHI*)

0. Introduction

One of the most challenging problems in complex differential geometry is the following conjecture of Frankel [3].

(F–n) *A compact Kaehler manifold M of dimension n with positive sectional (or more generally, positive holomorphic bisectional) curvature is biholomorphic to the complex projective space $\boldsymbol{P}^n(\boldsymbol{C})$.*

There are also algebraic counterparts:

(G–n) *A non-singular irreducible n-dimensional projective variety M with ample tangent bundle and the second Betti number* 1 *is isomorphic to $\boldsymbol{P}^n(\boldsymbol{C})$.*

(H–n) *A non-singular irreducible n-dimensional projective variety M with ample tangent bundle is isomorphic to $\boldsymbol{P}^n(\boldsymbol{C})$.*

The last (H–n) known as Hartshorne's Conjecture obviously implies (G–n). The first remarkable fact is that, for each n, Conjecture (G–n) implies (F–n); this is a consequence of the theorem of Bishop-Goldberg [1] and the celebrated theorem ("Every Hodge manifold is projective algebraic") of Kodaira [20]. (See also Goldberg-Kobayashi [9].)

We here give a historical sketch: (H–1) is straightforward from the fact that $\boldsymbol{P}^1(\boldsymbol{C})$ is the only compact Riemann surface with positive Euler number. Conjecture (F–2) was proved by Frankel and Andreotti [3], whereas Hartshorne [13] gave a purely algebraic proof of (H–2). Their proofs essentially depend on the classification of the rational algebraic surfaces.

Received June 30, 1976.
Revised March 31, 1977.
*) Supported by an Earle C. Anthony Fellowship at the University of California, Berkeley.

Some progress has been made on (H–3): Let M be a non-singular irreducible 3-dimensional projective variety with ample tangent bundle. Then,

(0–i): (Kobayashi-Ochiai [19]) *The group* $\mathrm{Aut}(M)$ *of biregular automorphisms of* M *satisfies*:

(1) $\dim_C \mathrm{Aut}(M) \geqq 7$.

(2) *M can be embedded into $\boldsymbol{P}^{N-1}(\boldsymbol{C})$ for some N in such a way that* $\mathrm{Aut}(M)$ *acts on M as a closed algebraic subgroup of $PGL(N, \boldsymbol{C})$.*

(2) follows from the ampleness of $-K_M$, whereas the proof of (1) is essentially an estimate of the Riemann-Roch formula for the tangent bundle.

$$\begin{aligned}\dim_C \mathrm{Aut}(M) &= \dim_C H^0(M, T(M)) = \chi(M, T(M)) \\ &= \left\{\frac{1}{2}(c_1^3 - 2c_1\cdot c_2 + c_3) + \frac{5}{24}c_1\cdot c_2\right\}[M] \geqq 7\,,\end{aligned}$$

where each c_i is the i-th Chern class of the tangent bundle of M.

(0–ii): (Iitaka [14]) *M is rational.*

As for (F–3), Kobayashi-Ochiai [19] reduced the proof of (F–3) to the existence of a compact Lie subgroup of $\mathrm{Aut}(M)$ of real dimension $\geqq 7$, which can be obtained if, for instance, there exists an Einstein-Kaehler metric on M (see Ochiai [24]).

In a series of papers we shall give a slightly different approach to (F–3), making an essential use of noncompact-group actions.

Consider the following three standard group actions on a non-singular irreducible 3-dimensional projective variety M with ample tangent bundle. In § 1, we shall show that the study of such standard actions will suffice to prove Conjecture (H–3).

i) $(\boldsymbol{G}_a)^3 \subseteq \mathrm{Aut}(M)$,

where $(\boldsymbol{G}_a)^3$ is a three-dimensional abelian unipotent group (as a complex Lie group, $\boldsymbol{C}^3$).

ii) $(\boldsymbol{G}_m)^3 \subseteq \mathrm{Aut}(M)$,

where $(\boldsymbol{G}_m)^3$ is a three-dimensional algebraic torus (as a complex Lie group, $(\boldsymbol{C}^*)^3$).

iii) $\operatorname{Aut}(M)$ contains a semi-simple algebraic group isogenous to either $SL(3; \boldsymbol{C})$ or $SL(2; \boldsymbol{C}) \times SL(2; \boldsymbol{C})$.

In this paper we shall study $\boldsymbol{C}^3$-actions on M, and prove Conjecture (G–3) in case i): $\boldsymbol{C}^3 \subseteq \operatorname{Aut}(M)$. The remaining cases will be treated in separate papers [21, 22, 23]. More precisely we prove the following:

THEOREM. *Let M be a 3-dimensional non-singular irreducible projective variety with ample tangent bundle and the second Betti number 1. If $\boldsymbol{C}^3$ acts on M holomorphically and effectively, then M is algebraically isomorphic to $\boldsymbol{P}^3(\boldsymbol{C})$.*

The proof depends essentially on the following two facts:

THEOREM A (Fujita [5], Kobayashi-Ochiai [16]). *Let M be a 3-dimensional irreducible non-singular projective variety with ample tangent bundle. Assume that, in $H^2(M)$ $(=H^2(M; \boldsymbol{Z})/$torsion classes), the first Chern class c_1 of the tangent bundle is written in the form:*

$$c_1 = r \cdot g \qquad \text{for some } 2 \leqq r \in \boldsymbol{Z} \text{ and some } g \in H^2(M) .$$

Then M is isomorphic to $\boldsymbol{P}^3(\boldsymbol{C})$, (cf. (5.4) below).

THEOREM B. *Let M be a non-singular irreducible 3-dimensional projective variety with ample tangent bundle and the second Betti number 1. Assume that there exists a section*

$$0 \neq s \in H^0(M, T(M))$$

whose zero locus contains a non-empty 2-dimensional component. Then M is isomorphic to $\boldsymbol{P}^3(\boldsymbol{C})$ (cf. (6.1) below).

In concluding this introduction, I wish to thank all those people who encouraged me and gave me suggestions, and in particular Professors S. Kobayashi, S. S. Roan and I. Satake who helped me again and again in the preparation of this paper. I was also stimulated by the recent work of T. Fujita, with whom I have had many valuable discussions by correspondence. One of his results (Theorem (5.4)) is crucial in our approach. (I understand that T. Fujita has also obtained an inequality similar to ours in Theorem (4.1).)

NOTATIONS AND CONVENTIONS.

(0.1) All varieties are defined over the complex number field $\boldsymbol{C}$.

(0.2) For every non-singular irreducible projective variety X,

i) $H^q(X)$ denotes $H^q(X; \boldsymbol{Z})$/torsion classes, and $H_q(X)$ denotes $H_q(X; \boldsymbol{Z})$/torsion classes.

ii) $\circ : H_*(X) \times H_*(X) \to H_*(X)$ denotes the intersection pairing which is dual to the cup product operation.

iii) $\mathrm{Div}(X)$ denotes the set of all divisors on X.

iv) For any closed subvariety Y of X, $[Y]$ denotes the homology class carried by Y (with multiplicity 1).

v) For every $F \in \mathrm{Div}(X)$,

(1) $\mathrm{supp}(F)$ denotes the support of F.

(2) $[F] \in H_{2n-2}(X)$ (where $\dim X = n$) denotes the algebraic cycle carried by the divisor F.

(3) $\mathcal{O}(F)$ denotes the line bundle over X associated with the divisor F.

(4) $c_1([F]) \in H^2(X)$ denotes the Poincaré dual of $[F]$ in X which is at the same time the first Chern class $c_1(\mathcal{O}(F))$ of the line bundle $\mathcal{O}(F)$. Note that for any F and F' in $\mathrm{Div}(X)$,

$$c_1([F]) + c_1([F']) = c_1([F + F']) .$$

vi) Given an embedding $i: X \hookrightarrow \boldsymbol{P}^N(\boldsymbol{C})$, and given a divisor $F \in \mathrm{Div}(X)$, we denote by $\deg_{\boldsymbol{P}^N}(F)$ the degree of the algebraic cycle $i_*([F])$ in $H_*(\boldsymbol{P}^N(\boldsymbol{C}))$, where i_* is the canonical homomorphism $i_* : H_*(X) \to H_*(\boldsymbol{P}^N(\boldsymbol{C}))$ induced from $i: X \hookrightarrow \boldsymbol{P}^N(\boldsymbol{C})$.

(0.3) For any vector bundle $\pi: E \to M$ over a non-singular irreducible projective variety M,

i) $\tilde{\pi}: E^* \to M$ denotes the dual vector bundle of $\pi: E \to M$.

ii) for each $p \in M$, E_p (resp. E_p^*) denotes the fibre $\pi^{-1}(p)$ (resp. $\tilde{\pi}^{-1}(p)$).

iii) $\mathrm{pr}: P(E^*) \to M$ denotes the associated projective bundle of E^* over M, which is, by definition, the quotient of E^* — (zero section) by the group $\boldsymbol{C}^* = \boldsymbol{C} - 0$ acting on E^* — (zero section) by complex (scalar) multiplication. $\pi': L(E^*) \to P(E^*)$ denotes the associated line bundle of the $\boldsymbol{C}^*$-bundle $\tilde{\pi}: (E^*$ — (zero section)) $\to P(E^*)$.

iv) for every $p \in M$, $P(E^*)_p$ denotes the fibre $\mathrm{pr}^{-1}(p)$ of the projective bundle $\mathrm{pr}: P(E^*) \to M$. Note that $P(E^*)_p$ is canonically identified with $P(E_p^*)$.

v) for each $q \in P(E^*)$, $L(E^*)_q$ denotes the fibre $\pi'^{-1}(q)$ of $\pi' : L(E^*) \to P(E^*)$ over q. One can see that $L(E^*)_q$ is regarded as the complex line ℓ_q in $E^*_{\mathrm{pr}(q)}$ corresponding to q by the canonical projection:

$$\begin{array}{ccc} E^*_{\mathrm{pr}(q)} - 0 & \longrightarrow & P(E^*_{\mathrm{pr}(q)}) = P(E^*)_{\mathrm{pr}(q)} \\ \cup & & \Psi \\ \ell_q - 0 & \longrightarrow & q \end{array}$$

vi) for any $D = \sum_{i=1}^{k} a_i \cdot D_i \in \mathrm{Div}\,(M)$ (where $a_i \in \boldsymbol{Z} - 0$, and D_i is a prime divisor on M), $\mathrm{pr}^{-1}(D)$ denotes $\sum_{i=1}^{k} a_i \cdot \mathrm{pr}^{-1}(D_i) \in \mathrm{Div}\,(P(E^*))$, where each $\mathrm{pr}^{-1}(D_i)$ is regarded as a prime divisor on $P(E^*)$. Note that:

(1) for any F and F' in $\mathrm{Div}\,(M)$, $\mathrm{pr}^{-1}(F + F') = \mathrm{pr}^{-1}(F) + \mathrm{pr}^{-1}(F')$.

(2) for any $F \in \mathrm{Div}\,(M)$, $c_1([\mathrm{pr}^{-1}(F)]) = \mathrm{pr}^*(c_1([F]))$.

(0.4) A vector bundle $\pi : E \to M$ is said to be *ample*, if $L(E^*)^{-1}$ is an ample line bundle over $P(E^*)$, (cf. Hartshorne [12]).

1. Reduction of the problem

In this section, making a detailed study of linear algebraic groups of dimension $\geqq 7$, we shall show that a consideration of four standard types of group actions on projective threefolds with ample tangent bundle will suffice to prove Conjecture (H–3).

(1.1) THEOREM. *Every linear algebraic group G of dimension $\geqq 7$ contains a closed subgroup which is isomorphic to one of the following four algebraic groups:*

i) *The 3-dimensional algebraic torus $(\boldsymbol{G}_m)^3$ $(=(\boldsymbol{C}^*)^3$, as a complex Lie group).*

ii) *An (8-dimensional) algebraic group which is isogenous to $SL(3; \boldsymbol{C})$.*

iii) *A (6-dimensional) algebraic group which is isogenous to $SL(2; \boldsymbol{C}) \times SL(2; \boldsymbol{C})$.*

iv) *The 3-dimensional connected unipotent abelian group $(\boldsymbol{G}_a)^3$ $(=\boldsymbol{C}^3$, as a complex Lie group).*

Proof. Without loss of generality, we may assume that G is connected. Now, by Levi decomposition of connected algebraic groups, there exists a semi-simple closed connected subgroup H of G such that:

$$G = H \cdot \mathrm{Rad}\,(G) \quad \text{and} \quad H \cap \mathrm{Rad}\,(G) = \text{a finite group},$$

where $\mathrm{Rad}\,(G)$ denotes the radical of G, that is, the maximal connected normal closed solvable subgroup of G. Then the following three cases are possible:

$$(1) \qquad \mathrm{rank}\,(H) \geqq 3\,,$$

$$(2) \qquad \mathrm{rank}\,(H) = 2\,,$$

$$(3) \qquad \mathrm{rank}\,(H) \leqq 1\,,$$

where $\mathrm{rank}\,(H)$ is the common dimension of the maximal tori in H.

Case (1). Since $\mathrm{rank}\,(H) \geqq 3$, H (and hence G) contains a closed subgroup isomorphic to $(\boldsymbol{G}_m)^3$.

Case (2). Since $\mathrm{rank}\,(H) = 2$, the classification table of semisimple algebraic groups says that H is isomorphic to one of the following four types of algebraic groups:

(a) An algebraic group of type A_2 (=an algebraic group which is isogenous to $SL(3; \boldsymbol{C})$).

(b) An algebraic group of type $A_1 \times A_1$ (=an algebraic group which is isogenous to $SL(2; \boldsymbol{C}) \times SL(2; \boldsymbol{C})$).

(c) An algebraic group of type B_2.

(d) An algebraic group of type G_2.

Therefore, noting the fact that:

(α) an algebraic group of type G_2 always contains an algebraic group of type $A_1 \times A_1$, and

(β) an algebraic group of type B_2 always contains an algebraic group of type $A_1 \times A_1$, we finally obtain:

In this Case (2), G contains a closed subgroup which is isogenous to either $SL(3; \boldsymbol{C})$ or $SL(2; \boldsymbol{C}) \times SL(2; \boldsymbol{C})$.

Case (3). If $\mathrm{rank}\,(H) = 0$, then $H = \{e\}$, i.e., G is a connected solvable subgroup of dimension $\geqq 7$. If $\mathrm{rank}\,(H) = 1$, then H is isogenous to $SL(2; \boldsymbol{C})$, and therefore, it follows that:

$$(\text{a Borel subgroup of } H) \cdot (\mathrm{Rad}\,(G))$$

is a connected solvable closed subgroup of G of dimension $\geqq 6$. Thus, we have:

In Case (3), G always contains a connected solvable closed subgroup (which is denoted by S) of dimension $\geqq 6$.

Now, by Chevalley decomposition, S is written as a semidirect product of two closed subgroups:

$$S = T \cdot U \quad \text{and} \quad T \cap U = \{e\} ,$$

where U is the unipotent group consisting of all unipotent elements in S, and T is an arbitrarily chosen maximal torus in S. Since $\dim S \geqq 6$, it follows that:

$$\text{(3.1)} \qquad \text{either} \quad \dim T \geqq 3 ,$$

$$\text{(3.2)} \qquad \text{or} \quad \dim U \geqq 4 .$$

In case of (3.1), T (and hence G) contains a closed subgroup isomorphic to $(\boldsymbol{G}_m)^3$, whereas in case of (3.2), $(\boldsymbol{G}_a)^3$ is shown to be contained in U (and hence in G) as a closed subgroup by the following lemma. (Thus, the proof of Theorem (1.1) is reduced to that of Lemma 1.)

LEMMA 1. *Every connected unipotent algebraic group U of dimension $\geqq 4$ contains a closed subgroup isomorphic to $(\boldsymbol{G}_a)^3$.*

Proof. Let L be the Lie algebra of U. Note that:

(1) By the unipotency of U, there is a one-to-one correspondence between the Lie subalgebras of L and the closed (=algebraic) subgroups of U.

(2) Every connected unipotent abelian 3-dimensional algebraic group is isomorphic to $(\boldsymbol{G}_a)^3$.

(3) L is nilpotent and $\dim L \geqq 4$.

By (1) and (2) above, the proof is reduced to showing the existence of a 3-dimensional abelian Lie subalgebra of L. By Lie's Theorem (cf. (3) above), there exists a chain of ideals of L:

$$0 = L_0 \subset L_1 \subset L_2 \subset L_3 \subset L_4 \subset \cdots \subset L_n = L ,$$

such that $\dim L_i = i$ $(i = 1, 2, \cdots, n)$. The nilpotency of L implies

i) $[L, L_1] \subsetneqq L_1$, i.e., $[L, L_1] = 0$.

ii) $[L/L_1, L_2/L_1] \subsetneqq L_2/L_1$, i.e., $[L, L_2] \subseteq L_1$.

Choose a C-basis $\{x_1, x_2, \cdots, x_n\}$ for L such that $\{x_1, x_2, \cdots, x_i\}$ is a C-basis for L_i, $i = 1, 2, \cdots, n$. Then, by i) above,

$$[x_1, x_2] = [x_1, x_3] = [x_1, x_4] = 0 .$$

By ii) above, for some linear combination $\alpha x_3 + \beta x_4 \neq 0$ of x_3 and x_4, we have:

$$[x_2, \alpha x_3 + \beta x_4] = 0 .$$

Therefore the Lie subalgebra $Cx_1 + Cx_2 + C(\alpha x_3 + \beta x_4)$ of L is 3-dimensional and abelian, which finishes the proof of Lemma 1, and hence that of Theorem (1.1).

Now, combining Theorem (1.1) above and (0–i) of Introduction, we obtain the following reduction of (H–3) to four standard cases.

(1.2) THEOREM. *Let M be a non-singular irreducible 3-dimensional projective variety with ample tangent bundle. Then we have one of the four following situations:*

i) $(G_m)^3$ *acts on M regularly and effectively.*

ii) $SL(3; C)$ *acts on M regularly and essentially effectively.*

iii) $SL(2; C) \times SL(2; C)$ *acts on M regularly and essentially effectively.*

iv) $(G_a)^3$ *acts on M regularly and effectively.*

2. Zeroes of a section to a vector bundle of rank $r \geqq 2$ over an irreducible non-singular projective variety

Throughout this section, let $\pi: E \to M$ be a vector bundle of rank $r \geqq 2$ over an n-dimensional irreducible non-singular projective variety M with $\dim H^0(M, E) > 0$.

(2.1) We have a natural isomorphism:

$$H^0(M, E) \cong H^0(P(E^*), L(E^*)^{-1}) .$$

(See, for instance, Kobayashi-Ochiai [17].) It is given by the 1–1 correspondence:

$$s \in H^0(M, E) \leftrightarrow s' \in H^0(P(E^*), L(E^*)^{-1})$$

which is determined by the following relation:

$$\langle s'(q), h \rangle = \langle s(\mathrm{pr}(q)), h \rangle ,$$

where, on the left-hand side, h is regarded as an element of $L(E^*)_q$, and on the right-hand side, h is regarded as an element of $E^*_{\mathrm{pr}(q)}$. (pr is the canonical projection: $P(E^*) \to M$.)

Remark 1. In the above, $L(E^*)_q$ is regarded as the complex line ℓ_q in $E^*_{\mathrm{pr}(q)}$ which corresponds to q by the canonical projection:

$$\begin{array}{ccc} E^*_{\mathrm{pr}(q)} - 0 & \longrightarrow & P(E^*_{\mathrm{pr}(q)}) = P(E^*)_{\mathrm{pr}(q)} \\ \cup & & \Psi \\ \ell_q - 0 & \longrightarrow & q \end{array}$$

Remark 2. In the above, let $0 \neq s \in H^0(M, E)$. Then $U = \{p \in M;\ s(p) \neq 0\}$ is an open dense subset of M. Defining a subset $\{C \cdot s\}$ of E^* (resp. $\{C \cdot s|_U\}$ of $E^*|_U$) by

$$\{C \cdot s\}^\perp = \bigcup_{p \in M} \{h \in E^*_p;\ \langle s(p), h\rangle = 0\}$$
$$\left(\text{resp. } \{C \cdot s_{|U}\}^\perp = \bigcup_{p \in U} \{h \in E^*_p;\ \langle s(p), h\rangle = 0\}\right)$$

we obtain:

(i) $\{C \cdot s\}^\perp = \{C \cdot s_{|U}\}^\perp \cup \tilde{\pi}^{-1}$ (zero locus of s),

where $\tilde{\pi}: E^* \to M$ is the canonical projection.

(ii) $\tilde{\pi}|_{\{C \cdot s_{|U}\}^\perp}: \{C \cdot s_{|U}\}^\perp \to U$ is a rank $(r-1)$ vector sub-bundle of $\tilde{\pi}|_{(E^*_{|U})}: E^*_{|U} \to U$.

(iii) (zero locus of s') $= P(\{C \cdot s_{|U}\}^\perp) \cup \mathrm{pr}^{-1}$ (zero locus of s).

where $P(\{C \cdot s_{|U}\}^\perp)$ is the projective bundle associated with the vector bundle $\{C \cdot s_{|U}\}^\perp$ over U.

(i) and (ii) are obvious. We explain (iii). Let $q \in P(E^*)$. Then $s'(q) = 0$ if and only if $\ell_q \subseteq \{C \cdot s\}^\perp$. Thus from (i) above, we obtain: $s'(q) = 0$ if and only if $q \in P(\{C \cdot s_{|U}\}^\perp) \cup \mathrm{pr}^{-1}(\{s = 0\})$.

(2.2) DEFINITION OF S, S' AND D_s. To each $0 \neq s \in H^0(M, E)$, we associate $S \in \mathrm{Div}\,(M)$ and S', $D_s \in \mathrm{Div}\,(P(E^*))$ by the following three steps:

Step 1. Let $s' \in H^0(P(E^*), L(E^*)^{-1})$ correspond to $s \in H^0(M, E)$ by the isomorphism: $H^0(P(E^*), L(E^*)^{-1}) \cong H^0(M, E)$ in (2.1). Since $L(E^*)^{-1}$ is a line bundle, we can define $S' \in \mathrm{Div}\,(P(E^*))$ by

S' = the zeroes of s' counted with appropriate multiplicities.

Step 2. Let $\{F_1, F_2, \cdots, F_k\}$ be the set of $(n-1)$-dimensional irreducible components of the zero locus of s. By (iii) of Remark 2 of (2.1),

$$\mathrm{pr}^{-1}\,(\text{zero locus of } s) \subseteq \text{zero locus of } s' \,,$$

and therefore,

$$\mathrm{pr}^{-1}\,(F_i) \subseteq \mathrm{supp}\,(S') \qquad \text{for } i = 1, 2, \cdots, k \,.$$

We define $S \in \mathrm{Div}\,(M)$ by

$$S = \sum_{i=1}^{k} \nu_i \cdot F_i \,,$$

where ν_i is the multiplicity of $\mathrm{pr}^{-1}\,(F_i)$ in S'. (If the zero locus of s contains no $(n-1)$-dimensional components, we simply put $S = 0$.) Clearly S is either 0 or an effective divisor.

Step 3. From Step 2,

$$\mathrm{pr}^{-1}\,(S) = \sum_{i=1}^{k} \nu_i \cdot \mathrm{pr}^{-1}\,(F_i) \,.$$

We define $D_s \in \mathrm{Div}\,(P(E^*))$ by

$$D_s = S' - \mathrm{pr}^{-1}\,(S) \,.$$

Then D_s is either 0 or an effective divisor which has no $\mathrm{pr}^{-1}\,(F_i)$-terms, $(i = 1, 2, \cdots, k)$. In particular,

$$\mathrm{codim}_{P(E^*)}\,\{\mathrm{supp}\,(D_s) \cap \mathrm{pr}^{-1}\,(\mathrm{supp}\,(S))\} \geqq 2 \,.$$

(2.2.1) DEFINITION. For each $0 \neq s \in H^0(M, E)$, we call

(i) $S \in \mathrm{Div}\,(M)$ *the divisor part of the zeroes of* s.

(ii) $S' \in \mathrm{Div}\,(P(E^*))$ *the divisor of zeroes of* s'.

(iii) $D_s \in \mathrm{Div}\,(P(E^*))$ *the difference divisor on* $P(E^*)$ *associated with* s. Note that $S' = \mathrm{pr}^{-1}\,(S) + D_s$ can be taken as the definition of D_s.

(2.3) *Horizontal-vertical decomposition of* S'.

(2.3.1) PROPOSITION. *For each* $0 \neq s \in H^0(M, E)$, *the difference divisor* D_s *satisfies*:

(i) D_s *is an effective divisor consisting of only one component, i.e.*, $\mathrm{supp}\,(D_s)$ *is irreducible.*

(ii) $\mathrm{pr}_{|\mathrm{supp}\,(D_s)} : \mathrm{supp}\,(D_s) \to M$ *is generically a* $\boldsymbol{P}^{r-2}(\boldsymbol{C})$*-bundle over* M. *In fact, if* $U = \{p \in M\,;\, s(p) \neq 0\}$, *then*

$$\mathrm{pr}_{|\mathrm{supp}\,(D_s)} : (\mathrm{pr}_{|\mathrm{supp}\,(D_s)})^{-1}(U) \to U$$

is a $P^{r-2}(C)$*-bundle over* U.

Remark 1. It is not hard to see that D_s is even a prime divisor. (We do not use this fact later.)

Remark 2. This proposition shows that: $S' = \mathrm{pr}^{-1}(S) + D_s$ gives a decomposition of S' into the "horizontal component" D_s and the "vertical component" $\mathrm{pr}^{-1}(S)$.

Proof of (2.3.1). We use the same notation as in Remark 2 of (2.1).

Step 1. By (iii) of Remark 2 of (2.1),

$$(\text{zero locus of } s') \cap \mathrm{pr}^{-1}(U) = P(\{C\cdot s|_U\}^{\perp}) .$$

On the other hand, $\mathrm{pr}^{-1}(S) \cap \mathrm{pr}^{-1}(U) = \emptyset$ and $D_s = S' - \mathrm{pr}^{-1}(S)$ imply

$$\mathrm{supp}\,(D_s) \cap \mathrm{pr}^{-1}(U) = \mathrm{supp}\,(S') \cap \mathrm{pr}^{-1}(U) .$$

Therefore, $P(\{C\cdot s|_U\}^{\perp}) = \mathrm{supp}\,(D_s) \cap \mathrm{pr}^{-1}(U) = (\mathrm{pr}|_{\mathrm{supp}(D_s)})^{-1}(U)$. This shows (ii).

Step 2. Note that: $M - U = \mathrm{supp}\,(S) \cup F$, for some F with $\mathrm{codim}_M F \geqq 2$. Therefore,

$$\mathrm{supp}\,(D_s) \cap \mathrm{pr}^{-1}(M - U) \subseteq \{\mathrm{supp}\,(D_s) \cap \mathrm{pr}^{-1}(\mathrm{supp}\,(S))\} \cup \mathrm{pr}^{-1}(F) ,$$

where 1) $\mathrm{codim}_{P(E^*)}\{\mathrm{pr}^{-1}(F)\} \geqq 2$, and
2) $\mathrm{codim}_{P(E^*)}\{\mathrm{supp}\,(D_s) \cap \mathrm{pr}^{-1}(\mathrm{supp}\,(S))\} \geqq 2$, (cf. Step 3 of (2.2)).

Thus, $\mathrm{codim}_{P(E^*)}\{\mathrm{supp}\,(D_s) \cap \mathrm{pr}^{-1}(M - U)\} \geqq 2$.

Step 3. By Steps 1 and 2, noting that $P(\{C\cdot s|_U\}^{\perp})$ is irreducible and 1-codimensional in $P(E^*)$, we obtain:

$$\mathrm{supp}\,(D_s) = (\text{closure of } P(\{C\cdot s|_U\}^{\perp}) \text{ in } P(E^*)) = \text{irreducible}.$$

Since $D_s \neq 0$ in $\mathrm{Div}\,(P(E^*))$, D_s is also an effective divisor, (cf. Step 3 of (2.2)). This finishes (i).

3. Positive polynomial $\Phi_k = \Phi_k(u_1, u_2, \cdots, u_k)$

In this section, we make a quick review of the polynomial Φ_k studied by several people. (See, for instance, Griffiths [10, 11], Gieseker [8], Kobayashi-Ochiai [17].) Also a slightly modified version of the formula in Gieseker [8; Lemma 1.8] will be given for our later purpose.

(3.1) DEFINITION. Let $u_1, u_2, \cdots, u_k$ be indeterminates, where k is an arbitrary positive integer. Consider the polynomial:

$$f_k(X) = X^k - u_1 X^{k-1} + u_2 X^{k-2} - \cdots + (-1)^k u_k ,$$

and split it into factors:

$$f_k(X) = (X - t_1)\cdot(X - t_2) \cdots (X - t_k) .$$

Now we define $\Phi_k = \Phi_k(u_1, u_2, \cdots, u_k) \in \boldsymbol{Q}[u_1, u_2, \cdots, u_k]$ by

$$\Phi_k = \sum_{q_1+q_2+\cdots+q_k=k} t_1^{q_1}\cdot t_2^{q_2} \cdots t_k^{q_k} , \qquad k > 0 ,$$

where the summation is taken over all non-negative integers $q_1, q_2, \cdots, q_k$ such that $q_1 + q_2 + \cdots + q_k = k$. Also put

$$\Phi_0 = 1 \in \boldsymbol{Z} .$$

Since the right-hand side is a symmetric polynomial of $t_1, t_2, \cdots, t_k$, the Φ_k above is a polynomial of $u_1, u_2, \cdots, u_k$.

(3.2) DEFINITION. Let $u_1, u_2, \cdots, u_{r+1}, \cdots$ be a sequence of indeterminates. Fix $r \in \boldsymbol{Z}_+$. Consider the subring $R_r = \boldsymbol{Z}[u_1, u_2, \cdots, u_r]$ of $R = \boldsymbol{Z}[u_1, u_2, \cdots, u_r, u_{r+1}, \cdots]$. Also put $R_0 = \boldsymbol{Z}$. Let $f_r(X)$ be the polynomial $X^r - u_1 X^{r-1} + u_2 X^{r-2} - \cdots + (-1)^r u_r$ in $R_r[X]$. For each $k \in \boldsymbol{Z}_+ \cup \{0\}$ (and $r \in \boldsymbol{Z}_+$), we define $\Phi_{k;r} = \Phi_{k;r}(u_1, u_2, \cdots u_r) \in R_r$ by

$$\Phi_{k;r} = \text{the coefficient of } X^{r-1} \text{ in the remainder of } X^{k+r-1} \text{ when divided by } f_r(X) .$$

(Put $\Phi_{0;0} = 1 \in \boldsymbol{Z} \subseteq R$.) Then the following are straightforward from the definition above:

i) $X^{k+r-1} \equiv (\Phi_{k;r}\cdot X^{r-1}) + \cdots \qquad (\text{mod.}\, f_r(X))$,

where the dots denote a polynomial in X of deg $\leqq r-2$ with coefficients in $R_r = \boldsymbol{Z}[u_1, u_2, \cdots, u_r]$.

ii) For any non-negative integer r, $\Phi_{0;r} = 1$.

iii) For any k, $r \in \boldsymbol{Z}_+$ satisfying $r \geqq k$,

$$\Phi_{k;r} = \Phi_{k;k}$$

iv) For any k, $r \in \boldsymbol{Z}_+$ satisfying $k \geqq r$,

$$\Phi_{k;r} = \Phi_{k;k}(u_1, u_2, \cdots, u_r, \underbrace{0, 0, \cdots, 0}_{k-r \text{ pieces}})$$

v) For any non-negative integer k,

$$\Phi_{k;k} \in R_k = \boldsymbol{Z}[u_1, u_2, \cdots, u_k] \subseteq \boldsymbol{Q}[u_1, u_2, \cdots, u_k] .$$

(3.3) LEMMA. *$\Phi_k = \Phi_{k;k}$ for all $k \geqq 0$.*

Proof. Since $\Phi_0 = 1 = \Phi_{0;0}$, we may assume $k \in \boldsymbol{Z}_+$. Since

$$\Phi_k, \Phi_{k;k} \in \boldsymbol{Q}[u_1, u_2, \cdots, u_k] ,$$

it suffices to show that

$$\Phi_k(u_1, u_2, \cdots, u_k) = \Phi_{k;k}(u_1, u_2, \cdots, u_k)$$

holds for all *complex numbers* $u_1, u_2, \cdots, u_k$. Let $t_1, t_2, \cdots, t_k \in \boldsymbol{C}$ be the roots of the polynomial: $X^k - u_1X^{k-1} + u_2X^{k-2} - \cdots + (-1)^k u_k\ (= f_k(X))$, where $u_1, \cdots, u_k$ are arbitrarily fixed complex numbers. Then

$$\begin{aligned}
\Phi_k(u_1, \cdots, u_k) &= \sum_{q_1+q_2+\cdots+q_k=k} t_1^{q_1}\cdot t_2^{q_2} \cdots t_k^{q_k} \\
&= \left[\begin{array}{l}\text{the coefficient of } z^{-1} \text{ in} \\ z^{k-1}\cdot \prod_{i=1}^{k}\left(1 + \dfrac{t_i}{z} + \left(\dfrac{t_i}{z}\right)^2 + \left(\dfrac{t_i}{z}\right)^3 \cdots\right)\end{array}\right] \\
&= \left[\begin{array}{l}\text{the coefficient of } z^{-1} \text{ in the Laurent series} \\ \text{of } z^{k-1}\cdot \prod_{i=1}^{k}\left(1 - \left(\dfrac{t_i}{z}\right)\right)^{-1} \quad \text{expanded for} \\ \rho_0 < |z| < +\infty, \text{ where } \rho_0 \text{ is large enough.}\end{array}\right] \\
&= \frac{1}{2\pi\sqrt{-1}}\int_{|z|=\rho} z^{k-1}\cdot \prod_{i=1}^{k}\left(1 - \left(\frac{t_i}{z}\right)\right)^{-1} dz \qquad \text{for any } \rho > \rho_0 , \\
&= \frac{1}{2\pi\sqrt{-1}}\int_{|z|=\rho} \frac{z^{2k-1}}{f_k(z)} dz , \qquad \text{for any } \rho > \rho_0 \\
&= \frac{1}{2\pi\sqrt{-1}}\int_{|z|=\rho} \frac{\Phi_{k;k}\cdot z^{k-1} + (\text{lower degree terms in } z)}{f_k(z)} dz \\
&\qquad\qquad\qquad\qquad\qquad \text{for any } \rho > \rho_0 .
\end{aligned}$$

Therefore, letting $\rho \to +\infty$, we obtain:

$$\Phi_k(u_1, u_2, \cdots, u_k) = \Phi_{k;k}(u_1, u_2, \cdots, u_k) \qquad \text{Q.E.D.}$$

Remark. From the above lemma and i–iv of (3.2), it follows that:

(3.3.1) For any $k \in \boldsymbol{Z}_+ \cup \{0\}$ and any $r \in \boldsymbol{Z}_+$ with $r \geqq k$,

$$X^{k+r-1} \equiv \Phi_k(u_1, u_2, \cdots, u_k) \cdot X^{r-1} + \text{lower degree in } X \quad \text{mod. } (X^r - u_1X^{r-1} + \cdots + (-1)^r u_r) .$$

(3.3.2) For any $k \in \boldsymbol{Z}_+$ and any $r \in \boldsymbol{Z}_+$ with $k \geqq r$,

$$X^{k+r-1} \equiv \Phi_k(u_1, \cdots, u_r, 0, \cdots, 0) \cdot X^{r-1} + \text{lower degree in } X \quad \text{mod. } (X^r - u_1X^{r-1} + \cdots + (-1)^r u_r) .$$

We here give Φ_k for the first few values of k:

(3.3.3)
$$\begin{aligned} \Phi_0 &= 1 \\ \Phi_1 &= u_1 \\ \Phi_2 &= u_1^2 - u_2 \\ \Phi_3 &= u_1^3 - 2u_1u_2 + u_3 . \end{aligned}$$

(3.4) We interpret the definition above in terms of the geometry of vector bundles. Let $\pi: E \to M$ be a vector bundle of rank $r \geqq 2$ over an n-dimensional non-singular irreducible projective variety M. Let $g \in H^2(P(E^*))$ denote the first Chern class $c_1(L(E^*)^{-1})$ of the line bundle $L(E^*)^{-1}$ over $P(E^*)$. For each $i = 1, 2, \cdots$, we denote by d_i the i-th Chern class $c_i(E)$ of the vector bundle E. Now we shall show the following integral formula (see Gieseker [8]):

(3.4.1) For any q-dimensional (irreducible) subvariety F (of multiplicity $= 1$) of M and any $\psi \in H^{2q-2k}(M)$ with $0 \leqq k \leqq q \leqq n$,

$$\{\mathrm{pr}^* (\psi) \cdot g^{k+r-1}\}[\mathrm{pr}^{-1} (F)] = \{\psi \cdot \Phi_k(d_1, d_2, \cdots, d_k)\}[F] ,$$

where $\mathrm{pr}: P(E^*) \to M$ is the canonical projection.

Proof. First note that, in $H^{2r}(P(E^*))$,

$$g^r - \mathrm{pr}^* (d_1) \cdot g^{r-1} + \mathrm{pr}^* (d_2) \cdot g^{r-2} - \cdots + (-1)^r \mathrm{pr}^* (d_r) = 0 .$$

From (3.3.1) and (3.3.2), it follows that, in $H^{2k+2r-2}(P(E^*))$,

$$g^{k+r-1} = \Phi_k(\mathrm{pr}^* (d_1), \cdots, \mathrm{pr}^* (d_k)) \cdot g^{r-1} + \begin{pmatrix} \text{a polynomial in } g \text{ of } \deg \leqq r-2 \\ \text{with coefficients in } \mathrm{pr}^* (H^*(M)) \end{pmatrix} .$$

Therefore, in $H^{2q+2r-2}(P(E^*))$,

$$\mathrm{pr}^*(\psi)\cdot g^{k+r-1} = \{\mathrm{pr}^*(\psi)\cdot\Phi_k(d_1, d_2, \cdots, d_k)\}\cdot g^{r-1} + \begin{pmatrix}\text{a polynomial in } g \text{ of } \deg \leqq r-2 \\ \text{with coefficients in } \mathrm{pr}^*(H^*(M))\end{pmatrix}.$$

We integrate both sides over $\mathrm{pr}^{-1}(F)$. Since

$$\int_{\text{fibre}} g^i = \begin{cases} 0 & \text{for } i \leqq r-2 \\ 1 & \text{for } i = r-1, \end{cases}$$

we obtain:

$$\{\mathrm{pr}^*(\psi)\cdot g^{k+r-1}\}[\mathrm{pr}^{-1}(F)] = \{\psi\cdot\Phi_k(d_1, d_2, \cdots, d_k)\}[F]. \qquad \text{Q.E.D.}$$

(3.5) DEFINITION. Let M be a non-singular irreducible projective variety. A class $\omega \in H^{2q}(M)(0 \leqq q \leqq \dim M)$ is said to be *numerically positive*, if $\omega[F] > 0$ for every q-dimensional irreducible subvariety F of M.

EXAMPLE 1 (Bloch-Gieseker [2]). Assume that $\pi: E \to M$ is an ample vector bundle of rank r over an n-dimensional irreducible non-singular projective variety M. Let $d_i = c_i(E)$ and $h = \text{minimum}\ (n, r)$. Then,

$$d_i \in H^{2i}(M) \qquad i = 0, 1, \cdots, h$$

are numerically positive.

EXAMPLE 2 (Fulton [7; Proposition 2]). Under the same assumption as in Example 1, if $\omega \in H^{2q-2k}(M)\ (0 \leqq k \leqq q \leqq n)$ is numerically positive, then

$$\omega\cdot\Phi_k(d_1, d_2, \cdots, d_k)$$

is also numerically positive. (In particular, $\Phi_k(d_1, d_2, \cdots, d_k)$ is numerically positive for $k = 0, 1, \cdots, n$.)

4. Inequalities associated with a section to an ample vector bundle of rank $r \geqq 2$

Combining the previous two sections, we now prove the following:

(4.1) THEOREM. *Let $\pi: E \to M$ be an ample vector bundle of rank $r \geqq 2$ over an irreducible non-singular n-dimensional projective variety M with* $\dim H^0(M, E) > 0$. *Let d_i denote the i-th Chern class of the vector bundle E, and Φ_k, $k = 0, 1, 2, \cdots$ denote the polynomials defined by § 3. Fix an arbitrary irreducible subvariety F of M and an arbitrary numerically* *positive cohomology class* $\omega \in H^{2q-2k}(M)$ *with* $1 \leqq k \leqq q = \dim F \leqq n$. *Then, for any non-zero section* $s \in H^0(M, E)$ *with* $F \not\subseteq$ *(zero locus of* s*), the corresponding* $S \in \mathrm{Div}(M)$ *(= the divisor part of the zeroes of* s*) satisfies the following inequality*:

$$\{\omega \cdot \Phi_k(d_1, d_2, \cdots, d_k)\}[F] > \{\omega \cdot \Phi_{k-1}(d_1, d_2, \cdots, d_{k-1})\}([F] \circ [S]) ,$$

where $[F] \circ [S] \in H_{2q-2}(M)$ *denotes the intersection of the homology classes* $[F] \in H_{2q}(M)$ *and* $[S] \in H_{2n-2}(M)$.

Proof. *Step* 1. Recall that in (2.2) we defined $S' \in \mathrm{Div}(P(E^*))$ and $D_s \in \mathrm{Div}(P(E^*))$ (associated with $0 \neq s \in H^0(M, E)$) by

$$S' = \text{the zeroes of } s'$$
$$D_s = S' - \mathrm{pr}^{-1}(S) ,$$

where s' is the element of $H^0(P(E^*), L(E^*)^{-1})$ corresponding to $s \in H^0(M, E)$, (cf. (2.1)). The first Chern class $g \in H^2(P(E^*))$ of the line bundle $L(E^*)^{-1}$ over $P(E^*)$ is given by $g = c_1([S'])$. Therefore

$$\begin{aligned} g &= c_1([\mathrm{pr}^{-1}(S) + D_s]) = c_1([\mathrm{pr}^{-1}(S)]) + c_1([D_s]) \\ &= \mathrm{pr}^* c_1([S]) + c_1([D_s]) . \end{aligned} \tag{1}$$

On the other hand, $L(E^*)^{-m}$ is very ample for a sufficiently large positive integer m. Therefore, $P(E^*)$ is embedded into some $\boldsymbol{P}^N(\boldsymbol{C})$ so that

$$g = \frac{1}{m} \cdot c_1([H]) \qquad (m \in \boldsymbol{Z}_+) , \tag{2}$$

where $H \in \mathrm{Div}(P(E^*))$ is a generic hyperplane section on $P(E^*)$.

Step 2. Now we make a computation. (For convenience, we write $\Phi_k(d_1, \cdots, d_k)$ and $\Phi_{k-1}(d_1, \cdots, d_{k-1})$ simply as Φ_k and Φ_{k-1} respectively.)

$$\begin{aligned} &(\omega \cdot \Phi_k)[F] - (\omega \cdot \Phi_{k-1})([F] \circ [S]) \\ &= \{\mathrm{pr}^*(\omega) \cdot g^{k+r-1}\}[\mathrm{pr}^{-1}(F)] - (\omega \cdot \Phi_{k-1})([F] \circ [S]) \qquad (\text{cf. } (3.4.1)) \\ &= \{\mathrm{pr}^*(\omega) \cdot g^{k+r-2}\}(\mathrm{pr}^* c_1([S]) + c_1([D_s]))[\mathrm{pr}^{-1}(F)] - \{\omega \cdot c_1([S]) \cdot \Phi_{k-1}\}[F] \\ &\hspace{28em} (\text{cf. } (1)) \\ &= \{\mathrm{pr}^*(\omega) \cdot g^{k+r-2} \cdot c_1([D_s])\}[\mathrm{pr}^{-1}(F)] \quad (\text{Apply } (3.4.1) \text{ to } \psi = \omega \cdot c_1([S]).) \\ &= m^{-k-r+2} \cdot \{\mathrm{pr}^*(\omega) \cdot c_1([H])^{k+r-2} \cdot c_1([D_s])\}[\mathrm{pr}^{-1}(F)] \qquad (\text{cf. } (2)) \\ &= m^{-k-r+2} \cdot \mathrm{pr}^*(\omega)([H_1] \circ [H_2] \circ \cdots \circ [H_{k+r-2}] \circ [D_s] \circ [\mathrm{pr}^{-1}(F)]) , \end{aligned}$$

where $H_1, H_2, \cdots, H_{k+r-2} \in \mathrm{Div}(P(E^*))$ are generic hyperplane sections on $P(E^*)$.

$$= m^{-k-r+2}\cdot\omega\{\mathrm{pr}_*([H_1]\circ[H_2]\circ\cdots\circ[H_{k+r-2}]\circ[D_s]\circ[\mathrm{pr}^{-1}(F)])\}\ .$$

Step 3. Since $F \not\subseteq$ (zero locus of s), it follows that

$$\mathrm{supp}\,(D_s) \cap \mathrm{pr}^{-1}(F) \neq \mathrm{pr}^{-1}(F)\ .$$

Hence, by the irreducibility of F, $\mathrm{supp}\,(D_s)$ and $\mathrm{pr}^{-1}(F)$ intersect properly. Let $U = \{p \in M\,;\, s(p) \neq 0\}$. Since $U \cap F$ is non-empty,

$$(\mathrm{pr}|_{\mathrm{supp}(D_s)})^{-1}(U \cap F) \xrightarrow{\mathrm{pr}} U \cap F$$

is a $\boldsymbol{P}^{r-2}(\boldsymbol{C})$-bundle over $U \cap F$, (cf. (ii) of (2.3.1)). Therefore, letting Y be the closure of $(\mathrm{pr}|_{\mathrm{supp}(D_s)})^{-1}(U \cap F)$ in $\mathrm{supp}\,(D_s) \cap \mathrm{pr}^{-1}(F)$, we have

$$\mathrm{pr}|_Y : Y \to F \tag{3}$$

is generically a $\boldsymbol{P}^{r-2}(\boldsymbol{C})$-bundle over F, so that Y is an irreducible component of $\mathrm{supp}\,(D_s) \cap \mathrm{pr}^{-1}(F)$ with $\mathrm{codim}_{\{\mathrm{pr}^{-1}(F)\}}\, Y = 1$. Let $e \in \boldsymbol{Z}_+$ be the multiplicity of Y in the intersection of cycles D_s and $\mathrm{pr}^{-1}(F)$ in $P(E^*)$. Then

$$[D_s]\circ[\mathrm{pr}^{-1}(F)] - e\cdot[Y] = \text{either } 0 \text{ or an effective cycle}\ .$$

Thus, noting that $H_1, H_2, \cdots, H_{k+r-2} \in \mathrm{Div}\,(P(E^*))$ in Step 2 are generic hyperplane sections and that ω is numerically positive, we have:

$$\begin{aligned}&\omega\{\mathrm{pr}_*([H_1]\circ[H_2]\circ\cdots\circ[H_{k+r-2}]\circ[D_s]\circ[\mathrm{pr}^{-1}(F)])\}\\ &\qquad \geqq e\cdot\omega\{\mathrm{pr}_*([H_1]\circ[H_2]\circ\cdots\circ[H_{k+r-2}]\circ[Y])\}\ .\end{aligned}$$

Step 4. First note that, by (3), Y is an irreducible $(q+r-2)$-dimensional subvariety of $P(E^*)$. Since $k \leqq q$, there are two cases.

Case 1. $k = q$: Then,

$$\begin{aligned}[H_1]\circ[H_2]\circ\cdots\circ[H_{k+r-2}]\circ[Y] &= (\deg_{P^N}(Y))\cdot[\text{a point}]\\ &= \text{an effective cycle}\ .\end{aligned}$$

Therefore, by the numerical positivity of ω,

$$\omega\{\mathrm{pr}_*([H_1]\circ[H_2]\circ\cdots\circ[H_{k+r-2}]\circ[Y])\} > 0\ .$$

Case 2. $k < q$: Then, for any $j < k + r - 2$,

$$\begin{aligned}\dim\,(H_1 \cap H_2 \cap \cdots \cap H_j \cap Y) &\geqq (q+r-2) - j\\ &= (q-k) + (k+r-2-j) \geqq 2\ .\end{aligned}$$

Therefore, by the irreducibility of F, applying Bertini's theorem step by step, we obtain:

i) First we can choose $(r-2)$ hyperplane sections $H_1, H_2, \cdots, H_{r-2} \in \mathrm{Div}\,(P(E^*))$, such that $H_1 \cap H_2 \cap \cdots H_{r-2} \cap Y$ is a q-dimensional irreducible subvariety of $P(E^*)$. Since $\mathrm{pr}|_Y : Y \to F$ is generically a $\boldsymbol{P}^{r-2}(\boldsymbol{C})$-bundle over F, (cf. (3)),

$$H_1 \cap H_2 \cap \cdots \cap H_{r-2} \cap Y \xrightarrow{\mathrm{pr}} F$$

is a subjective regular map with generically finite fibres, because of $\dim (H_1 \cap H_2 \cap \cdots \cap H_{r-2} \cap Y) = q = \dim F$.

ii) By i), we can choose k additional generic hyperplane sections $H_{1+r-2}, H_{2+r-2}, \cdots, H_{k+r-2} \in \mathrm{Div}\,(P(E^*))$, such that

$$H_1 \cap H_2 \cap \cdots \cap H_{k+r-2} \cap Y = \text{irreducible and } (q-k)\text{-dimensional}\,,$$
$$\dim \{\mathrm{pr}\,(H_1 \cap H_2 \cap \cdots \cap H_{k+r-2} \cap Y)\} = q-k\,.$$

Thus, $\mathrm{pr}_*\,([H_1] \circ [H_2] \circ \cdots \circ [H_{k+r-2}] \circ [Y])$ is an effective cycle.

Since ω is numerically positive, we have:

$$\omega\{\mathrm{pr}_*\,([H_1] \circ [H_2] \circ \cdots \circ [H_{k+r-2}] \circ [Y])\} > 0\,.$$

Therefore, in both cases 1 and 2,

$$\omega\{\mathrm{pr}_*\,([H_1] \circ [H_2] \circ \cdots \circ [H_{k+r-2}] \circ [Y])\} > 0\,.$$

Step 5. By Steps 2, 3, and 4, we obtain:

$$\{\omega \cdot \Phi_k(d_1, \cdots, d_k)\}[F] - \{\omega \cdot \Phi_{k-1}(d_1, \cdots, d_{k-1})\}([F] \circ [S]) > 0\,.$$

Q.E.D.

(4.2) COROLLARY. *In Theorem* (4.1) *above, we furthermore assume that* $F \cap \mathrm{supp}\,(S) \neq \phi$. *Then*

$$(d_1)^q[F] > \{(d_1)^{q-1}c_1([S])\}[F] > 0\,.$$

Proof. Note that $\{(d_1)^{q-1} \cdot c_1([S])\}[F] = (d_1)^{q-1}([F] \circ [S])$. Therefore it suffices to show:

i) $(d_1)^q[F] > (d_1)^{q-1}([F] \circ [S])$,

ii) $(d_1)^{q-1}([F] \circ [S]) > 0$.

Since E is an ample vector bundle over M, $d_1 = c_1(E) = c_1(\Lambda^r E)$ is represented by a positive definite (1,1)-form. Therefore,

$$(d_1)^{q-1} = \text{numerically positive} .$$

Now i) is straightforward from an application of Theorem (4.1) to $\omega = (d_1)^{q-1}$ and $k=1$.

ii) Note the following facts:

(1) F is irreducible,

(2) F is not contained in the zero locus of s,

(3) $F \cap \operatorname{supp}(S) \neq \phi$.

These imply that F and support (S) intersect properly, i.e.,

$$[F] \circ [S] = \text{an effective cycle} .$$

Thus, ii) follows from the numerical positivity of $(d_1)^{q-1}$. Q.E.D.

(4.2.1) *Remark.* When $F = M$, Corollary (4.2) is stated as follows: Let $\pi: E \to M$ be an ample vector bundle of rank $r \geqq 2$ over an irreducible non-singular n-dimensional projective variety M. Suppose E admits a non-zero section $s \in H^0(M, E)$ whose zero locus contains an $(n-1)$-dimensional component. Then the corresponding $S \in \operatorname{Div}(M)$ (= the divisor part of the zeroes of s) satisfies the following inequalities:

$$(d_1)^n[M] > \{(d_1)^{n-1} \cdot c_1([S])\}[M] > 0 , \qquad \text{where } d_1 = C_1(E) .$$

5. Theory of polarized varieties

Recently Fujita has developed a theory of polarized varieties introducing the notion of Δ-genus. Since some of his results have been unpublished, we briefly discuss the related part of his work and give a proof (due to Fujita) of a theorem which we shall need later. See, for reference, Fujita ([4][5][6]).

(5.1) DEFINITION. A *polarized variety* is a pair (M, L) consisting of an irreducible complete algebraic variety M (defined over $\boldsymbol{C}$) and an ample line bundle L over M. For a polarized variety (M, L) of $\dim M = n$, he defined three invariants, which are, when M is non-singular, given by the following formulas.

i) $\Delta(M, L) = n + c_1(L)^n[M] - \dim H^0(M, L)$.

This is called the *Δ-genus* of (M, L).

ii) $d(M, L) = c_1(L)^n[M]$.

iii) $g(M, L) = 1 + \frac{1}{2}\{c_1(K_M) + (n-1)c_1(L)\} \cdot c_1(L)^{n-1}[M]$,

(= "adjunction formula"), where K_M is the canonical bundle of M.

(5.2) Let $|L|$ denote the complete linear system of Cartier divisors associated with L, and $B_s|L|$ denote the set of base points in M of the linear system $|L|$. A basic tool of his theory is the following inequality:

$$\dim(B_s|L|) < \Delta(M, L) ,$$

where $\dim(B_s|L|) < 0$ means: $B_s|L| = \phi$. (See Fujita [4].)

(5.3) He also proved the following facts:

Let (M, L) be a polarized variety satisfying the following three conditions:

(1) M is non-singular,

(2) $\dim(B_s|L|) < 1$,

(3) $g(M, L) \geqq \Delta(M, L)$.

Then,

i) "Bertini-type theorem": If $d(M, L) \geqq 2\cdot\Delta(M, L) - 1$ and $\dim M \geqq 2$, then a general member of $|L|$ is irreducible and non-singular. (See, for non-singularity, Fujita [6]. Connectedness follows from Fujita [4; Lemma 6.1].)

ii) If $d(M, L) \geqq 2\cdot\Delta(M, L)$, then $B_s|L| = \phi$.

iii) If $d(M, L) \geqq 2\cdot\Delta(M, L) + 1$, then L is very ample.

Now we state the theorem we need.

(5.4) THEOREM. *Let M be a 3-dimensional irreducible non-singular projective variety with ample tangent bundle $T(M)$. We denote by c_i $(i = 1, 2, \cdots)$ the i-th Chern class of the tangent bundle $T(M)$. Assume that, in $H^2(M)$ $(= H^2(M; \boldsymbol{Z})/\text{torsion classes})$, c_1 is written in the form:*

$$c_1 = r\cdot g \qquad \text{for some } 2 \leqq r \in \boldsymbol{Z} \text{ and some } g \in H^2(M) .$$

Then M is isomorphic to $\boldsymbol{P}^3(\boldsymbol{C})$.

Remark. 1) In the above, the case $r \geqq 3$ is due to Kobayashi-Ochiai [16]. 2) The case $r = 2$ is due to Fujita. Note that the proof of 2) (which we are going to give now) is much harder than that of 1), although Fujita proved 2), stimulated by the work of Kobayashi and Ochiai.

Proof of Theorem (5.4). Assuming 1), we shall prove 2) of Remark above. First note that the ampleness of $T(M)$ implies that c_1 is represented by a positive definite (1,1)-form. Therefore, by Kobayashi

[15], $H_1(M, Z) = 0$. In particular,

$$H^1(M, \mathcal{O}^*) \cong H^2(M; Z) = H^2(M) .$$

By this isomorphism, we identify each line bundle on M with its first Chern class. Let L be the line bundle corresponding to $g \in H^2(M)$. Since

$$L = g = \tfrac{1}{2} \cdot c_1 ,$$

L is ample and (M, L) is a polarized variety.

Step 1. We compute three invariants of (M, L). Note that:

$$\begin{aligned} \Delta(M, L) &= 3 + L^3[M] - \dim H^0(M, L) \\ d(M, L) &= L^3[M] \\ g(M, L) &= \{\tfrac{1}{2}(-c_1 + 2L)L^2[M]\} + 1 \qquad \text{(See (5.1).)} \end{aligned}$$

Since $c_1 = 2 \cdot L$, we immediately obtain:

$$g(M, L) = 1 .$$

Since L is ample, we have:

$$d(M, L) \geqq 1 .$$

Now we compute $\Delta(M, L)$. The Riemann-Roch theorem for an algebraic threefold asserts that

$$\begin{aligned} \chi(M, L) &\left(= \sum (-1)^i \dim H^i(M, L)\right) \\ &= \left\{\frac{1}{6}L^3 + \frac{1}{4}L^2 \cdot c_1 + \frac{1}{12}L \cdot (c_1)^2 + \frac{1}{12}L \cdot c_2 + \frac{1}{24}c_1 \cdot c_2\right\}[M] , \\ \chi(M, \mathcal{O}) &\left(= \sum (-1)^i \dim H^i(M, \mathcal{O})\right) = \frac{1}{24}c_1 \cdot c_2[M] . \end{aligned}$$

Noting that $c_1 = 2 \cdot L$, we obtain:

$$(\sharp) \qquad \chi(M, L) = L^3[M] + 2 \cdot \chi(M, \mathcal{O}) .$$

Since $L - K_M = L + c_1 = 3L$, Kodaira's Vanishing Theorem says that:

$$\dim H^i(M, L) = 0 , \qquad i \geqq 1 .$$

Thus,

$$(\sharp\sharp) \qquad \chi(M, L) = \dim H^0(M, L) .$$

On the other hand, the negativity of $K_M = -2 \cdot L$ implies that

$$\dim H^i(M, \mathcal{O}) = 0 , \qquad i \geqq 1 .$$

Therefore, we get:

$$(\#\#\#) \qquad \chi(M, \mathcal{O}) = 1 .$$

Thus, from (♯), (♯♯) and (♯♯♯), we obtain:

$$\dim H^0(M, L) = L^3[M] + 2 ,$$

i.e.,

$$\Delta(M, L) = 1 .$$

Step 2. Consider the complete linear system $|L|$ of Cartier divisors associated with the line bundle L. By Step 1 and (5.2), we have:

$$\dim (B_s |L|) < 1 ,$$
$$g(M, L) = 1 = \Delta(M, L) ,$$
$$d(M, L) \geqq 1 = 2\cdot\Delta(M, L) - 1 .$$

Therefore, by (5.3), a general member S of $|L|$ is non-singular and irreducible.

Step 3. Consider the embedding: $S \subsetneq M$. Since $S \in |L|$, we have:

$$K_S = (K_M + L)_{|S} = (-c_1 + L)_{|S} = -L_{|S} ,$$

which is a negative line bundle over S. Therefore, by Kodaira's Vanishing Theorem,

i) $q(S)$ $(= \dim H^1(S, \mathcal{O})) = 0$,
ii) $P_2(S)$ $(= \dim H^0(S, 2K_S)) = 0$.

Thus, by Castelnuovo's criterion, S is a rational surface.

Step 4. We claim that S is a relatively minimal model. For contradiction, assume that there exists an exceptional curve C of the first kind on S. (i.e., $C \cong \boldsymbol{P}^1(\boldsymbol{C})$, and $(C, C)_S = -1$.) By Adjunction formula,

$$0 = \text{genus of } C = \tfrac{1}{2}\{(C, C)_S + K_S[C]\} + 1 .$$

Therefore,

$$K_S[C] = -1 .$$

Thus,

$$c_1[C] = 2\cdot L[C] = -2\cdot K_S[C] = 2 .$$

On the other hand, from the ampleness of $T(M)$ and $C \cong \boldsymbol{P}^1(\boldsymbol{C})$, we obtain:

$$c_1[C] \geqq 4 . \qquad \text{(Hartshorne; cf. [18])}$$

This is a contradiction.

Step 5. By Steps 3 and 4, S is a relatively minimal model of rational surfaces with negative canonical bundle. First note that all the relatively minimal models of rational surfaces are

$$\begin{gathered} \boldsymbol{P}^2(\boldsymbol{C}) , \\ \boldsymbol{P}^1(\boldsymbol{C}) \times \boldsymbol{P}^1(\boldsymbol{C}) , \\ F_n = \operatorname{Proj}\,(\mathcal{O}_{\boldsymbol{P}^1}(0) \oplus \mathcal{O}_{\boldsymbol{P}^1}(n)) , \qquad n = 2, 3, 4, \cdots . \end{gathered}$$

Since each F_n contains a curve e_n satisfying

$$e_n \cong \boldsymbol{P}^1(\boldsymbol{C}) \text{ and } (e_n, e_n)_{F_n} = -n , \qquad \text{(cf. Šafarevič [25])}$$

we have, by Adjunction formula,

$$0 = \text{genus of } e_n = \tfrac{1}{2}\cdot(-n + K_{F_n}[e_n]) + 1 .$$

Thus, $K_{F_n}[e_n] = n - 2$, and F_n $(n = 2, 3, \cdots)$ cannot have negative canonical bundle, i.e., $S = \boldsymbol{P}^2(\boldsymbol{C})$ or $\boldsymbol{P}^1(\boldsymbol{C}) \times \boldsymbol{P}^1(\boldsymbol{C})$.

Step 6. Since $S \in |L|$ and $L = \mathcal{O}([S])$ is ample, choosing a sufficiently large $\nu \in \boldsymbol{Z}_+$, we can embed $M \subsetneq \boldsymbol{P}^N(\boldsymbol{C})$, such that S $(= \operatorname{supp}(\nu\cdot S))$ is a hyperplane section on M in $\boldsymbol{P}^N(\boldsymbol{C})$. Since M is non-singular, $S \subsetneq M$ induces, by the Lefschetz Theorem on hyperplane sections, a surjective mapping:

$$(*) \qquad H_2(S; \boldsymbol{Z}) \to H_2(M, \boldsymbol{Z}) \to 0 \qquad \text{(exact)} .$$

Therefore, the following three cases are possible:

i) $S = \boldsymbol{P}^2(\boldsymbol{C})$, and the second Betti number $b_2(M)$ of M is 1.
ii) $S = \boldsymbol{P}^1(\boldsymbol{C}) \times \boldsymbol{P}^1(\boldsymbol{C})$, and $b_2(M) = 2$.
iii) $S = \boldsymbol{P}^1(\boldsymbol{C}) \times \boldsymbol{P}^1(\boldsymbol{C})$, and $b_2(M) = 1$.

Case i. Since $H_2(S; \boldsymbol{Z}) \cong \boldsymbol{Z}$, and $b_2(M) = 1$, it follows that (*) is an isomorphism:

$$H_2(S;\boldsymbol{Z}) \cong H_2(M;\boldsymbol{Z}) .$$

Since $H^2(M;\boldsymbol{Z}) = H^2(M)$ (i.e. $H^2(M;\boldsymbol{Z})$ is torsion-free), the isomorphism above induces:

$$H^2(M;\boldsymbol{Z}) \cong H^2(S;\boldsymbol{Z}) \qquad (\cong \boldsymbol{Z}) .$$

Note that, by this isomorphism, $L \in H^2(M;\boldsymbol{Z})$ is mapped to $L_{|S} \in H^2(S;\boldsymbol{Z})$.

$$\begin{aligned} L_{|S} &= -K_S \qquad \text{(cf. Step 3)} \\ &= 3\cdot \mathcal{O}_{\boldsymbol{P}^2}(1) . \end{aligned}$$

Therefore, L is also written as $3\cdot h$ for some $h \in H^2(M;\boldsymbol{Z})$. Thus

$$\begin{aligned} 9 &= (K_S)^2[S] \qquad (\text{because } S = \boldsymbol{P}^2(\boldsymbol{C})) \\ &= L^2[S] \qquad \text{(cf. Step 3)} \\ &= 27\cdot h^3[M] . \end{aligned}$$

This is in contradiction to $h^3[M] \in \boldsymbol{Z}$. Therefore, Case i) cannot happen.

Case ii. $S = \boldsymbol{P}^1(\boldsymbol{C}) \times \boldsymbol{P}^1(\boldsymbol{C})$ and $b_2(M) = 2$:
Since $H_2(S;Z) \cong \boldsymbol{Z} \oplus \boldsymbol{Z}$ and $b_2(M) = 2$, (*) is again an isomorphism:

$$H_2(S;\boldsymbol{Z}) \cong H_2(M;\boldsymbol{Z}) .$$

Since $H^2(M;\boldsymbol{Z})$ is torsion-free, this induces:

$$\begin{aligned} H^2(M;\boldsymbol{Z}) &\cong H^2(S;\boldsymbol{Z}) \qquad (\cong \boldsymbol{Z}\oplus\boldsymbol{Z}) \\ L &\leftrightarrow L_{|S} = -K_S \end{aligned}$$

Since $-K_S = -K_{(\boldsymbol{P}^1\times\boldsymbol{P}^1)}$ is divisible by 2 in $H^2(S;\boldsymbol{Z})$, L is also divisible by 2 in $H^2(M;\boldsymbol{Z})$. Therefore, $c_1\,(= 2\cdot L)$ is divisible by 4 in $H^2(M;\boldsymbol{Z})$. By Remark 1) (Kobayashi-Ochiai [16]), we have:

$$M \cong \boldsymbol{P}^3(\boldsymbol{C}) .$$

Case iii. $S = \boldsymbol{P}^1(\boldsymbol{C}) \times \boldsymbol{P}^1(\boldsymbol{C})$ and $b_2(M) = 1$:
Since $S = \boldsymbol{P}^1(\boldsymbol{C}) \times \boldsymbol{P}^1(\boldsymbol{C})$, the group $H_2(S;\boldsymbol{Z})$ is generated by the homology classes $[C]$ and $[C']$ carried by curves:

$$C = \{\text{a point}\} \times \boldsymbol{P}^1(\boldsymbol{C}) \subsetneq \boldsymbol{P}^1(\boldsymbol{C}) \times \boldsymbol{P}^1(\boldsymbol{C}) ,$$

and

$$C' = \boldsymbol{P}^1(\boldsymbol{C}) \times \{\text{a point}\} \subsetneq \boldsymbol{P}^1(\boldsymbol{C}) \times \boldsymbol{P}^1(\boldsymbol{C}) .$$

Note that (1) $$L_{|S}[C] = -K_S[C] = 2\ ,$$

and

$$(2) \qquad L_{|S}[C'] = -K_S[C'] = 2\ .$$

Let $[C]_M$ and $[C']_M$ denote the homology cycles in $H_2(M; \boldsymbol{Z})$ carried by the curves C and C' respectively. Recall that (*) is a surjective mapping:

$$\begin{gathered} H_2(S; Z)\ (= \boldsymbol{Z}[C] + \boldsymbol{Z}[C']) \to H_2(M; \boldsymbol{Z}) \to 0 \qquad \text{(exact)}\ . \\ [C] \mapsto [C]_M \\ [C'] \mapsto [C']_M \end{gathered}$$

From (1) and (2) above, we obtain:

$$L([C]_M) = L_{|S}[C] = 2 = L_{|S}[C'] = L([C']_M)\ .$$

Since $b_2(M) = 1$, this shows that:

$$[C]_M \equiv [C']_M \qquad \text{mod. (torsion classes)}\ .$$

On the other hand, by the surjectivity of the above mapping,

$$H_2(M; \boldsymbol{Z}) = \boldsymbol{Z}([C]_M) + \boldsymbol{Z}([C']_M)\ .$$

Therefore, noting that $H_2(M)\ (= H_2(M; \boldsymbol{Z})/\text{torsion classes}) \cong \boldsymbol{Z}$, we obtain, in $H^2(M)$,

$$[C]_M = [C']_M = (\text{a generator of } H_2(M))\ .$$

Thus $c_1([C]_M) = 2\cdot L([C]_M) = 4$ implies that c_1 is divisible by 4 in $H^2(M)$. Therefore, by Remark 1) (Kobayashi-Ochiai [16]),

$$M \cong \boldsymbol{P}^3(\boldsymbol{C})\ .$$

Q.E.D.

6. Statement and proof of the key theorem

We shall give a proof of the following key fact by combining the previous sections.

(6.1) THEOREM. *Let M be a non-singular irreducible 3-dimensional projective variety with ample tangent bundle $T(M)$ and the second Betti number $b_2(M) = 1$. Assume that there exists a section:*

$$0 \neq s \in H^0(M, T(M))$$

whose zero locus on M contains a (non-empty) 2*-dimensional component. Then M is isomorphic to* $\boldsymbol{P}^3(\boldsymbol{C})$.

Proof. Let $c_1 \in H^2(M)$ denote the first Chern class of the tangent bundle $T(M)$. Since $T(M)$ is ample, $c_1 \neq 0$ in $H^2(M)$. Note that $b_2(M) = 1$ implies

$$H^2(M)\ (= H^2(M\,; \boldsymbol{Z})/\text{torsion classes}) \cong \boldsymbol{Z}\ .$$

Therefore, we can choose a generator g of $H^2(M)$ such that

i) $H^2(M) = \boldsymbol{Z}\cdot g$,
ii) $c_1 = r\cdot g$, for some $r \in \boldsymbol{Z}_+$.

Now let $S \in \mathrm{Div}\,(M)$ denote the divisor part of the zeroes of s, (cf. (2.2.1)). Since $c_1([S]) \in H^2(M)$, there exists $t \in \boldsymbol{Z}$ such that

$$c_1([S]) = t\cdot g\ .$$

We apply Remark (4.2.1) to the tangent bundle $T(M)$. Then

$$(c_1)^3[M] > \{(c_1)^2\cdot c_1([S])\}[M] > 0\ .$$

Thus,

$$r^3\cdot g^3[M] > r^2\cdot t\cdot g^3[M] > 0\ .$$

Since $r \in \boldsymbol{Z}_+$, we have: $g^3[M] > 0$. Therefore,

$$r > t > 0\ , \qquad (r, t \in \boldsymbol{Z})\ .$$

Thus,

$$r \geqq 2\ ,$$

which implies, by Theorem (5.4),

$$M \cong \boldsymbol{P}^3(\boldsymbol{C}) \qquad \text{Q.E.D.}$$

7. C^3-actions on algebraic threefolds with ample tangent bundle and the second Betti number 1

First we get rid of "bad" actions, using the key theorem (6.1).

(7.1) PROPOSITION. *Let M be a* 3*-dimensional connected compact complex* *manifold on which a 3-dimensional connected abelian complex Lie group G acts holomorphically and effectively. Assume that one of the following two conditions is satisfied:*

i) *There is a 2-dimensional G-orbit.*
ii) *There is no 3-dimensional G-orbit.*

Then, there exists a non-trivial holomorphic vector field on M whose zero locus contains a 2-dimensional analytic subvariety of M.

Proof. i) Suppose M contains a 2-dimensional G-orbit $G\cdot p$, $p \in M$. Let $(G_p)^0$ denote the identity component of the isotropy subgroup G_p of G at p. Note that $(G_p)^0$ is a 1-dimensional closed complex Lie subgroup of G. Since G is abelian, $(G_p)^0$ acts trivially on $G\cdot p$. Let $\mathfrak{g}_0$ be the (1-dimensional) Lie algebra of $(G_p)^0$ with a generator $X \in \mathfrak{g}_0$. Let X' denote the non-zero holomorphic vector field on M associated with X. Since $(G_p)^0$ acts trivially on $G\cdot p$,

$$X'_{|(G\cdot p)} \equiv 0 .$$

Thus, the zero locus of X' contains a 2-dimensional G-orbit $G\cdot p$ (and hence contains its analytic closure in M). This finishes i).

ii) Let n denote the maximal dimension of the G-orbits in M. By i) above, it suffices to show that $n = 1$ implies the existence of a non-trivial holomorphic vector field on M whose zero locus contains a non-empty 2-dimensional component. Let $\{X_1, X_2, X_3\}$ be a basis of the Lie algebra $\mathfrak{g}$ of G, and X_1', X_2', X_3' be the corresponding holomorphic vector fields on M. Put $U = M -$ (zero locus of X_1'). Since $n = 1$, there exists a holomorphic function f on U, such that

$$X_2' = f\cdot X_1' \qquad \text{on} \quad U .$$

Note that, by the definition of U, we can finish ii) if we show that $M - U$ contains a 2-dimensional component. For contradiction, we assume

$$\dim (M - U) \leqq 1 .$$

Then, by Holomorphic Extension Theorem, f can be extended to a holomorphic function on M. Since M is compact, f is a constant function. Thus,

$$aX_1' - X_2' \equiv 0 \qquad \text{on } M \text{ for some } a \in \boldsymbol{C}\,.$$

By the effectiveness of the G-action, this implies $a \cdot X_1 - X_2 = 0$, which is a contradiction. Q.E.D.

(7.1.1) COROLLARY. *Let M be a 3-dimensional irreducible non-singular projective variety with ample tangent bundle and the second Betti number 1. Assume that a 3-dimensional connected abelian complex Lie group G acts on M holomorphically and effectively, satisfying one of the following two conditions*:

i) *There is a 2-dimensional G-orbit.*
ii) *There is no 3-dimensional G-orbit.*

Then, M is (algebraically) isomorphic to $\boldsymbol{P}^3(\boldsymbol{C})$.

Proof. This is straightforward from Theorem (6.1), Proposition (7.1) above, and *GAGA*.
Now we prove the main theorem:

(7.2) THEOREM. *Let M be a 3-dimensional irreducible non-singular projective variety with ample tangent bundle $T(M)$ and the second Betti number $b_2(M) = 1$. Assume that the complex Lie group $\boldsymbol{C}^3$ $(= \boldsymbol{C} \times \boldsymbol{C} \times \boldsymbol{C})$ acts on M holomorphically and effectively. Then M is (algebraically) isomorphic to $\boldsymbol{P}^3(\boldsymbol{C})$.*

Proof of Theorem (7.2). *Step* 1. Put $G = \boldsymbol{C}^3$. By (7.1.1), we may assume

i) there is no 2-dimensional G-orbit, and
ii) there is a 3-dimensional G-orbit.

Let $\{X_1, X_2, X_3\}$ be a basis for the Lie algebra $\mathfrak{g}$ of G, and X_1', X_2', X_3' be the corresponding holomorphic vector fields on M. Then

$$\begin{aligned}
(1)\quad \{p \in M\,;\, \dim(G\cdot p) \leqq 1\} &= \{p \in M\,;\, \dim(G\cdot p) \leqq 2\} \\
&= (\text{zero locus of } X_1' \wedge X_2' \wedge X_3')\,, \\
(2)\quad \{p \in M\,;\, \dim(G\cdot p) = 3\} &= M - (\text{zero locus of } X_1' \wedge X_2' \wedge X_3')\,,
\end{aligned}$$

where $X_1' \wedge X_2' \wedge X_3'$ is regarded as a section to the line bundle $\wedge^3 T(M)$ over M. Put $F =$ (zero locus of $X_1' \wedge X_2' \wedge X_3'$). Then, by ii) and (2) above, $M - F$ is a non-empty disjoint union of the 3-dimensional G-orbits. Since $M - F$ is connected, it consists of a single 3-dimensional G-orbit

$G \cdot p_0$, $(p_0 \in M)$. By the effectiveness of our G-action,

$$G \cdot p_0 \cong G/\{e\} \cong \boldsymbol{C}^3 .$$

Thus, by the compactness of M, $F = M - (G \cdot p_0)$ is non-empty. Therefore, from the definition of F, F is purely 2-dimensional, i.e.,

$$(\sharp) \qquad F = \bigcup_{i=1}^{r} F_i , \qquad r = \begin{pmatrix}\text{number of the irreducible} \\ \text{components of } F,\end{pmatrix}$$

where each F_i is a 2-dimensional irreducible subvariety of M.

Step 2. Since $M - F$ is non-singular, the Lefschetz duality theorem (see, for instance, Spanier [26; p. 297]) asserts that:

$$H^q(M, F; \boldsymbol{Z}) \cong H_{6-q}(M - F; \boldsymbol{Z}) .$$

Since $M - F = G \cdot p_0 \cong \boldsymbol{C}^3$, we obtain:

$$H^4(M, F; \boldsymbol{Z}) = 0 , \quad \text{and} \quad H^5(M, F; \boldsymbol{Z}) = 0 .$$

From the exact sequence:

$$0 = H^4(M, F; \boldsymbol{Z}) \to H^4(M; \boldsymbol{Z}) \to H^4(F; \boldsymbol{Z}) \to H^5(M, F; \boldsymbol{Z}) = 0 ,$$

it follows that

$$H^4(M; \boldsymbol{Z}) \cong H^4(F; \boldsymbol{Z}) .$$

By Poincaré duality, $b_2(M) = 1$ implies $b_4(M) = 1$. On the other hand, by $(\sharp)$, $b_4(F) = r$. Hence the isomorphism implies $r = 1$. Thus,

F is an *irreducible* closed subvariety of M.

Step 3. Let k be the maximal dimension of the G-orbits in F. By (1) of Step 1, k is either 0 or 1.

Case a. $k = 0$: Since $k = 0$, G acts trivially on F, and hence $X'_{1|F} \equiv 0$. Since $\dim F = 2$, Theorem (6.1) implies

$$M \cong \boldsymbol{P}^3(\boldsymbol{C}) .$$

Case b. $k = 1$: Consider $f = X'_1 \wedge X'_2 \wedge X'_3 \in H^0(M, \wedge\, {}^3T(M))$. Note that the divisor Zero $(f) \in$ Div (M) defined as the zeroes of f (counted with appropriate multiplicities) is written as:

$$\text{Zero}\,(f) = \nu \cdot F , \qquad \nu \in \boldsymbol{Z}_+ .$$

Let I_F denote the ideal sheaf ($\subseteq$ the structure sheaf $\mathcal{O}$ of the projective variety M) defining the subvariety F in M. Note that:

$$c_1(T(M)) = c_1(\wedge^3 T(M)) = \nu \cdot c_1([F]) \in H^2(M; \boldsymbol{Z}) .$$

Therefore, by Theorem (5.4), the proof is reduced to showing $\nu \geqq 2$ (i.e., $f \in H^0(M, I_F^2(\wedge^3 T(M)))$ under the assumption $k = 1$).

Step 4. We want to show $f \in H^0(M, I_F^2(\wedge^3 T(M)))$, assuming $k = 1$. Since $k = 1$, without loss of generality we may assume:

$$X'_{1|F} \not\equiv 0 .$$

Fix a point $p_1 \in F$ with $X'_1(p_1) \neq 0$, and choose an affine open neighborhood U of p_1 in M, such that

$$U \subseteq \{p \in M; X'_1(p) \neq 0\} .$$

Put $F' = F \cap U$. Since F' is non-empty and $I_{F'} = (I_F)_{|U}$, it suffices to show:

$$f_{|U} \in H^0(U, I_{F'}^2(\wedge^3 T(M)_{|U})) .$$

Let $p \in F'$. Since $X'_1(p) \neq 0$ and $k = 1$, it follows that:

$$X'_2(p) = a_p \cdot X'_1(p) \quad \text{and} \quad X'_3(p) = b_p \cdot X'_1(p) ,$$

for some $a_p, b_p \in \boldsymbol{C}$. F' being an affine algebraic subvariety of the affine set U, there exist, for (algebraic) sections $X'_1, X'_2, X'_3 \in H^0(M, T(M))$, regular functions $g_2, g_3 \in \boldsymbol{C}[F']$ on F', such that

$$X'_{2|F'} = g_2 \cdot X'_{1|F'} \quad \text{and} \quad X'_{3|F'} = g_3 \cdot X'_{1|F'} .$$

In other words, there exist regular functions $G_2, G_3 \in \boldsymbol{C}[U]$ on U such that:

$$X'_{2|U} - G_2 \cdot X'_{1|U} \in H^0(U, I_{F'}(T(M)_{|U}))$$

and

$$X'_{3|U} - G_3 \cdot X'_{1|U} \in H^0(U, I_{F'}(T(M)_{|U})) .$$

Thus,

$$\begin{aligned} f_{|U} &= (X'_{1|U}) \wedge (X'_{2|U}) \wedge (X'_{3|U}) \\ &= (X'_{1|U}) \wedge (X'_{2|U} - G_2 \cdot X'_{1|U}) \wedge (X'_{3|U} - G_3 \cdot X'_{1|U}) \\ &\in H^0(U, I_{F'}^2(\wedge^3 T(M)_{|U})) . \end{aligned}$$

Q.E.D.

(7.2.1) *Remark.* Theorem (7.2) is valid also for $(G_m)^3$-actions on M; this is an immediate consequence of Corollary (7.1.1), if we use the more or less known fact that every non-singular n-dimensional projective variety with an effective regular $(G_m)^n$-action always admits a q-dimensional orbit for any q with $0 \leqq q \leqq n$. In a forthcoming paper, however, we shall prove stronger results for n-dimensional non-singular projective varieties with $(G_m)^n$-actions and ample tangent bundle without the additional assumption on the second Betti number.

Added in proof: After the completion of this paper, (we are informed that) H. Sumihiro and S. Mori has succeeded in proving the equivalence of $(G\text{–}n)$ and $(H\text{–}n)$. Therefore their results, combined with our results (the proof of $(G\text{–}3)$), prove $(H\text{–}3)$.

REFERENCES

[1] R. L. Bishop, S. I. Goldberg, On the second cohomology group of a Kaehler manifold of positive curvature. Proc. Amer. Math. Soc. **16** (1965), 119–122.

[2] S. Bloch, D. Gieseker, The positivity of the Chern classes of an ample vector bundle. Inventiones Math. **12** (1971), 112–117.

[3] T. T. Frankel, Manifolds with positive curvature. Pacific J. Math. **11** (1961), 165–174.

[4] T. Fujita, On the structure of polarized varieties with Δ-genera zero. J. Fac. Sci. Univ. Tokyo **22** (1975), 103–115.

[5] ——, On the structure of certain types of polarized varieties, I & II. Proc. Japan Acad. **49** (1973), 800–802, and **50** (1974), 411–412.

[6] ——, Defining equations for polarized varieties. In: Complex Analysis and Algebraic Geometry (dedicated to K. Kodaira), edited by W. L. Baily and T. Shioda, To appear.

[7] W. Fulton, Ample vector bundles, Chern classes, and numerical criteria. Inventiones Math. **32** (1976), 171–178.

[8] D. Gieseker, P-ample bundles and their Chern classes. Nagoya Math. J. **43** (1971), 91–116.

[9] S. I. Goldberg and S. Kobayashi, On holomorphic bisectional curvature. J. Differential Geometry **1** (1967), 225–233.

[10] P. A. Griffiths, The extension problem in complex analysis-II; embeddings with positive normal bundle. Amer. J. Math. **88** (1966), 366–446.

[11] ——, Hermitian differential geometry, Chern classes, and positive vector bundles. In: Global Analysis, edited by D. C. Spencer and S. Iyanaga, Princeton Mathematical Series, No. **29**, Tokyo, 1969, 185–251.

[12] R. Hartshorne, Ample vector bundles. Publ. Math. IHES **29** (1966), 63–94.

[13] ——, Ample subvarieties of algebraic varieties. Lecture Notes in Math., No. 156, Springer Verlag, 1970.

[14] S. Iitaka, On some new birational invariants of algebraic varieties and their application to rationality of certain algebraic varieties of dimension 3. J. Math. Soc. Japan **24** (1972), 384–396.

[15] S. Kobayashi, On compact Kaehler manifolds with positive definite Ricci tensor. Ann. of Math. **74** (1961), 570–574.
[16] S. Kobayashi and T. Ochiai, Characterizations of complex projective spaces and hyperquadrics. J. Math. Kyoto Univ. **13** (1973), 31–47.
[17] ——, On complex manifolds with positive tangent bundle. J. Math. Soc. Japan **22** (1970), 499–525.
[18] ——, Compact homogeneous complex manifolds with positive tangent bundle. Differential Geometry in honor of K. Yano, Kinokuniya, Tokyo, 1972, 221–232.
[19] ——, Three-dimensional compact Kaehler manifolds with positive holomorphic bisectional curvature. J. Math. Soc. Japan **24** (1972), 465–480.
[20] K. Kodaira, On Kaehler varieties of restricted type. Ann. of Math. **60** (1954), 28–48.
[21] T. Mabuchi, Almost homogeneous torus actions on varieties with ample tangent bundle. To appear.
[22] ——, On the classification of essentially effective $SL(3; C)$-actions on algebraic threefolds. To appear.
[23] ——, On the classification of essentially effective $SL(2; C) \times SL(2; C)$-actions on algebraic threefolds. To appear.
[24] T. Ochiai, On compact Kaehler manifolds with positive holomorphic bisectional curvature. Proc. Symp. Pure Math., Vol. **27**, Part II, Amer. Math. Soc. 1975, 113–123.
[25] I. R. Šafarevič, Algebraic surfaces. Proc. Steklov Inst. Math. **75** (1965).
[26] E. H. Spanier, Algebraic topology. New York: McGraw Hill, 1966.

Department of Mathematics
University of California, Berkeley