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DIFFERENTIAL EQUATIONS AND AN ANALOG OF THE PALEY-WIENER THEOREM FOR LINEAR SEMISIMPLE LIE GROUPS

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§ 1. Introduction

Let G be a noncompact linear semisimple Lie group. Fix $G = KAN$ an Iwasawa decomposition of G. That is, K is a maximal compact subgroup of G, A is a vector subgroup with AdA consisting of semisimple transformations and A normalizes N, a simply connected nilpotent subgroup of G. Let M' denote the normalizer of A in K, M the centralizer of A in K, and $W = M'/M$ the restricted Weyl group of G. Fix θ a Cartan involution of G which leaves every element of K fixed and set $\bar{N} = \theta N$. We denote the Lie algebras of $G, K, A, N, \bar{N}$, and M respectively by $\mathfrak{G}, \mathfrak{K}, \mathfrak{A}, \mathfrak{N}, \bar{\mathfrak{N}}$, and $\mathfrak{M}$ respectively.

For $g \in G$ set $g = K(g) \exp H(g) n(g)$ where $K(g) \in K$, $H(g) \in \mathfrak{A}$, and $n(g) \in N$ and $\exp|_{\mathfrak{A}}$ is an isomorphism from $\mathfrak{A}$ to A with inverse log. Recall that $\lambda \in \mathfrak{A}^*$ is called a root if $\mathfrak{G}_\lambda = \{X \in \mathfrak{G} : [H, X] = \lambda(H)X \text{ for all } H \in \mathfrak{A}\} \neq \{0\}$ and λ is a positive root if $\mathfrak{G}_\lambda \subseteq \mathfrak{N}$. Let P denote the set of all positive roots and let L be the semilattice of all elements of $\mathfrak{A}^*$ of the form $\sum_{\lambda \in P} c_\lambda \lambda$ and c_λ is a nonnegative integer.

Let V be a finite dimensional vector space and let K act on V via the double representation τ. That is, for $v \in V$ and $k_1, k_2 \in K$

$$\tau(k_1, k_2) : v \longrightarrow \tau(k_1) \cdot v \cdot \tau(k_2)^{-1}.$$

Consider the C^∞ functions $f : G \to V$ for which $f(k_1 g k_2) = \tau(k_1) f(g) \tau(k_2)$ $(k_1, k_2 \in K)$. We denote these functions by $C^\infty(G, \tau)$ and we denote the C^∞-functions with compact support by $C_c^\infty(G, \tau)$ and the Schwartz functions in $C^\infty(G, \tau)$ by $\mathscr{C}(G, \tau)$.

Consider $f \in \mathscr{C}(G, \tau)$ and for $\nu \in \mathfrak{A}_C^*$ $m \in M$ set

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$$g_f(\nu)(m) = \int_A da \int_N f(man)e^{(\rho - i\nu)(\log a)}dn$$

where for $H \in \mathfrak{A}$ $\rho(H) = \frac{1}{2}\operatorname{tr} adH_{|\mathfrak{N}}$ and for $\omega \in \hat{M}$, set

$$\psi_f(\omega : \nu) = \int_M \chi_\omega(m')g_f(\nu)(m')dm' .$$

Now $\psi_f(\omega : \nu) \in V^M$ where $V^M = \{v \in V : \tau(m)v = v\tau(m)$ for all $m \in M\}$ and in fact $\psi_f(\omega : \nu) \in V^M(\omega)$ where $V^M(\omega) = E_\omega(V^M)$ and

$$E_\omega(v) = d_\omega \int_M \overline{\chi_\omega(m)}\tau(m)vdm .$$

In general for $A \in V^M$ we define the Eisenstein integral of Harish-Chandra by setting

$$E(A : \nu : x) = \int_K \tau(K(xk)) \circ A \circ \tau(k)^{-1}e^{(i\nu - \rho)(H(xk))}dk .$$

Remark. Our notation for the Eisenstein integral differs slightly from Harish-Chandra's Eisenstein integral only in that we shall have no need to specify the parabolic subgroup $P = MAN$ which defines the integral.

Part of the Plancherel formula of Harish-Chandra [6], [7] tells us that for $f \in \mathscr{C}(G, \tau)$ there is a function $f_A \in \mathscr{C}(G, \tau)$ where

$$f_A(x) = \sum_{\omega \in \hat{M}} \int_{\mathfrak{A}^*} E(\psi_f(\omega : \nu) : \nu : x)\mu(\omega : \nu)d\nu$$

and $F = f - f_A \in \mathscr{C}(G, \tau)$ with

$$\int_N F(gn)dn \equiv 0$$

where N is the unipotent radical of $P = MAN$. Moreover, the function $\mu : \hat{M} \times \mathfrak{A}_C^* \to C$ satisfies the following conditions:

1) $\nu \to \mu(\omega : \nu)$ is meromorphic on $\mathfrak{A}_C^*(\omega \in \hat{M})$;
2) $\nu \to \mu(\omega : \nu)$ is analytic and ≥ 0 on $\mathfrak{A}^*(\omega \in \hat{M})$; and,
3) For $s \in W$ $\mu(s\omega : s\nu) = \mu(\omega : \nu)$.

In the following we will say that a function $F \in \mathscr{C}(G, \tau)$ is a quasi-cusp form if

$$\int_N F(gn)dn \equiv 0 .$$

We denote the space of quasi-cusp forms by $\mathscr{C}_q(G, \tau)$.

The main result of this paper (Theorem 3.1) gives a weak analog of the classical Paley-Wiener theorem in characterizing the support of a function $f \in C_c^\infty(G, \tau)$ in terms of growth conditions on the "Fourier-Laplace transform" $\psi_f(\omega : \nu)$.

We first state some results concerning some estimates which we shall need in the proof of the Paley-Wiener theorem.

In Section 3 we prove our result which contains a rather ambiguous residue function which we treat somewhat further in Section 4. In Section 5 we apply our results to the study of some partial differential operators on G.

§ 2. Some estimates.

Let V be as in section one, let $A \in V^M$ and consider the Eisenstein integral $E(A : \nu : x)$. Let $\mathfrak{A}^+ = \{H \in \mathfrak{A} : \lambda(H) > 0$ for all $\lambda \in P\}$ and set $A^+ = \exp \mathfrak{A}^+$. Harish-Chandra in Warner [16] has given a useful expansion of $E(A : \nu : a)$ for $a \in A^+$ which we now describe.

For $a \in A^+$ and $s \in W$ there exist functions $c : W \times \mathfrak{A}_C^* \to \operatorname{End} V^M$ and $\Phi_s : A \times \mathfrak{A}_C^* \to \operatorname{End} V^M$ such that $E(A : \nu : a) = \sum_{s \in W} \Phi_s(a : \nu)(c(s : \nu)(A))$. Furthermore, we have that

$$\Phi_s(a : \nu) = \sum_{\mu \in L} \Gamma_\mu(is\nu - \rho) e^{(is\nu - \rho - \mu)(\log a)}$$

where for $\mu \in L$ $\nu \to \Gamma_\mu(is\nu - \rho)$ is a rational function with image in $\operatorname{End}(V^M)$. Here $\Gamma_0 = I$.

For $\lambda \in \mathfrak{A}^*$ there is an $H_\lambda \in \mathfrak{A}$ such that $\lambda(H) = B(H, H_\lambda)$ for all $H \in \mathfrak{A}$ where B is the Killing form of $\mathfrak{G}$. For $\nu \in \mathfrak{A}_c^*$ write $-i\nu = \xi + i\eta$ when $\xi, \eta \in \mathfrak{A}^*$. For $H_0 \in \mathfrak{A}$ set $T(H_0) = \{\nu \in \mathfrak{A}_c^* : H_\xi \in H_0 + \mathfrak{A}^+\}$. The Γ_μ's now satisfy the following

LEMMA 2.1 (Lemma 2.3 [13]). *Fix $H_0 \in \mathfrak{A}$ and $H_1 \in \mathfrak{A}^+$. Then there is a polynomial $p_{H_0}(\nu)$ and a polynomial $K(\nu) > 0$ depending on p_{H_0}, H_0 and H_1 such that*

$$\| p_{H_0}(\nu) \Gamma_\mu(i\nu - \rho) \| \leq K e^{\mu(H_1)} .$$

For the proof of this lemma we refer to [13]. We now need some estimates on the functions $c(s : \nu)$.

We say that for $a \in A^+$ $a \to \infty$ if $\|\log a\| = B(\log a, \log a)^{1/2} \to \infty$ and there is an $\varepsilon>0$ such that for all $\lambda \in P$ $\lambda(\log a) \geq \varepsilon \|\log a\|$. Then from Harish-Chandra [6], [7] we have for $A \in V^M$ and $\nu \in \mathfrak{A}^*$ that

$$\lim_{a\to\infty} (e^{\rho(\log a)}E(A:\nu:a) - \sum_{s\in W} c(s:\nu)(A)e^{is\nu(\log a)}) = 0 .$$

Again from Harish-Chandra [6], [7] we have that the map $\nu \to c(s:\nu) \in \mathrm{End}(V^M)$ is meromorphic and hence we see that if $\mathrm{Re}\, i\nu(\log a) > 0$ for all $a \in A^+$

$$\log_{a\to\infty} e^{(\rho - i\nu)(\log a)}E(A:\nu:a) = c(1:\nu)(A) .$$

Hence for $\mathrm{Re}\, i\nu(\log a) > 0$ and all $a \in A^+$ we obtain

$$c(1:\nu) = \int_{\overline{N}} A \circ \tau(K(\overline{n}))^{-1}e^{-(i\nu+\rho)(H(\overline{n}))}d\overline{n} .$$

More generally we obtain that if $\mathrm{Re}\, is\nu(\log a) > 0$ for all $a \in A^+$ and $s \in W$

$$\log_{a\to\infty} e^{(\rho - is\nu)(\log a)}E(A:\nu:a) = c(s:\nu)(A)$$

and in this case an elementary calculation yields

$$c(s:\nu)(A) = \tau(w)j_s^-(\nu) \circ A \circ j_s^+(\nu)\tau(w)^{-1} \qquad (w \in s)$$

where

$$j_s^+(\nu) = \int_{\overline{N}_1} e^{-(i\nu+\rho)H(\overline{n})}\tau(K(\overline{n}))^{-1}d\overline{n}$$

and

$$j_s^-(\nu) = \int_{\overline{N}_2} e^{(i\nu-\rho)H(\overline{n})}\tau(K(\overline{n}))d\overline{n}$$

with $\overline{N}_1 = \{\overline{n} \in \overline{N}: w\overline{n}w^{-1} \in \overline{N}\}$ and $\overline{N}_2 = \{\overline{n} \in \overline{N}: w\overline{n}w^{-1} \in N\}$.

We wish to apply estimates of the form found in Lemma 3.1 of [13]. To do so we first need a product formula for the functions $j_s^+(\nu)$ and $j_s^-(\nu)$ which may be attributed to Gindikin and Karpelevic [4] and Schiffmann [15]. A more general product formula has been obtained by Harish-Chandra [7].

Let $P_s^+ = \{\alpha \in P: s^{-1}\alpha>0\}$ and $P_s^- = \{\alpha \in P: s^{-1}\alpha < 0\}$. Then

$$\overline{\mathfrak{N}}_1 = \sum_{\alpha\in P_s^+} \mathfrak{G}_{-\alpha} \quad \text{and} \quad \overline{\mathfrak{N}}_2 = \sum_{\alpha\in P_s^-} \mathfrak{G}_{-\alpha}$$

and for $\alpha \in P$ where $\alpha/2 \in P$ let $\mathfrak{N}_\alpha = \mathfrak{G}_{-\alpha}+\mathfrak{G}_{-2\alpha}$. If $\alpha \in P_s^+$ set

$$j_\alpha^+(\nu) = \int_{\overline{N}_\alpha} e^{-(i\nu+\rho)(H(\overline{n}))}\tau(K(\overline{n}))^{-1}d\overline{n}$$

and if $\alpha \in P_s^-$ set

$$j_\alpha^-(\nu) = \int_{\overline{N}_\alpha} e^{(i\nu-\rho)H(\overline{n})}\tau(K(\overline{n}))d\overline{n} \ .$$

If $|P_s^+| = k$ and $|P_s^-| = \ell$ we may put an ordering on P_s^+ where $P_s^+ = \{\alpha_1, \cdots, \alpha_k\}$ on an ordering on P_s^- where $P_s^- = \{\lambda_1, \cdots, \lambda_\ell\}$ where $\alpha_i \leq \alpha_{i+1}$ and $\lambda_i \leq \lambda_{i+1}$ such that $j_s^+(\nu) = j_{\alpha_k}^+(\nu) \cdots j_{\alpha_1}^+(\nu)$ and $j_s^-(\nu) = j_{\lambda_\ell}^-(\nu) \cdots j_{\lambda_1}^-(\nu)$. The proof of this fact follows immediately from Gindikin-Karpelevic [4] or more precisely from the proof of their main theorem. From Lemma 3.2 of [13] we have the following lemma

LEMMA 2.2. *Given $\delta > 0$ there is an $R > 0$ and an integer $N > 0$ such that if $|\langle\nu, \alpha\rangle| > R$ and $|\arg\langle\nu, \alpha\rangle + \pi/2| \geq \delta$ for $\alpha \in P_s^+$ the matrix entries of $j_\alpha^+(\nu)^{-1}$ are bounded in absolute value by $|\langle\nu, \alpha\rangle|^N$. Hence there is an $R_1 > 0$ and an $N_1 > 0$ for which the matrix entries of $j_s^+(\nu)^{-1}$ are bounded in absolute value by $\pi_{\alpha\in P_s^+}|\langle\nu, \alpha\rangle|^{N_1}$ if $|\langle\nu, \alpha\rangle| > R$, and $|\arg\langle\nu, \alpha\rangle + \pi/2| \geq \delta$ for $\alpha \in P_s^+$. (Here $|\arg z| \leq \pi$.) Furthermore there is an $R' > 0$ and an integer $N' > 0$ for which the matrix entries of $j_s^-(\nu)^{-1}$ are bounded in absolute value by $\pi_{\alpha\in P_s^-}|\langle\nu, \alpha\rangle|^{N'}$ if $|\langle\nu, \alpha\rangle| > R'$ and $|\arg\langle\nu, \alpha\rangle - \pi/2| \geq \delta$ for $\alpha \in P_s^-$.*

Using the inner product on V^M we now compute the adjoint of $c(s:\nu)$ for $\nu \in \mathfrak{A}^*$. Fixing $w \in s$ as before and letting $B \in \operatorname{End} V^M$, we see that

$$c(s:\nu)^*(B) = (j_s^-(\nu))^*\tau(w)^{-1}B\cdot\tau(w)(j_s^+(\nu))^* \ .$$

Moreover, we see that $(j_s^-(\nu))^*$ is the limit of operators of the form

$$\int_{\overline{N}_2} e^{-(i\lambda+\rho)H(\overline{n})}\tau(K(\overline{n}))^{-1}d\overline{n}$$

where $\lambda \to \nu$ $(\nu \in \mathfrak{A}^*)$ and $(j_s^+(\nu))^*$ is the limit of operators of the form

$$\int_{\overline{N}_1} e^{(i\lambda-\rho)(H(\overline{n}))}\tau(K(\overline{n}))d\overline{n}$$

where $\lambda \to \nu$ $(\nu \in \mathfrak{A}^*)$.

We now compute the adjoint of $c(s:\nu)$ for $\nu \in \mathfrak{A}^*$. For $w \in s$ and $B \in V^M$ we see that

$$c(s:\nu)^*(B) = j_s^-(\nu)^* \circ \tau(w)^{-1} \circ B \circ \tau(w) \circ j_s^+(\nu)^* .$$

For $\lambda \in \mathfrak{A}_C^*$ let

$$J_s^-(\lambda) = \int_{\bar{N}_2} e^{-(i\lambda+\rho)(H(\bar{n}))}\tau(K(\bar{n}))^{-1}d\bar{n}$$

and

$$J_s^+(\lambda) = \int_{\bar{N}_1} e^{(i\lambda-\rho)(H(\bar{n}))}\tau(K(\bar{n}))d\bar{n}$$

and denote their meromorphic continuations by the same symbols. Then $(j_s^-(\nu))^* = J_s^-(\nu)$ and $(j_s^+(\nu))^* = J_s^+(\nu)$. Letting $\tilde{C}(s:\lambda)(B) = J_s^-(\lambda)\tau(w)^{-1}B\tau(w) \times J_s^+(\lambda)$ we see that the function $\lambda \to \tilde{C}(s:\lambda)$ is defined meromorphically and for $\nu \in \mathfrak{A}^*$ $\tilde{C}(s:v) = c(s:\nu)^*$. It is a trivial fact to see that $J_s^-(\nu) = j_{\lambda_1}^+(\nu) \cdots j_{\lambda_\ell}^+(\nu)$ and $J_s^+(\nu) = j_{\alpha_1}^-(\nu) \cdots j_{\alpha_k}^-(\nu)$ where the α_i and λ_j are as before.

We conclude this section with the following observation. Suppose f is a holomorphic function on C^n and suppose that f satisfies the following estimate. There are constants C and $A > 0$ and an integer $N > n$ for which

$$|f(\vec{z})| \le C(1 + \|\vec{z}\|)^{-N}e^{A\|\mathrm{Im}\,\vec{z}\|}$$

where $\|\vec{z}\| = (\langle\vec{z},\vec{z}\rangle)^{1/2}$ and for $\vec{z} = \vec{x} + i\vec{y}$ with $\vec{x},\vec{y} \in \boldsymbol{R}^n$ $\mathrm{Im}\,\vec{z} = \vec{y}$.

Suppose $m > 0$ is an integer and let $c_1, \cdots, c_n$, $\lambda \in \boldsymbol{C}$. We assume that $\{\vec{z}: c_1z_1 + \cdots + c_nz_n - \lambda = 0\} \cap \boldsymbol{R}^n = \emptyset$. Let $g(\vec{z}) = (\vec{c}\cdot\vec{z} - \lambda)^{-m} f(\vec{z})$ where $\vec{c} = (c_1, \cdots, c_n)$. Then the following formula holds.

$$\begin{aligned}
&\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, \cdots, x_n)dx_1 \cdots dx_n \\
&\quad = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1 + iy, x_1, \cdots, x_n)dx_1 \cdots dx_n \\
&\quad - 2\pi i \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathrm{Res}_z \Big(g(z, x_2, \cdots, x_n), \\
&\qquad\qquad \frac{\lambda - c_2x_2 - \cdots - c_nx_n}{c_1}\Big)dx_2 \cdots dx_n
\end{aligned}$$

The above observation is useful since the singularities of the function $\nu \to \Gamma_\mu(is\nu - \rho)$ $(\mu \in L)$ and $\nu \to c(s:\nu)^{-1}$ have their singularities on hyperplanes and are meromorphic with polynomial growth.

§ 3. A Paley-Wiener theorem

We now describe our analog of the classical Paley-Wiener theorem.

We suppose first that $f \in C_c^\infty(G, \tau)$ and $f(g) = 0$ for $\sigma(g) > A$ where if $g = k_1 a k_2$ with $k_1, k_2 \in K$ and $a \in A$ $\sigma(g) = (B(\log a, \log a))^{1/2}$ or we say $f \in C_A^\infty(G, \tau)$. Observe that the map $\nu \to \psi_f(\omega : \nu)$ is holomorphic and satisfies

(1) For $N > 0$ an integer there is a constant C_N such that

$$\|\psi_f(\omega : \nu)\| \leq C_N(1 + \|\nu\|)^{-N} e^{A\|\mathrm{Im}\,\nu\|} .$$

(2) For $s \in W$ we have

$$c(s : \nu)(\psi_f(\omega : \nu)) = c(1 : s\nu)(\psi_f(s\omega : s\nu)) .$$

We now derive a third condition which is satisfied by the function $\nu \to \psi_f(\omega : \nu)$ for $\omega \subset \tau_{|M}$. We have that

$$f_A(g) = \sum_{\omega \in \hat{M}} \int_{\mathfrak{A}^*} E(\psi_f(\omega : \nu) : \nu : g)\mu(\omega : \nu) d\nu .$$

Moreover, picking an $\eta \in \mathfrak{A}^*$ with $\|\eta\|$ small and with no $\nu \to \Gamma_\mu(is(\nu + i\eta) - \rho)$ $(\mu \in L)$ having a singularity for any $\nu \in \mathfrak{A}^*$ we have

$$f_A(g) = \sum_{\omega \in \hat{M}} \int_{\mathfrak{A}^*} E(\psi_f(\omega : \nu + i\eta) : \nu + i\eta : g)\mu(\omega : \nu + i\eta) d\nu$$

and by Lemma 2.1 we have for $a \in A^+$

$$f_A(a) = \sum_{s \in w} \sum_{\omega \in \hat{M}} \sum_{\mu \in L} \int_{\mathfrak{A}^* + i\eta} \Gamma_\mu(is\nu - \rho) c(s : \nu)(\psi_f(\omega : \nu))\mu(\omega : \nu) e^{(is\nu - \rho - \mu)(\log a)} d\nu$$

The Maass-Selberg relations of Harish-Chandra [6], [7] state that

$$\|c(s : \nu)(\psi_f(\omega : \nu))\|^2 = \|\tilde{C}(s : \nu)(\psi_f(\omega : \nu))\|^2 = \mu(\omega : \nu)^{-1} d_\omega \|\psi_f(\omega : \nu)\|^2$$

for $\nu \in \mathfrak{A}^*$. Hence we have $\mu(\omega : \nu)^{-1} d_\omega = c(s : \nu)\tilde{C}(s : \nu)_{|V^{M}(\omega)}$. Thus,

$$\mu(\omega : \nu) c(s : \nu)(\psi_f(\omega : \nu)) = d_\omega \tilde{C}(s : \nu)^{-1}(\psi_f(\omega : \nu)) .$$

For $H \in \mathfrak{A}$ and $s \in W$ consider the tube $T(s, H) = \{\nu \in \mathfrak{A}_C^* : -H_{\mathrm{Im}\, s\nu} \in \mathfrak{A}^+ + H\}$. Then the following lemma now follows from Lemmas 2.1 and 2.2

LEMMA 3.1. *Given $H_\eta \in \mathfrak{A}$ and $s \in W$ there are a finite number of hyperplanes $F_1, \cdots, F_r$ in $\mathfrak{A}_C^*$ which intersect $T(s, H_\eta)$ and for which the functions $\nu \to \Gamma_\mu(is\nu - \rho)$ $(\mu \in L)$ and $\nu \to \tilde{C}(s : \nu)^{-1}$ are analytic on $T(s, H)$ $\sim (F_1 \cup \cdots \cup F_r)$. Furthermore, there is a $C > 0$ such that $\{\nu : -\langle \mathrm{Im}\,\nu, \alpha \rangle > C$ for all $\alpha \in P\}$ $T(s, H)$ $F_i = \emptyset$ for all $1 \leq i \leq r$.*

Now setting for $s \in W$ and $a \in A^+$,

$$f_{A,s}(a) = \sum_{\mu\in\hat{M}} \sum_{\mu\in L} d_\omega \int_{\mathfrak{A}^*+i\eta} \Gamma_\mu(is\nu - \rho)\tilde{C}(s:\nu)^{-1}(\psi_f(\omega:\nu))d\nu \ .$$

Using our remarks at the end of Section 2, we see that $f_{A,s}(a) = \mathrm{Res}_s(f)(a) + f_{\epsilon,s}(a)$ where $\mathrm{Res}_s(f)(a)$ is a residue integral over the imaginary part of the hyperplanes $F_1, \cdots, F_r$ and

$$f_{\epsilon,s}(a) = \sum_{\omega\in\hat{M}} \sum_{\mu\in L} \int_{\mathrm{Im}\ \nu=\lambda} \Gamma_\mu(is\nu - \rho)\tilde{C}(s:\nu)^{-1}(\psi_f(\omega:\nu))e^{(is\nu-\rho-\mu)(\log a)}d\nu$$

with $-H_\lambda \in \mathfrak{A}^+$ and $\|\lambda\| > C$. By the standard method used in the classical Paley-Wiener theorem we see that $f_{\epsilon,s}(a) = 0$ if $\sigma(a) > A$. Letting $\mathrm{Res}(f) = \sum_{s\in w} \mathrm{Res}_s(f)$ and $f_\epsilon = \sum_{s\in w} f_{\epsilon,s}$ and using the Plancherel formula we now see that there is an $F \in \mathscr{C}_q(G, \tau)$ such that

(3) $f = f_\epsilon + \mathrm{Res}(f) + F$

and $\mathrm{Res}\, f(a) + F(a) = 0$ for $a \in A^+$ with $\sigma(a) > A$.

Now for $A > 0$ let $\mathscr{P}(A, \tau)$ be the space of all functions $F: \hat{M} \times \mathfrak{A}_C^* \to V$ such that $F(\omega:\nu) \equiv 0$ if $\omega \not\subset \tau_{|M}$ and F satisfies the following conditions.

I) $\nu_N(F) = \sup_{\omega,\nu}(1 + \|\nu\|)^N e^{-A|\mathrm{Im}\,\nu|}\|F(\omega:\nu)\| < \infty$

II) $c(s:\nu)(F(\omega:\nu)) = c(1:s\nu)(F(s\omega:s\nu))$

III) The function

$$f(g) = \sum_{\omega\in\hat{M}} \int_{\mathfrak{A}^*} E(F(\omega:\nu):\nu:g)\mu(\omega:\nu)d\nu$$

differs from a function in $C_c^\infty(G, \tau)$ by a function H in $\mathscr{C}_q(G, \tau)$. Moreover, for g regular $f(g) = \mathrm{Res}\, f(g) + f_\epsilon(g)$ with $f_\epsilon(g) = 0$ for $\nabla(g) > A$.

THEOREM 3.1. *A function $f \in C^\infty(G, \tau)$ is in $C_A^\infty(G, \tau) + \mathscr{C}_q(G, \tau)$ if and only if its Fourier-Laplace transform is in $\mathscr{P}(A, \tau)$.*

Proof. It is clear that if $f \in C_A^\infty(G, \tau) + \mathscr{C}_q(G, \tau)$ its Fourier-Laplace transform is in $\mathscr{P}(A, \tau)$.

Suppose $0 \neq F \in \mathscr{P}(A, \tau)$. By Theorem 3.1 of Arthur [1] we have that

$$f(g) = \sum_{\omega\in\hat{M}} \int_{\mathfrak{A}^*} E(F(\omega:\nu):\nu:g) \qquad \mu(\omega:\nu)d\nu \not\equiv 0 \ .$$

By Lemma 2.2 of [13] we have that $f \notin \mathscr{C}_q(G, \tau)$. By assumption there is an $H \in \mathscr{C}_q(G, \tau)$ for which $f\text{-}H \in C_c^\infty(G, \tau)$. However our arguments in obtaining 3) guarantee that $0 \neq f - H \in C_A^\infty(G, \tau)$. This completes our proof.

COROLLARY 1. *A function $f \in C_c^\infty(G, \tau)$ is in $C_A^\infty(G, \tau)$ if and only if for every integer $N > 0$ there is a $C_N > 0$ such that*

$$\|\psi_f(\omega : \nu)\| \le C_N(1 + \|\nu\|)^{-N} e^{A\|\mathrm{Im}\,\nu\|} .$$

COROLLARY 2. *Let $\mathscr{P}(\tau)$ be the union of all $\mathscr{P}(A, \tau)$. Then a function $f \in C^\infty(G, \tau)$ is in $C_c^\infty(G, \tau) + \mathscr{C}_q(G, \tau)$ if and only if its Four ier-Laplace transform is in $\mathscr{P}(\tau)$.*

§ 4. The function Res f

We inject here a few remarks concerning the function Res f where $f \in C_A^\infty(G, \tau)$. Although we have strong reason to believe that Res f extends to a function in $\mathscr{C}_q(G, \tau)$ and thus f_ε extends to a function in $C_A^\infty(G, \tau)$ we can only establish this for some special cases which we describe in this section. We first give a more detailed description of Res f.

Let P denote the set of positive restricted roots and let $\Delta = \{\alpha_1, \cdots, \alpha_\ell\}$ be the simple restricted roots in P. Let $\{\lambda_1, \cdots, \lambda_\ell\} = {}'\Delta$ be dual to Δ (i.e. $2(\langle\lambda_i, \alpha_j\rangle / \langle\alpha_j, \alpha_j\rangle) = \delta_{ij}$). For $F \subset \Delta$ let ${}^cF = \Delta \sim F$ and let ${}'F \subset {}'\Delta$ be dual to F and ${}^\circ F$ dual to ${}^cF$. Let $\mathfrak{A}(F)$ ($\mathfrak{A}({}^\circ F)$) be the linear span of $\{H_\alpha : \alpha \in F\}$ ($\{H_\alpha : \alpha \in {}^\circ F\}$) and set $A(F) = \exp \mathfrak{A}(F)$ ($A({}^\circ F) = \exp \mathfrak{A}({}^\circ F)$). Observe that if $H \in \mathfrak{A}$ $H = H_1 + H_2$ where $H_1 \in \mathfrak{A}(F)$ and $H_2 \in \mathfrak{A}({}^\circ F)$ and this decomposition is unique. Furthermore, if $H \in \mathfrak{A}^+$ $H = H_1 + H_2$ where $H_1 \in \mathfrak{A}(F)^+ = \{H \in \mathfrak{A}(F) : \alpha(H) > 0 \text{ for all } \alpha \in F\}$ and $H_2 = \sum c_\lambda H_\lambda$ where the sum is over ${}^\circ F$ and each $c_\lambda > 0$. (It is easy to see that the converse holds only when $F = \Delta$ or $F = \emptyset$). Now for $a \in A^+$ we set $a = a_1 a_2$ where $H = \log a$ and $a_i = \exp H_i$ as above.

Continuing our integration process described at the end of Section 2 and allowing F to vary we see that the function Res f is a finite sum of functions of the form

$$\tilde{\eta}_\nu(a) = \tilde{\eta}_\nu(a_1, a_2) = \sum_{\mu \in L} \eta_{\nu-\mu}(a_1) e^{(i\nu - \rho - \mu)(\log a_2)}$$

where $\eta_{\nu-\mu}(a_1) \in \mathrm{End}\,(V^M)$, $-H_{\mathrm{Im}\,\nu} \in \mathfrak{A}^+$, L is the semilattice described in Section 2, the series converges absolutely for $a \in A^+$ and $\tilde{\eta}_\nu(a) = 0$ for $\sigma(a_1) > \mathrm{A}$ as do all $\eta_{\nu-\mu}$'s.

The following lemma is an immediate consequence of this expansion.

LEMMA 4.1. *If* Res $f(a) = 0$ *for all* $a \in A^+$ *with* $\sigma(a) > C$ *then* Res $f = 0$.

THEOREM 4.1. *If G has split rank one* $\operatorname{Res} f$ *extends to a (quasi) cusp form. If G has only one conjugacy class of Cartan subgroup* $\operatorname{Res} f = 0$.

Proof. The case where G has split rank one has been treated in [13] and the case where G has only one conjugacy class of Cartan subgroup follows from Lemma 4.1.

COROLLARY. *Suppose G has split rank one or has only one conjugacy class of Cartan subgroup. Then if $f \in C_c^\infty(G, \tau)$ $f = f_\varepsilon$.*

§ 5. Applications to differential equations

Let $U(\mathfrak{G})$ be the complexified enveloping algebra of $\mathfrak{G}$ and let $U(\mathfrak{G})^{\mathfrak{K}}$ be the centralizer of $\mathfrak{K}$ in $U(\mathfrak{G})$. If $f \in C^\infty(G)$ and $X \in \mathfrak{G}$ set $Xf(g) = (d/dt)f(\exp -tXg)|_{t=0}$ and extend this action to all of $U(\mathfrak{G})$. Let $\mathscr{E}'(G)$ denote the distributions with compact support.

In [14] a sufficient condition for $D \in U(\mathfrak{G})^{\mathfrak{K}}$ to be injective as an operator $D: \mathscr{E}'(G) \to \mathscr{E}'(G)$ was established. In this section we prove the converse of this result. We first recall the definition of the principal series.

Let $\omega: M \to Gl(H)$ be an irreducible unitary representation of M and let $\nu \in \mathfrak{A}_C^*$. ω and ν define a representation $V_{\omega,\nu}$ of the group $MAN = B$ on H by setting $V_{\omega,\nu}(man) = e^{(i\nu+\rho)(\log a)}\omega(m)$ $(m \in M,\ a \in A,\ n \in N)$. Now let $H^{\omega,\nu}$ be the set of all measurable functions $f: G \to H$ such that:

1) $f(gp) = V_{\omega,\nu}(p)^{-1}f(g)$ $(g \in G,\ p \in B)$; and,

2) $\int_K \|f(k)\|^2\, dk = \|f\|^2 < \infty$.

Now $H^{\omega,\nu}$ becomes a Hilbert space with inner product

$$(u, v) = \int_K (u(k), v(k))dk$$

and left translation induces a representation $\pi_{\omega,\nu}$ of G on $H^{\omega,\nu}$ and we call the pairs $(\pi_{\omega,\nu}, H^{\omega,\nu})$ the principal series of G. Let $K^{\omega,\nu}$ denote the K-finite vectors of $H^{\omega,\nu}$. Observe that $\pi_{\omega,\nu}$ induces a representation of $U(\mathfrak{G})$ on $X^{\omega,\nu}$ and that as a K-module $X^{\omega,\nu}$ is isomorphic to the space $X(\omega) = \{u: K \to H: u$ is left K-finite and $u(km) = \omega(m)^{-1}u(k)$ for all $k \in K$, $m \in M\}$. We abuse notation and identify $X^{\omega,\nu}$ with $X(\omega)$.

We now restate Lemma 3.1 of [14]. (Injectivity criterion) Suppose

$D \in U(\mathfrak{G})^{\mathfrak{K}}$. Suppose for no $\omega \in \hat{M}$ is there a finite dimensional subspace $U \subseteq X(\omega)$ such that $\pi_{\omega,\nu}(D): U \to U$ and $\det \pi_{\omega,\nu}(D)|_U = 0$ for all ν. Then $D: \mathscr{E}'(G) \to \mathscr{E}'(G)$ is injective.

Observe that $\pi_{\omega,\nu}$ defines a linear map

$$\pi_{\omega,\nu}: C_c^\infty(G, \tau) \longrightarrow L(H^{\omega,\nu},\ V \otimes H^{\omega,\nu})$$

by setting

$$\pi_{\omega,\nu}(f)u = \int_G f(x)\pi_{\omega,\nu}(x)udx \qquad (f \in C_c^\infty(G, \tau),\ u \in H^{\omega,\nu}) .$$

If we set $\theta_{\omega,\nu}(f) = \sum_{i=1} (\pi_{\omega,\nu}(f)u_i, u_i)$ where $\{u_i : i \geqq 1\}$ is an orthonormal basis of $H_{\omega,\nu}$ we obtain by a simple calculation that $\theta_{\omega,-\nu}(\ell(x)^{-1}f) = E(\psi_f(\omega:\nu):\nu:x)$ where $\ell(x)$ $(r(x))$ denotes left (right) translation by x. (Although the Eisenstein integral may be obtained from a distribution on G our treatment here is useful in the study of differential equations.)

We may now select $u_1, \cdots, u_d$ an orthonormal set of vectors in $H^{\omega,-\nu}$ such that

$$\begin{aligned}\theta_{\omega,-\nu}(\ell(x)^{-1}Df) &= \theta_{\omega,-\nu}(r(x)Df) \\ &= \sum_{i=1}^{d} (\pi_{\omega,-\nu}(D)\pi_{\omega,-\nu}(r(x)f)u_i, u_i)\end{aligned}$$

where for $h \in C_c^\infty(G)$

$$(\pi_{\omega,-\nu}(h)u_i, u_i) = \int_G h(x)(\pi_{\omega,-\nu}(x)u_i, u_i)dx .$$

We now prove the converse of the injectivity criterion.

Suppose that $D \in U(\mathfrak{G})^{\mathfrak{K}}$ and for $\omega_0 \in \hat{M}$ we have a finite dimensional K-invariant subspace $U \subseteq X(\omega_0)$ such that $\pi_{\omega_0,\nu}(D): U \to U$ and $\det \pi_{\omega_0,\nu}(D)|_U = 0$ for all $\nu \in \mathfrak{A}_C^*$. Without loss of generality we may assume that $\pi_{\omega_0,\nu}(D) \equiv 0$ on U. Let τ be the representation of K on U and let $V = \operatorname{End} U$ and extend τ to a double representation of K on V.

Now let $F: \hat{M} \times \mathfrak{A}_C^* \to V^M$ be such that $F(\omega:\nu) = 0$ if $\omega \neq s\omega_0$ for some $s \in W$. Suppose also that F satisfies conditions I, II and III of Section 3. Set

$$f(x) = \sum_{\omega \in \hat{M}} \int_{\mathfrak{A}^*} E(F(\omega:\nu):\nu:x)\mu(\omega:\nu)dy .$$

There is an $H \in \mathscr{C}_q(G, \tau)$ such that $f + H \in C_c^\infty(G, \tau)$. Also a simple calculation yields

$$Df(x) = \sum_{\omega \in \hat{M}} \int_{\mathfrak{A}^*} E(\pi_{\omega,-\nu}(D) \circ F(\omega : \nu) : \nu : x) u(\omega : \nu) d\nu$$

and thus $Df = 0$ and if $G = f + H$ we see that $DG \in \mathscr{C}_q(G, \tau) \cap C_c^\infty(G, \tau)$ and by [14] $DG = 0$. Hence we have proved

THEOREM 5.1. *Suppose $D \in U(\mathfrak{G})^{\mathfrak{K}}$. $D: \mathscr{E}'(G) \to \mathscr{E}'(G)$ is injective if and only if for no $\omega \in \hat{M}$ is there a finite dimensional subspace $U \subset X(\omega)$ such that $\pi_{\omega,\nu}(D): U \to U$ and $\det \pi_{\omega,\nu}(D)|_U = 0$ for all $\nu \in \mathfrak{A}_C^*$.*

For $r > 0$ let $V_r(0) = \{g \in G : \sigma(g) \leq r\}$

THEOREM 5.2 (P-convexity). *Suppose $D \in U(\mathfrak{G})^{\mathfrak{K}}$ satisfies the injectivity criterion. Suppose $T \in \mathscr{E}'(G)$ and* $\operatorname{supp} DT \subseteq V_r(0)$. *Then* $\operatorname{supp} T \subseteq V_r(0)$.

Proof. By convoluting with functions in $C_c^\infty(G)$, we see that it suffices to prove this result for $T = f \in C_c^\infty(G)$. Furthermore, it suffices to assume that $f(x) = L(F(x))$ where $F \in C_c^\infty(G, \tau)$, $V = \operatorname{End} U$, U is a K-finite space of functions on K, $L \in V^*$ and τ is the double representation induced on V by left translation on U.

By hypothesis for all $N > 0$ there is a C_N such that

$$|\psi_{DF}(\omega : \nu)| \leq C_N(1 + \|\nu\|)^{-N} e^{r \|\operatorname{Im} \nu\|}$$

but as $\psi_{DF}(\omega : \nu) = \pi_{\omega,-\nu}(D)\psi_F(\omega : \nu)$ we have that $\psi_F(\omega : \nu)$ satisfies the same growth conditions. Thus, as $F \in C_c^\infty(G, \tau)$ we have $\operatorname{supp} F \subseteq V_r(0)$ and hence $\operatorname{supp} f \subseteq V_r(0)$.

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