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MAPPINGS OF NONPOSITIVELY CURVED MANIFOLDS

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1. Introduction.

In recent papers with S. S. Chern [3] and T. Ishihara [4], the author studied both the volume—and distance—decreasing properties of harmonic mappings thereby obtaining real analogues and generalizations of the classical Schwarz-Ahlfors lemma, as well as Liouville's theorem and the little Picard theorem. The domain M in the first case was the open ball with the hyperbolic metric of constant negative curvature, and the target was a negatively curved Riemannian manifold with sectional curvature bounded away from zero. In this paper, it is shown that M may be taken to be any complete Riemannian manifold of non-positive curvature.

THEOREM 1. *Let $f: M \to N$ be a harmonic K-quasiconformal mapping of Riemannian manifolds of dimensions m and n, respectively. If M is complete, and* (a) *the sectional curvatures of M are nonpositive and bounded below by a negative constant $-A$, and* (b) *the sectional curvatures of N are bounded above by the constant $-((m-1)/(k-1))kAK^4$, $k = \min(m, n)$, then f is distance-decreasing. If $m = n$ and* (b) *is replaced by the condition* (b′) *the sectional curvatures of N are bounded away from zero by $-AK^4$, then f is volume-decreasing.*

Thus, even in the 1-dimensional case, that is, even when M is a Riemann surface, the theorem is a generalization of Schwarz's lemma. P. J. Kiernan [8] assumed the ratio of distances attained its maximum on M in order to achieve this.

By assuming f is a mapping of bounded dilatation of order K (see [6]), a more general result may be obtained.

The concept of a K-quasiconformal mapping of equidimensional manifolds, $m = n > 2$, was introduced by Lavrentiev, Markusevic and Kreines

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in 1938, but it did not receive serious attention until the mid fifties. This notion was subsequently extended in [5] to include the cases $m \neq n$.

The proof of the theorem is inspired by the technique used so successfully to obtain the generalized Schwarz-Ahlfors lemma, as well as the real analogues and generalizations of Liouville's theorem and Picard's first theorem (see [5], § 5), viz., the manifold M is exhausted by convex open submanifolds defined in terms of the "distance from a point" function. This function is continuous and, in fact, convex since the sectional curvature of M is nonpositive.

2. Harmonic mappings and curvature.

We begin by reviewing the theory of harmonic mappings as found in [3]. Let ds_M^2 and ds_N^2 be the Riemannian metrics of M and N, respectively. Then, locally,

$$ds_M^2 = \omega_1^2 + \cdots + \omega_m^2\,, \qquad ds_N^2 = \omega_1^{*2} + \cdots + \omega_n^{*2}$$

where the ω_i and ω_a^* are linear differential forms in M and N, respectively. (In the sequel, the range of indices $i, j, k, \cdots = 1, \cdots, m$, and $a, b, c, \cdots = 1, \cdots, n$.) The structure equations in M are

$$d\omega_i = \sum_j \omega_j \wedge \omega_{ji}\,,$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,\ell} R_{ijk\ell}\omega_k \wedge \omega_\ell\,.$$

The Ricci tensor R_{ij} is defined by

$$R_{ij} = \sum_k R_{ikjk}$$

and the scalar curvature by

$$R = \sum_i R_{ii}\,.$$

Similar equations are valid in N, where we will denote the corresponding quantities in the same notation with asterisks.

Let $f: M \to N$ be a C^∞ mapping, and

$$f^*\omega_a^* = \sum_i A_i^a \omega_i$$

where f^* is the pull-back mapping, that is the dual of the tangent mapping f_*. If $e_1, \cdots, e_m$ and $f_1, \cdots, f_n$ are orthonormal bases of the tangent spaces $T_x(M)$ and $T_{f(x)}(N)$, respectively, then

$$(f_*)_x e_i = \sum_a A_i^a e_a^* .$$

It is evident that

$$\|f_*\|^2 = \sum_{a,i} (A_i^a)^2$$

is an upper bound for the ratio function of distances on M and N, respectively (see § 3 for the definition of the norm).

Later on we will drop f^* in such formulas when its presence is clear from the context.

The covariant differential of A_i^a is defined by

$$DA_i^a \equiv dA_i^a + \sum_j A_j^a \omega_{ji} + \sum_j A_i^b \omega_{ba}^* = \sum_j A_{ij}^a \omega_j \text{ (say)} \tag{2.1}$$

with

$$A_{ij}^a = A_{ji}^a . \tag{2.2}$$

The mapping f is called *harmonic* if

$$\sum_i A_{ii}^a = 0 . \tag{2.3}$$

Taking the exterior derivative of (2.1), and employing the structure equations in M and N, we obtain

$$\sum_j DA_{ij}^a \wedge \omega_j = -\frac{1}{2} \sum_{j,k,\ell} A_j^a R_{jik\ell} \omega_k \wedge \omega_\ell - \frac{1}{2} \sum_{b,c,d} A_i^b R_{bacd}^* \omega_c^* \wedge \omega_d^* \tag{2.4}$$

where

$$DA_{ij}^a \equiv dA_{ij}^a + \sum_b A_{ij}^b \omega_{ba}^* + \sum_k A_{kj}^a \omega_{ki} + \sum_k A_{ik}^a \omega_{kj} = \sum_k A_{ijk}^a \omega_k \text{ (say)} .$$

From (2.4)

$$A_{ijk}^a - A_{ikj}^a = -\sum_\ell A_\ell^a R_{\ell ikj} - \sum_{b,c,d} A_i^b A_k^c A_j^d R_{bacd}^* . \tag{2.5}$$

By (2.2) and (2.5), the laplacian

$$\varDelta A_i^a \equiv \sum_k A_{ikk}^a = \sum_k A_{kik}^a = \sum_k A_{kki}^a + \sum_j A_j^a R_{ji} - \sum_{b,c,d,k} R_{bacd}^* A_k^b A_k^c A_i^d$$

is easily calculated. For a harmonic mapping

$$\varDelta A_i^a = \sum_j A_j^a R_{ji} - \sum_{b,c,d,k} R_{bacd}^* A_k^b A_k^c A_i^d . \tag{2.6}$$

Put $u = \|f_*\|^2$. Then

$$du = \sum_j u_j \omega_j \tag{2.7}$$

where

$$u_j = 2 \sum_{a,i} A_i^a A_{ij}^a . \tag{2.8}$$

Taking the exterior derivative of (2.7), we get

$$\sum_k \left(du_k - \sum_i u_i \omega_{ki} \right) \wedge \omega_k = 0 .$$

We may therefore set

$$du_k - \sum_i u_i \omega_{ki} = \sum_j u_{kj} \omega_j$$

where $u_{jk} = u_{kj}$. Thus, from (2.8),

$$u_{kj} = 2 \sum_{a,i} A_{ik}^a A_{ij}^a + 2 \sum_{a,i} A_i^a A_{ikj}^a .$$

For a harmonic mapping, (2.5) yields the laplacian

$$\begin{aligned} \frac{1}{2} \Delta u \equiv \frac{1}{2} \sum_j u_{jj} = \sum_{a,i,j} (A_{ij}^a)^2 + \sum_{a,i,j} R_{ij} A_i^a A_j^a \\ - \sum_{\substack{a,b,c,d \\ i,j}} R^*_{abcd} A_i^a A_j^b A_i^c A_j^d . \end{aligned} \tag{2.9}$$

Let $A^a = (A_1^a, \cdots, A_m^a)$ and $A_i = (A_i^1, \cdots, A_i^n)$ be local vector fields in M and N, respectively. Then, locally, $\sum \|A^a\|^2 = \sum \|A_i\|^2 = \|f_*\|^2$. If M is pinched, that is, if there are constants C_1 and C_2 such that

$$C_1 \leq \text{sectional curvature of } M \leq C_2 ,$$

then it is easily checked that

$$(m-1) C_1 \|f_*\|^2 \leq \sum R_{ij} A_i^a A_j^a \leq (m-1) C_2 \|f_*\|^2 .$$

Let $\|A_i \wedge A_j\|$ denote the area of the parallelogram spanned by A_i and A_j at each point. Then,

$$\sum_{i<j} \|A_i \wedge A_j\|^2 = \| \wedge^2 f_* \|^2 .$$

The last term in formula (2.9) may be expressed as

$$\sum R^*_{abcd} A_i^a A_j^b A_i^c A_j^d = 2 \sum_{i<j} R^*(A_i, A_j) \|A_i \wedge A_j\|^2$$

where $R^*(A_i, A_j)$ denotes the sectional curvature of N along the section spanned by A_i and A_j at each point. Hence, if the sectional curvature of N is bounded above by a nonpositive constant $-B$, we obtain

$$-\sum R^*_{abcd} A^a_i A^b_j A^c_i A^d_j \geq 2B \,\| \wedge^2 f_* \|^2 .$$

3. *K*-quasiconformal mappings.

Let V_1 and V_2 be Euclidean vector spaces over the reals of dimensions m and n, respectively, and let $A: V_1 \to V_2$ be a linear mapping. Let $e_1, \cdots, e_m$ and $f_1, \cdots, f_n$ be orthonormal bases of V_1 and V_2, respectively. If $p \leqq \min(m, n)$, A may be extended to the linear mapping $\wedge^p A: \wedge^p V_1 \to \wedge^p V_2$ given by

$$\wedge^p A(e_{i_1} \wedge \cdots \wedge e_{i_p}) = Ae_{i_1} \wedge \cdots \wedge Ae_{i_p}$$

where $1 \leq i_1 < i_2 < \cdots < i_p \leq m$.

Denoting the dual space of V_1 by V_1^*, $\wedge^p A$ may be regarded as an element of $\wedge^p V_1^* \otimes \wedge^p V_2$, the space of $\wedge^p V_2$-valued p-forms. Set $Ae_i = \Sigma A^a_i f_a$, put $I \equiv (i_1, \cdots, i_p)$ with $1 \leq i_1 < i_2 < \cdots < i_p \leq m$, $J \equiv (a_1, \cdots, a_p)$ with $1 \leq a_1 < \cdots < a_p \leq n$, and let D^J_I denote $\det(A^{a_\alpha}_{i_\beta})$, where the i_β are the components of I and the a_α are the components of J. Moreover, let

$$e_I = e_{i_1} \wedge \cdots \wedge e_{i_p} , \qquad f_J = f_{a_1} \wedge \cdots \wedge f_{a_p} , \qquad \theta^I = \theta^{i_1} \wedge \cdots \wedge \theta^{i_p}$$

where $\theta^1, \cdots, \theta^m$ is the dual basis of $e_1, \cdots, e_m$. Then,

$$\wedge^p A = \sum D^J_I \theta^I \otimes f_J ,$$

the sum being taken over all possible I and J.

The inner products on V_1 and V_2 induce an inner product $\langle\, ,\rangle$ on $\wedge^p V_1^* \otimes \wedge^p V_2$, and a norm $\|\wedge^p A\|$ is then defined by

$$\| \wedge^p A \|^2 = \sum_I \langle \wedge^p A(e_I), \wedge^p A(e_I) \rangle .$$

Set $G = {}^tAA$. Then,

$$\| \wedge^p A \|^2 = \operatorname{trace} \wedge^p G , \qquad p \leq \min(m, n) .$$

In the sequel, we assume rank $A = k$. Then, $k \leq \min(m, n)$ and rank $G = k$. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > \lambda_{k+1} = \cdots = \lambda_m = 0$ be the eigenvalues of G. If $p \leq k$, trace $\wedge^p G$ is the p-th elementary symmetric function of the positive eigenvalues of G, that is

$$\operatorname{trace} \wedge^p G = \sum_{i_1<\cdots<i_p}^{k} \lambda_{i_1} \cdots \lambda_{i_p} .$$

From Newton's inequalities, we therefore obtain

$$\text{(3.1)} \quad \left[\|\wedge^p A\|^2 \Big/ \binom{k}{p}\right]^{1/p} \geq \left[\|\wedge^q A\|^2 \Big/ \binom{k}{q}\right]^{1/q}, \qquad 1 \leq p < q \leq k .$$

Assume now that A has maximal rank k. By an orthogonal transformation A is transformed to a diagonal matrix with entries $\gamma_i = \lambda_i^{1/2}, i = 1, \cdots, k$. Let S^{k-1} be the unit sphere of dimension $k-1$ in V_1. Then $A(S^{k-1})$ is an ellipsoid of dimension $k-1$ in V_2. For a given constant $K \geq 1$, A is said to be *K-quasiconformal* if the ratio of the largest to the smallest axes of the ellipsoid $A(S^{k-1})$ is less than K. Since $\gamma_1 \geq \cdots \geq \gamma_k > 0$, A is K-quasiconformal if and only if $\gamma_1/\gamma_k \leq K$ or $\lambda_1/\lambda_k \leq K^2$. As $\|\wedge^p A\|^2$ is the p-th elementary symmetric function of $\lambda_1 \geq \cdots \geq \lambda_k > 0, p \leq k$, we then obtain

$$\left[\|\wedge^p A\|^2 \Big/ \binom{k}{p}\right]^{1/p} \leq K^2 \left[\|\wedge^q A\|^2 \Big/ \binom{k}{q}\right]^{1/q}, \qquad 1 \leq p < q \leq k$$

if A is K-quasiconformal.

Let $f: M \to N$ be a C^∞ mapping. Then, the norm $\|\wedge^p f_*\|$ may be regarded as the "ratio function of intermediate volume elements" of M and N. In particular, $\|\wedge^k f_*\|$ is the ratio of volume elements when $k = m = n$, where $k = \operatorname{rank} f$. If $\operatorname{rank} f_* = k$ everywhere, then

$$\text{(3.2)} \quad \left[\|\wedge^p f_*\|^2 \Big/ \binom{k}{p}\right]^{1/p} \geq \left[\|\wedge^q f_*\|^2 \Big/ \binom{k}{q}\right]^{1/q}, \qquad 1 \leq p < q \leq k .$$

Let f be a C^∞ mapping of maximal rank and $K \geq 1$. Then, f is *K-quasiconformal* if at each $x \in M$, $(f_*)_x$ is a K-quasiconformal linear mapping of $T_x(M)$ into $T_{f(x)}(N)$.

LEMMA 3.1. *If f is K-quasiconformal, then*

$$\left[\|\wedge^p f_*\|^2 \Big/ \binom{k}{p}\right]^{1/p} \leq K^2 \left[\|\wedge^q f_*\|^2 \Big/ \binom{k}{q}\right]^{1/q}, \qquad 1 \leq p < q \leq k .$$

4. Proof of Theorem 1.

Let $d\tilde{s}^2_M$ be a Riemannian metric on M conformally related to ds^2_M. Then, there is a function $p > 0$ on M such that $d\tilde{s}^2_M = p^2 ds^2_M$. Let $\tilde{u} = \Sigma(\tilde{A}^a_i)^2 = p^{-2}\Sigma(A^a_i)^2$, and let $\tilde{\Delta}$ be the laplacian associated with $d\tilde{s}^2_M$. Then

$$\begin{aligned}\frac{1}{2}\tilde{\Delta}\tilde{u} &= \sum (\tilde{A}^a_{ij})^2 + \sum \tilde{A}^a_i \tilde{A}^a_{ijj} \\ &= \sum (\tilde{A}^a_{ij})^2 + \sum \tilde{R}_{ij}\tilde{A}^a_i\tilde{A}^a_j - \sum R^*_{abcd}\tilde{A}^a_i\tilde{A}^b_j\tilde{A}^c_i\tilde{A}^d_j \\ &\quad + p^{-4}\sum A^a_i[A^a_{jji} - 2A^a_{jj}p_i + (m-2)A^a_{ji}p_j \\ &\qquad + (m-2)A^a_j(p_{ji} - 2p_jp_i)]\end{aligned}$$

where p_i is given by $d\log p = \sum p_i\omega_i$, and $p_{ij} = p_{ji}$ is defined by

$$\sum p_{ij}\omega_j = dp_i - \sum p_j\omega_{ij} \ . \tag{4.1}$$

If f is harmonic with respect to (ds^2_M, ds^2_N), then

$$\begin{aligned}\frac{1}{2}\tilde{\Delta}\tilde{u} &= \sum (\tilde{A}^a_{ij})^2 + \sum \tilde{R}_{ij}\tilde{A}^a_i\tilde{A}^a_j - \sum R^*_{abcd}\tilde{A}^a_i\tilde{A}^b_j\tilde{A}^c_i\tilde{A}^d_j \\ &\quad + (m-2)p^{-4}[\sum A^a_iA^a_{ij}p_j + \sum A^a_iA^a_j(p_{ij} - 2p_ip_j)] \ .\end{aligned}$$

Let $\tilde{u}$ attain its maximum at x. Then at x,

$$d\tilde{u} = 2p^{-2}\sum[\sum A^a_iA^a_{ij} - p_j\sum (A^a_i)^2]\omega_j = 0 \ ,$$

so

$$\sum A^a_iA^a_{ij} = p_j\sum (A^a_i)^2 \ ,$$

and

$$\sum A^a_iA^a_{ij}p_j + \sum A^a_iA^a_j(p_{ij} - 2p_ip_j) = \sum A^a_iA^a_j[p_{ij} + \delta_{ij}\sum (p_k)^2 - 2p_ip_j]$$

at x.

LEMMA 4.1. *Let $f: M \to N$ be harmonic with respect to (ds^2_M, ds^2_N), and let $\tilde{u}$ attain its maximum at $x \in M$. If the symmetric matrix function*

$$X_{ij} = p_{ij} + \delta_{ij}\sum (p_k)^2 - 2p_ip_j$$

is positive semi-definite everywhere on M, then

$$-\sum R^*_{abcd}\tilde{A}^a_i\tilde{A}^b_j\tilde{A}^c_i\tilde{A}^d_j \le -\sum \tilde{R}_{ij}\tilde{A}^a_i\tilde{A}^a_j$$

at x.

Assume now that M is simply connected. Let y be a point of M and denote by $d(x, y)$ the distance-from-y function. Then

$$t(x) = (d(x, y))^2 , \qquad x \in M$$

is C^∞ and convex on M (see [2]). The function

$$\tau(x) = d(x, y)$$

is also convex, but it is only continuous on M. It is, however, C^∞ in $M - \{y\}$. The convex open submanifolds

$$M_\rho = \{x \in M \mid t(x) < \rho\}$$

of M exhaust M, that is $M = \bigcup_{\rho<\infty} M_\rho$.

The nonnegative function

$$v_\rho = \log \frac{\rho}{\rho - t}$$

is a C^∞ convex function on M_ρ, that is its hessian

$$(v_\rho)_{ij} = \frac{1}{(\rho - t)^2} t_i t_j + \frac{1}{\rho - t} t_{ij} ,$$

where t_i is given by $dt = \Sigma t_i \omega_i$ and t_{ij} is its covariant derivative (see (4.1)), is positive semi-definite. Observe that $v_\rho \to \infty$ on the boundary ∂M_ρ of M_ρ, and for x fixed, $v_\rho(x) \to 0$ as $\rho \to \infty$.

Consider the metric $d\tilde{s}^2 = e^{2v_\rho} ds^2$ on M_ρ. Then,

$$\tilde{u} = e^{-2v_\rho} u = \left(\frac{\rho - t}{\rho}\right)^2 u$$

is nonnegative and continuous on the closure $\bar{M}_\rho$ of M_ρ and vanishes on ∂M_ρ. Since $\bar{M}_\rho$ is compact, $\tilde{u}$ has a maximum in M_ρ. We compute the matrix X_{ij} when $p = e^{v_\rho}$. It is easily seen that $p_i = (v_\rho)_i$ (the right hand side being given by $dv_\rho = \Sigma (v_\rho)_i \omega_i$), and $p_{ij} = (v_\rho)_{ij}$, so that

$$\begin{aligned} X_{ij} &= (v_\rho)_{ij} + \delta_{ij} \sum (v_\rho)_k^2 - 2(v_\rho)_i (v_\rho)_j \\ &= \frac{1}{\rho - t} t_{ij} + \frac{1}{(\rho - t)^2} [\delta_{ij} \sum (t_k)^2 - t_i t_j] . \end{aligned}$$

Since the function $t(x)$ is convex, the matrix X_{ij} is positive semi-definite, so from Lemma 4.1

$$-\sum R^*_{abcd}\tilde{A}_i^a\tilde{A}_j^b\tilde{A}_i^c\tilde{A}_j^d \leq -\sum \tilde{R}_{ij}\tilde{A}_i^a\tilde{A}_j^a \ .$$

The relation between $\tilde{R}_{ij}$ and R_{ij} is given by

$$e^{2v_\rho}\tilde{R}_{ij} = R_{ij} - \frac{m-2}{\rho - t}t_{ij} - \frac{1}{\rho - t}\Big(\Delta t + \frac{m-1}{\rho - t}\langle dt, dt\rangle\Big)\delta_{ij} \ ,$$

from which

$$\begin{aligned} \sum \tilde{R}_{ij}\tilde{A}_i^a\tilde{A}_j^a = \Big(\frac{\rho - t}{\rho}\Big)^2 &\sum R_{ij}\tilde{A}_i^a\tilde{A}_j^a \\ &- \frac{\rho - t}{\rho^2}(m-2)\sum t_{ij}\tilde{A}_i^a\tilde{A}_j^a - \frac{\rho - t}{\rho^2}\Delta t\,\|f_*\|_\rho^2 \\ &- \frac{m-1}{\rho^2}\langle dt, dt\rangle\|f_*\|_\rho^2 \ . \end{aligned} \tag{4.2}$$

To see this, let $\{\tilde{\omega}_i\}$ be an orthonormal coframe such that $\tilde{\omega}_i = p\omega_i$. Then,

$$\begin{aligned} d\tilde{\omega}_i &= dp \wedge \omega_i + p\,d\omega_i \\ &= dp \wedge \omega_i + \sum p\omega_j \wedge \omega_{ji} \\ &= \frac{1}{p}dp \wedge \tilde{\omega}_i + \sum \tilde{\omega}_j \wedge \omega_{ji} \\ &= d\log p \wedge \tilde{\omega}_i + \sum \tilde{\omega}_j \wedge \omega_{ji} \ . \end{aligned}$$

Now, we know

$$d\log p = dv_\rho = \sum (v_\rho)_j\omega_j \ .$$

Hence,

$$\begin{aligned} d\tilde{\omega}_i &= \sum (v_\rho)_j\omega_j \wedge \tilde{\omega}_i + \sum \tilde{\omega}_j \wedge \omega_{ji} \\ &= \sum \tilde{\omega}_j \wedge (\omega_{ji} + (v_\rho)_j\omega_i) \\ &= \sum \tilde{\omega}_j \wedge \{\omega_{ji} + ((v_\rho)_j\omega_i - (v_\rho)_i\omega_j)\} \ . \end{aligned}$$

Thus, we obtain

$$\tilde{\omega}_{ji} = \omega_{ji} + (v_\rho)_j\omega_i - (v_\rho)_i\omega_j \ .$$

Substituting this in $\frac{1}{2}\sum \tilde{R}_{ijk\ell}\tilde{\omega}_k \wedge \tilde{\omega}_\ell = \sum \tilde{\omega}_{ik} \wedge \tilde{\omega}_{kj} - d\tilde{\omega}_{ij}$, gives

$$\begin{aligned} &\frac{1}{2}\sum \tilde{R}_{ijk\ell}\tilde{\omega}_k \wedge \tilde{\omega}_\ell \\ &\qquad = \sum (\omega_{ik} + (v_\rho)_i\omega_k - (v_\rho)_k\omega_i) \wedge (\omega_{kj} + (v_\rho)_k\omega_j - (v_\rho)_j\omega_k) \end{aligned}$$

$$
\begin{aligned}
&- (d\omega_{ij} + d(v_\rho)_i \wedge \omega_j + (v_\rho)_i d\omega_j - d(v_\rho)_j \wedge \omega_i - (v_\rho)_j d\omega_i) \\
&= \sum \omega_{ik} \wedge \omega_{kj} - d\omega_{ij} \\
&\quad - \sum (d(v_\rho)_i + (v_\rho)_k \omega_{ki}) \wedge \omega_j + \sum (d(v_\rho)_j + (v_\rho)_k \omega_{kj}) \wedge \omega_i \\
&\quad + \sum (v_\rho)_i (v_\rho)_k \omega_k \wedge \omega_j + \sum (v_\rho)_k (v_\rho)_j \omega_i \wedge \omega_k - \sum (v_\rho)_k^2 \omega_i \wedge \omega_j \\
&= \frac{1}{2} \sum \Big[R_{ijk\ell} - (v_\rho)_{ik}\delta_{j\ell} + (v_\rho)_{i\ell}\delta_{jk} + (v_\rho)_{jk}\delta_{i\ell} - (v_\rho)_{j\ell}\delta_{ik} \\
&\quad + (v_\rho)_i (v_\rho)_k \delta_{j\ell} - (v_\rho)_i (v_\rho)_\ell \delta_{jk} - (v_\rho)_j (v_\rho)_k \delta_{i\ell} \\
&\quad + (v_\rho)_j (v_\rho)_\ell \delta_{ik} - \sum_h (v_\rho)_h^2 (\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}) \Big] \omega_k \wedge \omega_\ell .
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
p^2 \tilde{R}_{ijk\ell} = R_{ijk\ell} &- (v_\rho)_{ik}(v_\rho)_{j\ell} + (v_\rho)_{i\ell}\delta_{jk} + (v_\rho)_{jk}\delta_{i\ell} \\
&- (v_\rho)_{j\ell}\delta_{ik} + (v_\rho)_i (v_\rho)_k \delta_{j\ell} - (v_\rho)_i (v_\rho)_\ell \delta_{jk} \\
&- (v_\rho)_j (v_\rho)_k \delta_{i\ell} + (v_\rho)_j (v_\rho)_\ell \delta_{ik} - \sum_h (v_\rho)_h^2 (\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}) .
\end{aligned}
$$

LEMMA 4.2. *For each ρ, there exists a positive constant $\varepsilon(\rho)$ such that the inequality*

$$
-\sum \tilde{R}_{ij} \tilde{A}_i^\alpha \tilde{A}_j^\alpha \leq [(m-1)A + \varepsilon(\rho)]\tilde{u}
$$

holds on M_ρ. Moreover, $\varepsilon(\rho) \to 0$ as $\rho \to \infty$.

Proof. Since $\langle dt, dt\rangle = 4\tau^2\langle d\tau, d\tau\rangle = 4t$, the last term on the right hand side of (4.2) tends to zero as $\rho \to \infty$. The lemma will follow if we can show that $\Delta\tau$ is bounded as $\tau \to \infty$. For, $\Delta t = 2\tau\Delta\tau + 2\langle d\tau, d\tau\rangle = 2(t^{\frac{1}{2}}\Delta\tau + 1)$. Under the circumstances $(\rho - t)\Delta t/\rho^2$ will tend uniformly to zero. Moreover, since the matrix t_{ij} is positive semi-definite, the quadratic form $\sum t_{ij}\tilde{A}_i^\alpha \tilde{A}_j^\alpha \leq \lambda_0((\rho - t)/\rho)^2\tilde{u}$, where λ_0 is the least upper bound of the largest eigenvalues of t_{ij} on M_ρ.

To see that $\Delta\tau$ is bounded as $\tau \to \infty$, observe that the level hypersurfaces of τ are spheres S with y as center. The hessian $D^2\tau$ of τ can be identified with the second fundamental form of those spheres, extended to be 0 in the normal direction. For, the value of $D^2\tau$ on a vector v is the second derivative of τ along the geodesic generated by v. Along a geodesic from y, τ is linear, so the second derivative is 0. This shows that $D^2\tau$ is 0 on the normals to the spheres. One way of viewing the second fundamental form is as follows. On the tangent space $T_x(S)$ we define a function $\delta(v)$ to be the signed distance from $\exp_x(v)$ to S. Then, the second fundamental form h is the hessian of δ at 0, where $T_0(T_x(S))$

is identified with $T_x(S)$ in the usual way, that is,

$$h(w, w) = \frac{d^2}{dt^2}(0)(\delta(tw)) , \qquad w \in T_x(S) .$$

But, for $S = \tau^{-1}(r)$, the signed distance to S is simply $\tau - r$, so $\frac{d^2}{dt^2}(0)$ $\cdot(\delta(tw))$ is just the second derivative of $\tau - r$ along the geodesic $t \to \exp_x(tw)$. Since r is constant, this is just $D^2\tau(w, w)$. It follows that $\Delta\tau = \operatorname{trace} D^2\tau = \operatorname{trace} h = (m - 1)\cdot$ mean relative curvature of S.

If the curvature $K \geq a^2$ [in fact, if the Ricci curvature $\geq (m-1)a^2$], then from [1; pp. 247–255]

$$\Delta\tau \leq (m - 1)a\frac{\cos a\tau}{\sin a\tau} .$$

If we put $a^2 = -\alpha^2$, then

$$\Delta\tau \leq (m - 1)\alpha \coth \alpha\tau .$$

It is now clear that $\Delta\tau$ is bounded as $\tau \to \infty$.

To complete the proof of the theorem, Lemmas 4.1 and 4.2 imply

$$-\sum R^*_{abcd}\tilde{A}^a_i\tilde{A}^b_j\tilde{A}^c_i\tilde{A}^d_j \leq [(m - 1)A + \varepsilon]\tilde{u}$$

at x where $\varepsilon \to 0$ as $\rho \to \infty$. Let $\| \wedge^p f_* \|_\rho$ denote the norm of $\wedge^p f_*$ with respect to $d\tilde{s}^2$. Then, if the sectional curvature of N is bounded above by a negative constant $-B$,

$$2B \| \wedge^2 f_* \|^2_\rho \leq [(m - 1)A + \varepsilon]\cdot \|f_*\|^2_\rho$$

at x, where $\varepsilon \to 0$ as $\rho \to \infty$. It follows from Lemma 3.1 that

$$\|f_*\|^2_\rho \leq \frac{kK^4}{B(k - 1)}[(m - 1)A + \varepsilon]$$

everywhere on M_ρ. Since this inequality holds for every ρ and $\lim_{\rho\to\infty} \|f_*\|^2_\rho = \|f_*\|^2$, we conclude that

$$\|f_*\|^2 \leq k\left(\frac{m - 1}{k - 1}\right)\frac{A}{B}K^4 .$$

The first part of the theorem follows by taking $B = ((m - 1)/(k - 1))$ $\cdot kAK^4$. Applying the inequality (3.2) we conclude that

$$\| \wedge^p f_* \|^{2/p} \leq \frac{m-1}{k-1} \binom{k}{p}^{1/p} \frac{A}{B} K^4 .$$

Putting $k = m = n$ and $B = AK^4$, the volume-decreasing statement is obtained. The assumption of simple connectedness is clearly not essential.

By taking $M = E^m$ with the standard flat metric the above proof quickly yields the following real version and generalization of Liouville's theorem as well as Picard's first theorem originally obtained in [4]. However, the definition of K-quasiconformality must be slightly revised to allow for the possibility that f_* vanish at each point x of M.

THEOREM 2. *Let N be an n-dimensional Riemannian manifold with negative sectional curvature bounded away from zero. Then, if $f: E^m \to N$ is a harmonic quasiconformal mapping, it is a constant.*

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