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ON THE SEMISIMPLICITY OF THE ALGEBRA ASSOCIATED TO A POLARIZED ALGEBRAIC VARIETY

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§ 1. Introduction.

Let V be a compact nonsingular algebraic variety of dimension n with a Hodge structure ω and let $H^{i,i}(V,C)$ be the subgroup of $2i$-th cohomology group $H^{2i}(V,C)$ represented by harmonic (i,i)-forms on V with respect to ω.

We denote

$$\mathfrak{H}^{i,i}(V,Q) = H^{i,i}(V,C) \cap H^{2i}(V,Q) ,$$
$$\mathfrak{H}(V,Q) = \bigoplus_{i=0}^{n} \mathfrak{H}^{i,i}(V,Q) .$$

Then $\mathfrak{H}(V,Q)$ forms a commutative associative algebra over Q. We denote by L and Λ the linear operators acting on the cohomology group $H^*(V,C)$ as follows

$$L\phi = \omega\cdot\phi ,$$
$$\Lambda\phi = i(\omega)\cdot\phi , \qquad (\phi \in H^*(V,C))$$

where $i(\omega)$ means the inner product of ω with ϕ.

Recently H. Morikawa introduced a symmetric binary composition $\circ$ in $\mathfrak{H}^{1,1}(V,Q)$ defined by the equation

$$\phi\circ\psi = \tfrac{1}{2}\{\Lambda\phi\cdot\psi + \Lambda\psi\cdot\phi - \Lambda(\phi\cdot\psi)\} . \qquad (\phi, \psi \in \mathfrak{H}^{1,1}(V,Q))$$

He remarked that if V is a polarized abelian variety, the Q-(not necessarily associative) algebra $\mathfrak{H}^{1,1}(V,Q)$ is canonically isomorphic to the Jordan algebra of symmetric elements in $\mathrm{End}_Q(V)$ with respect to the involution induced by the polarization (Cf. [4]).

In this paper, using formulae in Kähler geometry, we shall prove the following theorems that show the semisimplicity of the algebra $(\mathfrak{H}^{1,1}(V,Q), \circ)$.

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THEOREM 1. *Let V be a compact nonsingular algebraic variety of dimension n with a Hodge structure ω. Let $\circ$ be a binary composition in $\mathfrak{H}^{1,1}(V, \boldsymbol{Q})$ defined by*

$$\phi \circ \psi = \tfrac{1}{2}\{\Lambda\phi\cdot\psi + \Lambda\psi\cdot\phi - \Lambda(\phi\cdot\psi)\}\ , \tag{1.1}$$

and let $(\ ,\ )$ be a symmetric bilinear form given by

$$(\phi, \psi) = \Lambda(\phi \circ \psi)\ . \qquad (\phi, \psi \in \mathfrak{H}^{1,1}(V, \boldsymbol{Q})) \tag{1.2}$$

Then the algebra $(\mathfrak{H}^{1,1}(V, \boldsymbol{Q}), \circ)$ is commutative and has ω as its unity element. And the symmetric bilinear form $(\ ,\ )$ satisfies

$$(\phi \circ \psi, \tau) = (\phi, \psi \circ \tau)\ , \tag{1.3}$$

$$(\phi, \phi) > 0 \quad \textit{for } \phi \neq 0\ . \qquad (\phi, \psi, \tau \in \mathfrak{H}^{1,1}(V, \boldsymbol{Q})) \tag{1.4}$$

REMARK 1. A symmetric bilinear form for an arbitrary (not necessarily associative) algebra satisfying (1.3) is called a trace form.

DEFINITION 1. Let $\mathfrak{A}$ be an algebra. An ideal $\mathfrak{B}$ of $\mathfrak{A}$ is simple, by definition, if there is no ideal of $\mathfrak{A}$ contained in $\mathfrak{B}$ and different from (0) and $\mathfrak{B}$. An algebra $\mathfrak{A}$ is simple if the ideal $\mathfrak{A}$ is simple.

DEFINITION 2. For an algebra $\mathfrak{A}$ we call it semisimple if it is decomposed into a direct sum of simple ideals.

THEOREM 2. *The algebra $(\mathfrak{H}^{1,1}(V, \boldsymbol{Q}), \circ)$ is semisimple so that $\mathfrak{H}^{1,1}(V, \boldsymbol{Q})$ is uniquely expressible as a direct sum*

$$\mathfrak{H}^{1,1}(V, \boldsymbol{Q}) = \mathfrak{H}_1 + \cdots + \mathfrak{H}_k\ , \tag{1.5}$$

of simple ideals $\mathfrak{H}_i$.
Corresponding to this decomposition, the Hodge structure ω is decomposed

$$\omega = \omega_1 + \cdots + \omega_k\ , \tag{1.6}$$

with

$$\omega_i \circ \omega_j = 0 \qquad \textit{for } i \neq j\ ,$$
$$\omega_i \circ \omega_i = \omega_i\ .$$

Theorem 2 follows from the next general theorem (Cf, [3]).

THEOREM 3. *Let $(\mathfrak{A}, \circ)$ be an algebra of finite dimension satisfying*
(1) *there is a nondegenerate trace form $(\ ,\ )$ defined on $\mathfrak{A}$.*

(2) $\mathfrak{B}^2 \neq 0$ *for every ideal* $\mathfrak{B} \neq 0$ *of* $\mathfrak{A}$.
Then $\mathfrak{A}$ *is uniquely decomposed into a direct sum*

$$\mathfrak{A} = \mathfrak{A}_1 + \cdots + \mathfrak{A}_j ,$$

of simple ideals $\mathfrak{A}_i$.

But in our case the trace form is positive definite so the proof of Theorem 2 is easy as we shall see in § 3.

§ 2. Some formulae in Kähler geometry.

First of all, let us recall the fundamental formulae and theorems in Kähler geometry which will be used for the proofs of Theorem 1 and Theorem 2 (Cf, [1]).

We need following formulae between the operators L and Λ;

$$[L, \Lambda] = H = \sum_{i=0}^{2n} (i-n)P_i , \tag{2.1}$$

where P_i is the projection map on the i-th factor.

$$\begin{aligned} &[L, H] = -2L , \qquad [\Lambda, H] = 2\Lambda , \\ &\Lambda L^r - L^r \Lambda = \sum_{\substack{i,j \\ 0 \leqslant j \leqslant r-1}} (n-i) L^{r-1} P_{i-2j} . \end{aligned} \tag{2.2}$$

Denoting by $H^i(V, C)_0$ the i-th primitive cohomology group $\{\phi \in H^i(V, C) \mid \Lambda\phi = 0\}$, we have a criterion of primitivity;

$$H^i(V, C)_0 = \{\phi \in H^i(V, C) \mid \omega^{n-i+1}\phi = 0\} , \tag{2.3}$$

and Lefschetz decomposition theorem;

$$\begin{aligned} H^i(V, C) &= H^i(V, C)_0 + \cdots + L^r H^{i-2r}(V, C)_0 \\ r &\leq \left[\frac{i}{2}\right] \qquad \text{for } 0 \leq i \leq n , \end{aligned} \tag{2.4}$$

$$\begin{aligned} H^i(V, C) &= L^{i-n} H^{2n-i}(V, C)_0 + \cdots + L^{i-n+r} H^{2n-i-2r}(V, C)_0 \\ r &\leq \left[\frac{2n-i}{2}\right] \qquad \text{for } n < i \leq 2n . \end{aligned}$$

Putting

$$Q(\phi, \psi) = (-1)^{i(i+1)/2} \int_V \omega^{n-i} \cdot \phi \cdot \psi \qquad \text{for } \phi, \psi \text{ in } H^i(V, C)_0 ,$$

Q is symmetric bilinear form for i even and is an alternating bilinear form for i odd. For either case Q is nondegenerate. Moreover we have

$$Q(H_0^{i-r,r}, H_0^{s,i-s}) = 0 \qquad \text{for } r \neq s\,, \tag{2.5}$$

$$(\sqrt{-1})^i(-1)^{i+r}Q(H_0^{i-r,r}, H_0^{r,i-r}) > 0 \qquad \text{positive definite.} \tag{2.6}$$

LEMMA 1. *Using the notations above, we have*

$$L\Lambda\phi = \Lambda\phi\cdot\omega\,, \tag{2.7}$$

$$\Lambda L\phi = (n-2)\phi + \Lambda\phi\cdot\omega\,, \tag{2.8}$$

$$\Lambda\omega = n = \dim V\,. \qquad (\phi \in \mathfrak{H}^{1,1}(V, \boldsymbol{Q})) \tag{2.9}$$

Proof. From the formulae (2.1) and (2.2) between the operators L, Λ and H, it follows that

$$\begin{aligned} L\Lambda\phi &= \Lambda\phi\cdot L1 = \Lambda\phi\cdot\omega\,,\\ \Lambda L\phi &= (-H + L\Lambda)\phi = (n-2)\phi + \Lambda\phi\cdot\omega\,,\\ \Lambda\omega &= \Lambda L1 = (-H + L\Lambda)1 = -H1 = n\,. \end{aligned}$$

PROPOSITION 1.

$$\phi\circ\psi = \psi\circ\phi\,, \tag{2.10}$$

$$\phi\circ\omega = \omega\circ\phi = \phi\,. \qquad (\phi, \psi \in \mathfrak{H}^{1,1}(V, \boldsymbol{Q})) \tag{2.11}$$

Proof. From Lemma 1 and the definition (1.1) of the composition $\circ$, we have the commutativity (2.10) and

$$\begin{aligned} \phi\circ\omega &= \tfrac{1}{2}\{\Lambda\omega\cdot\phi + \Lambda\phi\cdot\omega - \Lambda(\phi\cdot\omega)\}\\ &= \tfrac{1}{2}\{n\phi + \Lambda\phi\cdot\omega - \Lambda L\phi\}\\ &= \phi\,. \end{aligned}$$

The equation (2.11) implies that the Hodge structure ω is the unity element of the algebra $(\mathfrak{H}^{1,1}(V, \boldsymbol{Q}), \circ)$.

We denote by $B_2(\ ,\ )$ and $B_3(\ ,\ ,\ )$ respectively a bilinear form and a trilinear form given by

$$\begin{aligned} B_2(\phi, \psi)\omega^n &= \phi\cdot\psi\cdot\omega^{n-2}\,,\\ B_3(\phi, \psi, \tau)\omega^n &= \phi\cdot\psi\cdot\tau\cdot\omega^{n-3}\,, \qquad (\phi, \psi, \tau \in \mathfrak{H}^{1,1}(V, \boldsymbol{Q})) \end{aligned}$$

Integrating both sides of the above first equation over V, we have

$$\int_V B_2(\phi, \psi)\omega^n = \int_V \phi\cdot\psi\cdot\omega^{n-2} ,$$

and

$$B_2(\phi, \psi) = \frac{1}{I(\omega)}\int_V \phi\cdot\psi\cdot\omega^{n-2} , \tag{2.12}$$

where

$$I(\omega) = \int_V \omega^n > 0 .$$

Similarly we have

$$B_3(\phi, \psi, \tau) = \frac{1}{I(\omega)}\int_V \phi\cdot\psi\cdot\tau\cdot\omega^{n-3} . \tag{2.13}$$

$B_2(,)$ and $B_3(, ,)$ are symmetric forms and by virture of (2.3), (2.5) and (2.6), we have

$$B_2(\omega, \omega) = 1 , \tag{2.14}$$

$$B_2(\phi, \omega) = B_2(\omega, \phi) = 0 \qquad \text{for primitive } \phi \text{ in } \mathfrak{H}^{1,1}(V, \boldsymbol{Q}) , \tag{2.15}$$

$$B_2(\phi, \phi) < 0 \qquad \text{for nonzero primitive } \phi \text{ in } \mathfrak{H}^{1,1}(V, \boldsymbol{Q}) , \tag{2.16}$$

These formulae will give the positive definiteness of the bilinear form (,) defined in Theorem 1.

LEMMA 2. *Let ϕ, ψ, τ be in $\mathfrak{H}^{1,1}(V, \boldsymbol{Q})$. Then we have*

$$\Lambda L^n 1 = nL^{n-1}1 = n\omega^{n-1} , \tag{2.17}$$

$$\Lambda\phi = nB_2(\phi, \omega) , \tag{2.18}$$

$$B_2(\Lambda(\phi\cdot\psi), \omega) = 2(n-1)B_2(\phi, \psi) , \tag{2.19}$$

$$B_2(\Lambda(\phi\cdot\psi), \tau) = nB_2(\phi, \psi)B_2(\tau, \omega) + (n-2)B_3(\phi, \psi, \tau) , \tag{2.20}$$

$$\Lambda^2(\phi\cdot\psi) = 2n(n-1)B_2(\phi, \psi) . \tag{2.21}$$

Proof. By the formulae (2.2), we have

$$\Lambda L^n 1 = L^n\Lambda 1 + \sum_{r=0}^{n-1}(n-2r)L^{n-1}1 = nL^{n-1}1 = n\omega^{n-1} .$$

Since

$$\Lambda(\phi\cdot\omega^{n-1}) = \Lambda(B_2(\phi,\omega)\omega^n) = B_2(\phi,\omega)\Lambda L^n 1 = nB_2(\phi,\omega)\omega^{n-1}\ ,$$

and

$$\begin{aligned}\Lambda(\phi\cdot\omega^{n-1}) &= \Lambda L^{n-1}\phi = L^{n-1}\Lambda\phi + \sum_{r=0}^{n-2}(n-2-2r)L^{n-2}\phi = \Lambda\phi L^{n-1}1\\ &= \Lambda\phi\cdot\omega^{n-1}\ ,\end{aligned}$$

comparing the coefficients of ω^{n-1} in $nB_2(\phi,\omega)\omega^{n-1}$ and $\Lambda\phi\cdot\omega^{n-1}$, we have (2.18).
Comparing the coefficients of ω^n of the following equations;

$$\begin{aligned}B_2(\Lambda(\phi\cdot\psi),\omega)\omega^n &= \Lambda(\phi\cdot\psi)\omega^{n-1} = L^{n-1}\Lambda(\phi\cdot\psi)\\ &= \Lambda L^{n-1}\phi\psi - \sum_{r=0}^{n-2}(n-4-2r)L^{n-2}\phi\psi\\ &= 2(n-1)B_2(\phi,\psi)\omega^n\ ,\end{aligned}$$

and

$$\begin{aligned}B_2(\Lambda(\phi\cdot\psi),\tau)\omega^n &= \Lambda(\phi\cdot\psi)\cdot\tau\omega^{n-2} = L^{n-2}\Lambda(\phi\cdot\psi)\cdot\tau\\ &= \left\{\Lambda L^{n-2}\phi\psi - \sum_{r=0}^{n-3}(n-4-2r)L^{n-3}\phi\psi\right\}\cdot\tau\\ &= nB_2(\phi,\psi)\omega^{n-1}\tau + (n-2)\omega^{n-3}\phi\psi\tau\\ &= \{nB_2(\phi,\psi)B_2(\tau,\omega) + (n-2)B_3(\phi,\psi,\tau)\}\omega^n\ ,\end{aligned}$$

we have (2.19) and (2.20).
By (2.18) and (2.19), we have

$$\Lambda^2(\phi\cdot\psi) = nB_2(\Lambda(\phi\cdot\psi),\omega) = 2n(n-1)B_2(\phi,\psi)$$

and the proof of Lemma 2 is completed.

§ 3. The proofs of Theorem 1 and Theorem 2.

By Proposition 1, the former part of Theorem 1 that the algebra $\mathfrak{H}^{1,1}(V,Q)$ is commutative and ω is the unity element is proved. Hence we prove that the symmetric bilinear form (,) is a trace form (1.3) and is positive definite (1.4).

If at least one of ϕ, ψ and τ is ω, since ω is the unity element, (1.3) holds. So considering the Lefschetz decomposition, we may assume that they are all primitive.
Then

$$(\phi\circ\psi)\circ\tau = \tfrac{1}{4}\{-\Lambda^2(\phi\cdot\psi)\cdot\tau + \Lambda(\Lambda(\phi\cdot\psi)\cdot\tau)\}\ ,$$

and from (1.2), (2.15), (2.20) and (2.21), we have

$$
\begin{aligned}
(\phi \circ \psi, \tau) &= \Lambda((\phi \circ \psi) \circ \tau) = \tfrac{1}{4}\Lambda^2(\Lambda(\phi \cdot \psi) \cdot \tau) \\
&= \tfrac{1}{2} n(n-1)(n-2) B_3(\phi, \psi, \tau) .
\end{aligned}
$$

On the other hand we have

$$
\begin{aligned}
(\phi, \psi \circ \tau) &= \Lambda(\phi \circ (\psi \circ \tau)) = \tfrac{1}{4}\Lambda^2(\phi \cdot \Lambda(\psi \cdot \tau)) \\
&= \tfrac{1}{2} n(n-1)(n-2) B_3(\phi, \psi, \tau) .
\end{aligned}
$$

This shows (1.3).

Now we prove (1.4). From (1.1), (1.2), (2.18) and (2.21), it follows

$$
\begin{aligned}
(\phi, \psi) &= \Lambda(\phi \circ \psi) = \tfrac{1}{2}\{\Lambda\phi \cdot \Lambda\psi + \Lambda\psi \cdot \Lambda\phi - \Lambda^2(\phi \cdot \psi)\} \\
&= \Lambda\phi \cdot \Lambda\psi - \tfrac{1}{2}\Lambda^2(\phi \cdot \psi) \\
&= n^2 B_2(\phi, \omega) B_2(\psi, \omega) - n(n-1) B_2(\phi, \psi) .
\end{aligned}
$$

We choose a base $\{e_0, \cdots, e_r\}$ of $\mathfrak{H}^{1,1}(V, Q)$ such that

$$
\begin{aligned}
&e_0 = \omega , \\
&e_i : \ \text{primitive for } 1 \leq i \leq r ,
\end{aligned}
$$

and express the bilinear forms $n^2 B_2(\phi, \omega) B_2(\psi, \omega)$, $n(n-1) B_2(\phi, \psi)$, and (ϕ, ψ) by matrices with respect to this base.

Then by virture of (2.14), (2.15) and (2.16), we have

$$
\left(n^2 B_2(e_i, \omega) B_2(e_j, \omega) \right) = \left(\begin{array}{c|c} n^2 & 0 \\ \hline 0 & 0 \end{array} \right) ,
$$

and

$$
\left(n(n-1) B_2(e_i, e_j) \right) = \left(\begin{array}{c|c} n(n-1) & 0 \\ \hline 0 & (*) \end{array} \right) ,
$$

where the matrix $(*)$ is negative definite.

So the matrix

$$
\left((e_i, e_j) \right) = \left(\begin{array}{c|c} n & 0 \\ \hline 0 & -(*) \end{array} \right) ,
$$

is positive definite. The proof of Theorem 1 is completed.

We prove Theorem 2. Let $\mathfrak{H}_1$ be a simple ideal of $\mathfrak{H}^{1,1}(V, \boldsymbol{Q})$. Putting

$\mathfrak{H}_1^{\perp} = \{\phi \in \mathfrak{H}^{1,1}(V, \boldsymbol{Q}) \mid (\phi, \psi) = 0$ for every ψ in $\mathfrak{H}_1\}$, $\mathfrak{H}_1^{\perp}$ is also an ideal of $\mathfrak{H}^{1,1}(V, \boldsymbol{Q})$, since the bilinear form $(\,,\,)$ is a trace form. Moreover taking an element ϕ in $\mathfrak{H}_1 \cap \mathfrak{H}_1^{\perp}$, we have

$$(\phi, \phi) = 0 ,$$

and

$$\phi = 0 ,$$

because the bilinear form $(\,,\,)$ is positive definite. Hence the algebra $\mathfrak{H}^{1,1}(V, \boldsymbol{Q})$ is decomposed into

$$\mathfrak{H}^{1,1}(V, \boldsymbol{Q}) = \mathfrak{H}_1 + \mathfrak{H}_1^{\perp} .$$

Repeating this method, we obtain the decomposition (1.5) such that

$$\mathfrak{H}^{1,1}(V, \boldsymbol{Q}) = \mathfrak{H}_1 + \cdots + \mathfrak{H}_k .$$

Let $\mathfrak{H}$ be any simple ideal of $\mathfrak{H}^{1,1}(V, \boldsymbol{Q})$. Then for each ideal $\mathfrak{H}_i$, it follows

$$\mathfrak{H} \cap \mathfrak{H}_i = 0 ,$$

or

$$\mathfrak{H} \cap \mathfrak{H}_i \neq 0 .$$

In case $\mathfrak{H} \cap \mathfrak{H}_i \neq 0$, it follows

$$\mathfrak{H} \cap \mathfrak{H}_i = \mathfrak{H} = \mathfrak{H}_i ,$$

because $\mathfrak{H}$ and $\mathfrak{H}_i$ are both simple ideals. From this the uniqueness of the decomposition (1.5) follows. The proof of Theorem 2 is completed.

Finally we present two problems. Let D be an ample divisor whose chern class is ω. Then corresponding to the decomposition (1.6) of ω, D can be written as follows

$$D = D_1 + \cdots + D_k ,$$

where

$$D_i = \sum_j q_{ij} D_{ij} \qquad (q_{ij} \in \boldsymbol{Q}) ,$$

(D_{ij} is a cycle of codimension one) and

$$c(D_i) = \omega_i \ .$$

Multiplying D by a suitable integer, we may assume that q_{ij} is an integer for all i, j.

PROBLEM 1. When we write D as above, is each divisor D_i effective?
If Problem 1 is affirmative, we can consider the following problem.

PROBLEM 2. We denote

$$V_i = \operatorname{Proj}\left(\bigoplus_{m=0}^{\infty} L(mD_i)\right) \qquad \text{for } 1 \leq i \leq k \ ,$$

(Cf, [5]).
Then, are there any mappings from V to $V_1 \times \cdots \times V_k$?

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